МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
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# THEORY OF ELASTICITY

Навчальний посібник

*Рекомендовано Вченою радою НУК*

ВИДАВНИЦТВО
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Викладено англійською мовою базові поняття та залежності класичної теорії пружності. Додатково розглянуто плоску задачу, чисте кручення, застосування чисельних та експериментальних методів в теорії пружності.

Призначено для студентів кораблебудівних, машинобудівних та інших спеціальностей.

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**ABSTRACT**

In the lectures “Theory of Elasticity” the linear or so called classic theory of elasticity is reviewed. There are basic conceptions and definitions are explained in frames necessary to study the structural mechanics within the course of structural mechanics offered by the National University of Shipbuilding Named After Admiral Makarov for shipbuilding specialties. The subject “Theory of Elasticity” initiates the sequence of strength disciplines taught in the University. The course is designed for 30 academic hours and consists of 15 lectures. It is designated for students of 3-rd courses of shipbuilding specialties.

## SYLLABUS OF THE COURSE "THEORY OF ELASTICITY"

| # | Lecture name | Lecture content |
|---|---|---|
| | **LECTURE 1.** INTRODUCTION TO STRUCTURAL MECHANICS AND TO BASIC IDEAS OF THE THEORY OF ELASTICITY | What Does Structural Mechanics Deal With? The Brief Description and the General Scheme of the Strength Analyses of a Ship Hull and its Structures. Basic Concepts, Definitions and Denotations in the Theory of Elasticity. Initial Remarks and Terms. Properties of Elastic Bodies |
| | **LECTURE 2.** CONCEPTIONS ABOUT STRESSES, DISPLACEMENTS AND STRAINS | Factors Which Cause Strain-Stress State of Elastic Body. Stresses. Elastic Displacements and Strains. Stress and Strain Tensors |
| | **LECTURE 3.** EQUILIBRIUM EQUATIONS IN THE THEORY OF ELASTICITY | Equilibrium Equations of the Whole Body. Equilibrium Equations of Elementary Tetrahedron. Equilibrium Equations of Elementary Parallelepiped |
| | **LECTURE 4.** PHYSICAL AND GEOMETRICAL RELATIONS | Hook's Law for Isotropic Bodies. Cauchy Geometrical Relations Between Strains and Displacements. Compatibility Conditions Expressed in Terms of Strains. Definition of Displacements for Known Strains. Additional Kinematic Relations for Multiply Connected Bodies. General Equations of Geometrically Nonlinear Theory of Elasticity |

| # | Lecture name | Lecture content |
|---|---|---|
| | **LECTURE 5.** DIRECT AND INVERSE PROBLEMS OF THE THEORY OF ELASTICITY | Direct and Inverse Problems of the Theory of Elasticity. Direct Problem. Inverse Problem of the Theory of Elasticity. Saint-Venant Principle. Half-Inverted Saint-Venant Method. Superposition Principle |
| | **LECTURE 6.** SIMPLE PROBLEMS OF THE THEORY OF ELASTICITY | Pure Bending of Prismatic Beam. Torsion of Round Prismatic Bar |
| | **LECTURE 7.** RELATIONS FOR STRESS STATE IN POINT | Stress State at Rotation of Axes. Stress State in Inclined Elemental Areas. Principal Planes and Principal Stresses. Stress Ellipsoid. Maximum Shear Stresses and Its Planes. Octahedron Stresses and Intensity of Stresses |
| | **LECTURE 8.** RELATIONS FOR STRAIN STATE IN POINT | Transformation of Displacements and Strains at Rotation of Coordinate Axes. Principal Linear and Maximum Shear Strains. Octahedron Strain and Intensity of Strains. Deformation of Volume and Form. Specific Elastic Energy of a Body. Basic Theories of Strength and Plasticity |
| | **LECTURE 9.** PURE TORSION OF PRISMATIC BARS. BASICS | Initial Explanations. Principal Equations of the Theory of Pure Torsion. Boundary Conditions. Torsion of Thin-Walled Simply Connected Compound Bars |
| 0 | **LECTURE 10.** ADDITIONAL QUESTIONS OF PURE TORSION | Strains and Displacements. Circulation of Shear Stresses for Multiply Connected Bars. Common Algorithm for Calculation of Pure Torsion. Torsion of Thin-Walled Multiply Connected Bars |
| 1 | **LECTURE 11.** TWO-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY IN RECTANGULAR COORDINATES | Basic Concepts of Plane Strain-Stress State. Major Equations and Relations. Plane Strain. Schemes of Solution of Direct Problem of Plane Strain-Stress State. Stress Function. Plain Problem of the Theory of Elasticity in Polynomials |

| # | Lecture name | Lecture content |
|---|---|---|
| 2 | **LECTURE 12.** TWO-DIMENTIONAL PROBLEM IN TRIGONOMETRICAL SERIES. EFFECTIVE WIDTH OF WIDE BEAM FLANGES | Solution of Two-Dimensional Problem for Rectangular Plates in Trigonometric Series. Effective Width of Wide Beam Flanges |
| 3 | **LECTURE 13.** TWO-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY IN POLAR COORDINATES | Basic Relations of Plane Strain-Stress State in Polar Coordinates. Stress Concentrations in Plates With Holes |
| 4 | **LECTURE 14.** NUMERICAL METHODS IN THE THEORY OF ELASTICIY | The General Description of Structures Strength Analysis Methods. Ritz Method. Generalized Bubnov-Galerkin Method. The Finite-Difference Method. The Finite Element Method |
| 5 | **LECTURE 15.** EXPERIMENTAL METHODS IN THE THEORY OF ELASTICIY | Basic Information about Experimental Methods of Explorations. Polarization-Optical Method. Electrotensometry. String Tensometers. Inductive Tensometers. Brittle Coating Method. Moire Fringe Method |

# LECTURE 1. INTRODUCTION TO STRUCTURAL MECHANICS AND TO BASIC IDEAS OF THE THEORY OF ELASTICITY

**What Does Structural Mechanics Deal With?** Structural mechanics of ships is the basic shipbuilding scientific discipline that deals with the problems of the **strength** and **rigidity** of ship hulls and warships and hull structures and their **stability** problem additionally. Also structural mechanics deals with problems of **fatigue**, **vibration** and **shock** loads of ship hull in general and ship hull constructions in particulars. The last three courses are studied in special parts of structural mechanics. Influence of temperature, radiation, electromagnetic fields, chemistry factors (for example corrosion) etc. can be considered.

The static **strength** of the ship hull and its structures we should understand as the ability of solids to resist fracture (separation into parts) under the action of operational and other loads that arise during the life of the solid.

The required **rigidity** of the ship hull and its structures assumes limitation of deformations in these structures. Such deformations don't have to block carriage of assigned ship functions in the operation process.

**Stability** of a structure is its ability to support a given load (which the structure can experience during its life) without appearance of a sudden change in its configuration or shape (i.e. without collapse).

**Fatigue** is connected with ability of constructions to keep its wholeness under the cyclic external loads for a given number of cycles. If the fatigue strength failed there are cracks appear which may grow and to cause failure of the construction. Failure of fatigue strength is the reason of many accidents with ships.

Calculation of **vibration** is mainly connected with fatigue strength but not only. For vibration calculations we calculate generally free vibration frequencies of constructions and other factors optionally to avoid resonance. Resonance cause increasing of displacements and as consequence increasing of stresses and strains. Resonance may cause: fatigue strength failure both for the whole ship hull and for separate ship hull constructions; threat for crew healthy.

**Shock** loads calculations are also important. Shock loads may cause serious damages of ship hull shells and cargo (even loss of ship). Shock loads are caused by slamming in natural conditions of exploitation, but also shock loads are connected with anthropogenic factors: such as detonation, ship collisions, etc. Shock loads caused by slamming cause great percentage of ship accidents.

**The Brief Description and the General Scheme of the Strength Analyses of a Ship Hull and its Structures**. There are several simplifying model bodies for the solid deformable bodies: rods, plates, shells (see fig. 1.1). These simplifying models have its kinds, divisions etc. It's important to know where to apply one model approximation or another. For example the same solid body may be considered as a rod and as a plate. There are definitions for rods, plates and shells, but sometimes given solid deformable body may be considered to be a plate and a rod simultaneously, etc.

A ship hull is a complex solid three-dimensional structure. Shell plating provides water impermeability of a ship. It is a thin covering that have top deck, sides and bottom. To provide rigidity (and in some cases static strength) of ship hull for a given external loads (water pressure for example) it is reinforced by framing. Framing connected with shell form shared system called grillage.

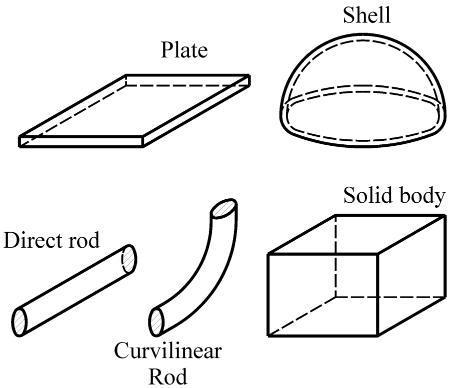



Fig. 1.1 – Simplifying models of solid bodies

All the continuous shell plating is usually separated into discrete flat plates by framing which can be reviewed independently from each other (for the simplest approaches). The form of those discrete plates is usually rectangular, but also other forms are possible. So called “support contour” for these rectangular plates (areas) is mutually perpendicular beams (framing) affixed to ship hull shell.

Plane of plates may experience two kinds of deformation: in its own plane (tension-compression) and in orthogonal directions to own plane (deflections caused by bending or/and buckling). Deformations in own plane are caused by interaction with adjacent areas. Deformations in orthogonal directions are caused mainly by orthogonal pressure or compressive external loads acting along own plane, which cause loss of stability. Stiffeners and other beams of the framing also have two kinds of deformations: axial ones caused by axial forces; deflections caused by orthogonal loads acting into plates or by compressive axial forces, which cause buckling (loss of stability).

In fact for most of ship types ship hull length is more than in 5 times than its side height. When a ship hull is loaded both on still water and by waves, it experience bending in centerline plane (C. P.). Thus we can consider the ship hull as the beam with variable cross-section along the ship length and without any supports as a first approximation when the common strength of hull girder is investigated (see fig. 1.2).

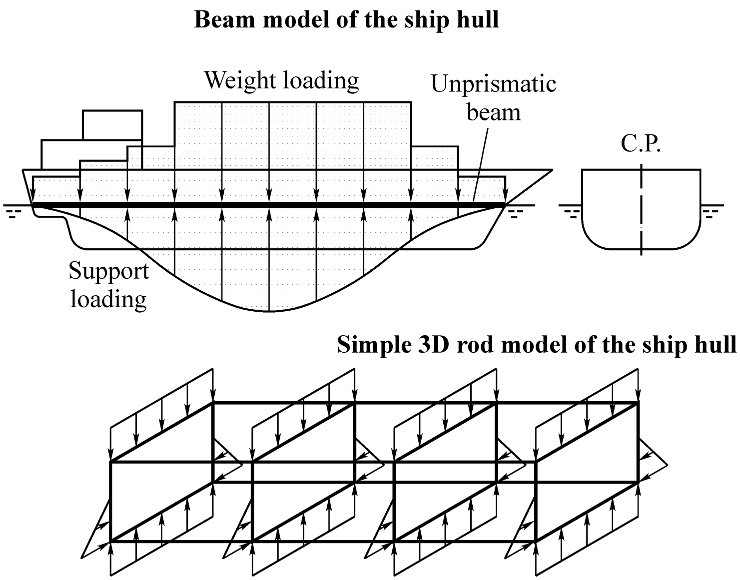



Fig. 1.2 – Several models of the same ship hull

The same ship hull may be considered as reinforced shell, 3*D* rod grillage or several 2*D* flat grillages, 3*D* solid body, etc (see fig. 1.2). These methods may have different complexity. Flat areas of shell can be considered as plates.

Considering the ship hull to be $3D$ solid body is the most precise approximation in strength estimation, but very labor-consuming and on reality almost is not used. Considering the ship hull to be reinforced shell gives (almost) the same results for its strain-stress as for ship hull as $3D$ solid body. That's why it's often used in practical calculations. The adequacy of results of strain-stress state applying rod models for ship hull or its parts ($3D$ or $2D$ rod models, different grillages) is less, but appropriate for imprecise strength estimations when we don't need particulars. Rod systems provide the least labor-consuming and the fastest results in comparison with methods described above. Combining different physical approximations we shall have complex reliable model in strength estimation.

Strength problems can be solved using different mathematical models and approaches. The simplest discipline which allows us to calculate strength is "Strength of materials". But this discipline is based on simple models and assumptions. Methods of "Strength of materials" are insufficient to solve complex problems which take place in structural mechanics of ships.

On the contrary the discipline "The theory of elasticity" allows us to calculate static strength very particular. But for engineering calculations the discipline "The theory of elasticity" is very complex and connected with using of complex mathematical computations. But engineers have to know "The theory of elasticity" and to be able to solve simple tasks for this discipline to understand what is going on in structural mechanics.

In structural mechanics we mostly operate with rods, plates and shells, but sometimes solid body models are applied for very complex geometries and loads.

In order strength disciplines have to be studied in next sequence: “Strength of materials”’!“The theory of elasticity”’!“Structural mechanics” (rod systems, plates, shells).

**Basic Concepts, Definitions and Denotations in the Theory of Elasticity.** Mechanic of solid deformable body (MSDB) is the science which explores strain-stress state of real bodies under external forces and temperature field. General assumptions of MSDB are: *continuity* and *elasticity*.

***Continuity*** is ability of solid bodies to fill in by material all available volume of the body without emptiness. The material of the body is continuum which have no discrete (molecular or atomic) structure.

***Elasticity*** is ability of solid bodies to recreate their initial sizes and forms after external loadings are removed. Thus initial state of body is suggested to be unstressed. This state is usually named *natural*.

There are three general branches of MSDB: theory of elasticity, theory of plasticity and theory of creep.

***Theory of elasticity*** is science which explores behavior of solid deformable bodies in which there are only elastic strains.

**Initial Remarks and Terms.** Further we shall use Cartesian coordinated system $Oxyz$ for getting of common relations in mechanics of solid deformable body. Coordinates of internal body points are denoted as $x, y, z$. But coordinates of outer body surface points are denoted as $\overline{x}$, $\overline{y}$, $\overline{z}$ .

***Cross section*** of the body is any mental surface which divides body into two pieces. For every section there are two appropriate coinciding inside surfaces. These surfaces belong to two pieces of a body and situated on both different sides of the body section.

***Outer normal vector*** $\nu$ is the direction which is orthogonal to outer or to some internal surface of body in given point and coming out. Direction of outer normal vector usually are set by direction cosines $l = \cos(\vec{\nu},\wedge \vec{x})$, $m = \cos(\vec{\nu},\wedge \vec{y})$, $n = \cos(\vec{\nu},\wedge \vec{z})$. *Moreover all vectors which are presented within this work are set by direction cosines*. Sometimes vectors are denoted by bold font.

***Outer plane*** is a piece of outer surface of body. Inside plane is a piece of some internal surface (section) of body.

***Elementary plane*** is an infinitely small plane. Elementary plane can always be considered to be plane. Position of an elementary plane is set by coordinates of one of points which belong to it, and orientation in space is set by outer normal vector $\boldsymbol{\nu}(l; m; n)$. All quantities related to elementary plane with given normal vector are denoted with index ν.

***Fiber*** is a continuous aggregate (totality) of material points situated along some line.

***Linear element*** of body is an infinitely small segment of fiber. It can be considered direct.

***Elementary volume*** is infinitely small piece of body, which have infinitely small sizes and volume. There are two kinds of elementary volumes: elementary parallelepiped and elementary tetrahedron. A body consists on these two elementary volumes. Outer surface is usually approximated by elementary tetrahedrons but inside volume consists on numerous of elementary parallelepipeds. All faces of elementary parallelepiped and internal faces of elementary tetrahedron are always parallel to coordinate planes and mutually orthogonal to each other.

**Properties of Elastic Bodies.** There are four common properties of ideal elastic bodies in frames of classic theory of elasticity:

**1. Homogeneity** means that elastic properties in all points of a body along some arbitrary direction are the same. Cutout specimens in the same direction in different points of such body under the tension-compression state have the same strains under the same loadings.

**2. Isotropy** means that elastic properties of specimens defined/cutout in locality of some point are the same in all directions. Combining properties of homogeneity and isotropy we can consider that in an ideally elastic body elastic properties of specimen don’t depend on that in what point of a body and in what direction it was defined.

**3. Physical linearity** means that there is a direct dependency between strains and stresses which appear in a body. For example when we change stresses in $n$ times, the stresses also change in $n$ times. If stresses disappear strains also disappear.

**4. Geometrical linearity** or relative rigidity means that relative linear strains and shear strains of linear elements are small in comparison with one; also absolute linear displacements of a body points are small in comparison with sizes of this body.

# LECTURE 2. CONCEPTIONS ABOUT STRESSES, DISPLACEMENTS AND STRAINS

**Factors Which Cause Strain-Stress State of Elastic Body.** There are several factors which causes strain-stress state: external forces and temperature field (see fig. 2.1). External forces divide into two kinds: external surface forces and external volume forces. In the theory of elasticity they usually use projections of forces into coordinate axes. Projections of full intensity of surface external forces $\boldsymbol{p}_\nu$ into coordinate axes $x$, $y$, $z$ are denoted as $p_{\nu x}$, $p_{\nu y}$, $p_{\nu z}$ (N/m$^2$). Surface forces are functions of coordinates of body surface points: $p_{\nu x} = p_{\nu x}(\bar{x}, \bar{y}, \bar{z})$, $p_{\nu y} = p_{\nu y}(\bar{x}, \bar{y}, \bar{z})$, $p_{\nu z} = p_{\nu z}(\bar{x}, \bar{y}, \bar{z})$. Projections of full intensity of external volume forces $\boldsymbol{R}$ into coordinate axes $x$, $y$, $z$ are denoted as $X$, $Y$, $Z$ (N/m$^3$). Volume external forces are also functions of body points: $X = X(x, y, z)$, $Y = Y(x, y, z)$, $Z = Z(x, y, z)$. Surface external forces act only to surface points of a body, so these forces act only for elementary tetrahedrons. But volume external forces act for every point of a body. Volume forces act both on elementary tetrahedrons and elementary parallelepipeds.

Temperature field usually is set as a function which depends on coordinates of body points $T = T(x, y, z)$, $K$. Temperature field acts for every point of a body, but it cause changes of linear strains only. Linear strains depend on temperature linearly.

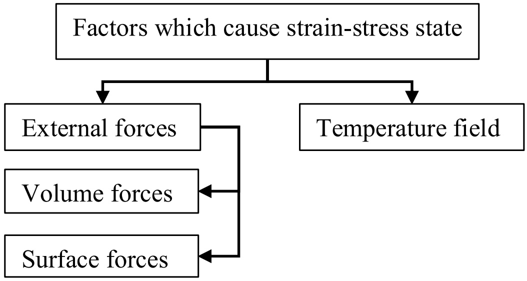



Fig. 2.1 – Factors which cause strain-stress state

**Stresses.** For example we have a body loaded by external forces and temperature field (see fig. 2.2).

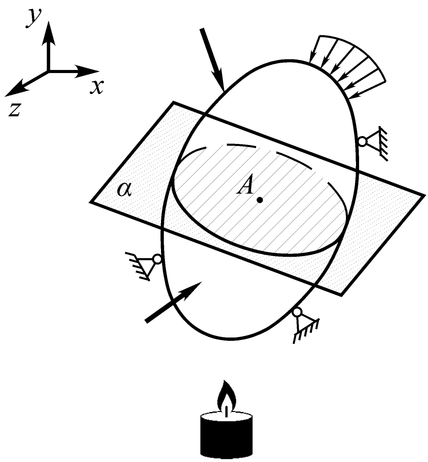



Fig. 2.2 – Body loaded by external loadings
and temperature field

These factors cause deformation of a body. Let's cut mentally our body by some plane cross plane α taking some point $A$ on the plane which belong to one of pieces of a body. Within elementary plane defined in locality of the point $A$ there is an internal force.

Directions of a given elementary plane and internal force are different. Relation of the magnitude of this internal force to the area of elementary plane is called full stresses $\boldsymbol{p}_\mathbf{v}$, *Pa* (see fig. 2.3). So stresses are measure of intensity of internal forces.

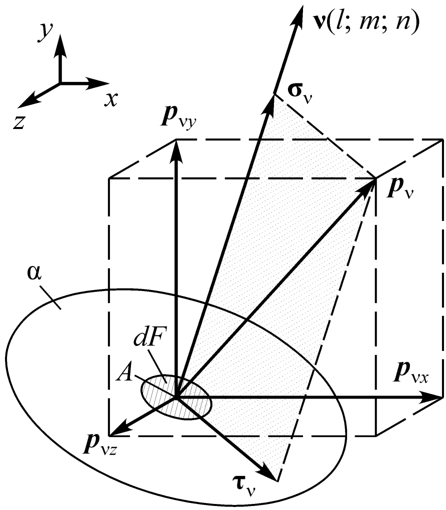



Fig. 2.3 – For explanation of stresses

There are three projections of full stresses $\boldsymbol{p}_\mathbf{v}$ into coordinate axes *x*, *y*, *z*: $\boldsymbol{p}_\mathbf{vx}$, $\boldsymbol{p}_\mathbf{vy}$, $\boldsymbol{p}_\mathbf{vz}$. Projection of full stresses into normal direction vector ν of elementary plane is called full normal stresses and denoted as $\boldsymbol{\sigma}_\mathbf{v}$. Projection of full stresses into cross section plane α is called full shear stresses $\boldsymbol{\tau}_\mathbf{v}$. Full stresses $\boldsymbol{p}_\mathbf{v}$, full normal stresses $\boldsymbol{\sigma}_\mathbf{v}$ and full shear stresses $\boldsymbol{\tau}_\mathbf{v}$ are connected by the next relations

$$p_\nu^2 = p_{\nu x}^2 + p_{\nu y}^2 + p_{\nu z}^2 = \sigma_\nu^2 + \tau_\nu^2, \qquad (2.1)$$

$$\sigma_\nu = p_{\nu x} l + p_{\nu y} m + p_{\nu z} n, \qquad (2.2)$$

$$\tau_\nu^2 = p_\nu^2 - \sigma_\nu^2. \qquad (2.3)$$

Direction cosines of full stresses can be defined by relations

$$l_p = \frac{p_{\nu x}}{p_\nu}; \quad m_p = \frac{p_{\nu y}}{p_\nu}; \quad n_p = \frac{p_{\nu z}}{p_\nu}. \qquad (2.4)$$

Let's review coordinate planes of some elementary parallelepiped to explain stress components on it. On every such plane there are known full stresses $\boldsymbol{p}_\nu$ as it is shown on the fig. 2.4.

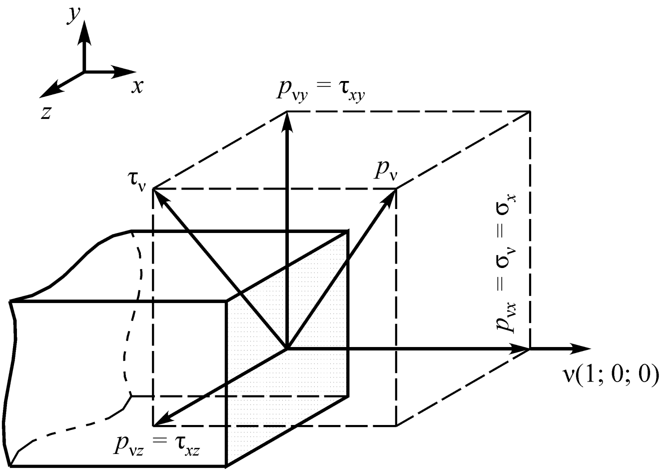



Fig. 2.4 – Stresses in plane of elementary tetrahedron

On the example of coordinate plane of elementary parallelepiped (see fig. 2.4) which is perpendicular to coordinate axis $x$ with direction cosines $\nu(1; 0; 0)$ we shall show how to get components of stress tensor of elementary parallelepiped.

Projecting full stresses $\boldsymbol{p}_\nu$ into normal vector $\boldsymbol{\nu} = \boldsymbol{x}$ we shall get full normal stresses $\boldsymbol{\sigma}_\nu = \boldsymbol{\sigma}_x$. Projecting full stresses $\boldsymbol{p}_\nu$ into plane perpendicular to $\boldsymbol{\nu} = \boldsymbol{x}$ we shall get full shear stresses $\boldsymbol{\tau}_\nu = \boldsymbol{\tau}_x$. Thereafter projecting $\boldsymbol{\tau}_\nu = \boldsymbol{\tau}_x$ into axes $y$ and $z$ we shall have components/projections of full shear stresses into coordinate axes accordingly $\boldsymbol{\tau}_{xy}$ and $\boldsymbol{\tau}_{xz}$.

If to review all coordinate planes of elementary parallelepiped defined in locality of some point we shall have ordered system of stresses on planes of this parallelepiped (see fig. 2.5). Index of normal stresses shows the direction of that normal stresses and the direction of normal vector of plane in which normal stresses are situated. First index of shear stresses shows the direction of normal vector of plane in which shear stresses act. But the second index of shear stresses shows its direction. For normal stresses there is the next sign rule: normal stresses have to be considered to be positive if they cause tension of elementary parallelepiped along given axis. The rule for shear stresses we shall explain at the example of $\tau_{xy}$: if *directions* of *normal vector* of plane in which shear stresses act *and* appropriate coordinate *axis x* of this normal vector are the same (opposite), for the shear stresses to be positive its direction (the 2-nd index) have to fit (to be opposite) to direction of axis *y*.

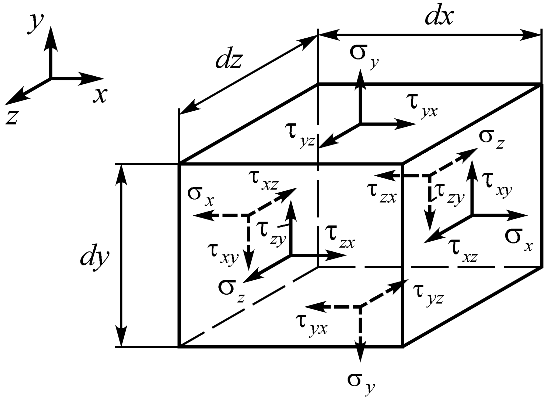



Fig. 2.5 – For explaining of sign rule for stresses

Stresses are functions which depends on coordinates of body points: $\sigma_x = \sigma_x(x, y, z)$, $\sigma_y = \sigma_y(x, y, z)$, $\sigma_z = \sigma_z(x, y, z)$, $\tau_{xy} = \tau_{xy}(x, y, z)$, $\tau_{yz} = \tau_{yz}(x, y, z)$, $\tau_{zx} = \tau_{zx}(x, y, z)$.

**Elastic Displacements and Strains.** For example we have some body loaded by external forces and temperature field (see fig. 2.2). These factors cause deformation of body. Deformation means movements of points. Distance between start position of some point of body before deformation and end position of this point after deformation is called full displacement. We shall denote full displacement vector as **ρ**. There are three projections of full displacement **ρ** into three coordinate axes $x$, $y$, $z$: $u = u\ (x, y, z)$, $v = v\ (x, y, z)$, $w = w\ (x, y, z)$, m.

On the fig. 2.6 there is full displacement **ρ** and its projections into coordinate axes are shown. Point $A_1$ is an end position of the point $A$ after deformation. Distance between points $A$ and $A_1$ is full displacement **ρ** of the point $A$. Full displacement as a vector has its module (or magnitude) and direction cosines, which can be found using next expressions

$$l_\rho = \frac{u}{\rho}; \quad m_\rho = \frac{v}{\rho}; \quad n_\rho = \frac{w}{\rho}. \tag{2.5}$$

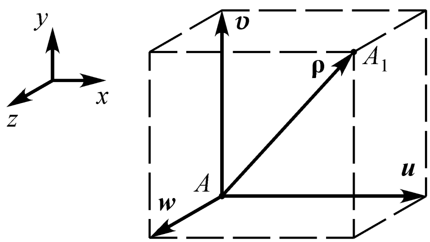



Fig. 2.6 – Projections of displacement in some point $A$ of a body

Also there are other cinematic parameters of a body. One of them is relative linear strain. Let's review what that is. For example we have some linear element $AB$ in a body (see fig. 2.7) before deformation. After deformation points $A$ and $B$ move and take new positions $A_1$ and $B_1$ appropriately. The difference between end length of some linear element after deformation and start length of the same element before deformation is called absolute *linear strain*. Absolute linear strain can be calculated using expression

$$\Delta l_{AB} = A_1B_1 - AB\text{, } m, \tag{2.6}$$

where $A_1B_1$ – length of linear element $AB$ after deformation (length of vector $\boldsymbol{A_1B_1}$); $AB$ – length of linear element $AB$ before deformation (length of vector $\boldsymbol{AB}$). Points $A_1$ and $B_1$ are new positions of points $A$ and $B$ appropriately after deformation.

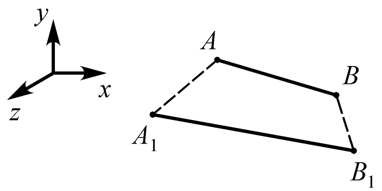



Fig. 2.7 – For explaining of linear strains

Relative linear strain is the relation of absolute linear strain to initial length of a linear element before deformation. It can be calculated using expression (see fig. 2.7)

$$\varepsilon_{AB} = \frac{\Delta l_{AB}}{AB} = \frac{A_1B_1 - AB}{AB}. \tag{2.7}$$

Let's review what angular (shear) strains are. For example we have two mutually orthogonal linear elements before deformation $AB \perp BC$ (see fig. 2.8). After deformation these linear elements change its lengths and directions. The angle between these linear elements will be another than orthogonal. The change of initial orthogonal angle between two mutually orthogonal linear elements after deformation called angular strain.

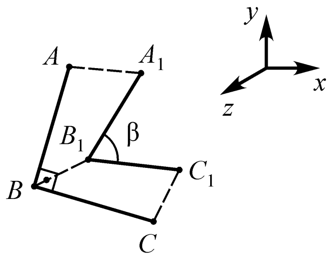



Fig. 2.8 – For explaining of angular strains

Angular strains can be computed using formula

$$\gamma_{AB,A_1B_1} = \frac{\pi}{2} - \beta \text{, rad.} \tag{2.8}$$

As a conclusion at the scheme presented on the fig. 2.9 the classification of strains is presented

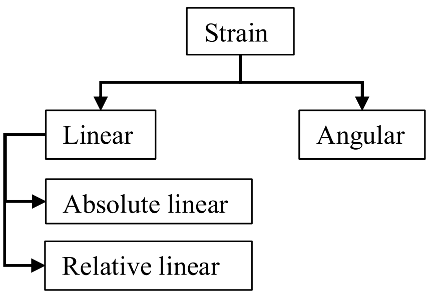



Fig. 2.9 – Classification of strains

There are projections of linear and angular strains into coordinate axes $x$, $y$, $z$ and coordinate planes. These projections are functions depending on coordinates of body points: $\varepsilon_x = \varepsilon_x(x, y, z)$, $\varepsilon_y = \varepsilon_y(x, y, z)$, $\varepsilon_z = \varepsilon_z(x, y, z)$, $\gamma_{xy} = \gamma_{xy}(x, y, z)$, $\gamma_{yz} = \gamma_{yz}(x, y, z)$, $\gamma_{zx} = \gamma_{zx}(x, y, z)$.

**Stress and Strain Tensors.** So in every point of a body we can define elementary parallelepiped or elementary tetrahedron. At faces of such elementary volumes there are stresses. Stresses cause appearance of strains. To describe strain-stress state in point there are usually tensors are used. There are two kinds of tensors: stress tensors and strain tensors.

Stress tensor $T_\sigma$ can be presented as

$$T_\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}. \tag{2.9}$$

Strain tensor $T_\varepsilon$ can be presented as

$$T_\varepsilon = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{z.} \end{bmatrix}. \tag{2.10}$$

There is one general property of stress and strain tensors: these tensors are symmetric relatively main diagonal owing to the *reciprocity law for shearing stresses* (*law of twinning shear stresses*) (see lecture 3). So in every point we have six independent stresses and strains.

# LECTURE 3. EQUILIBRIUM EQUATIONS IN THE THEORY OF ELASTICITY

As it was told above a body consists of numerous of elementary parallelepipeds and elementary tetrahedrons. If a body is affected by factors which cause strain-stress state (external loadings and temperature field) there are internal forces of elasticity appear in that body by which body resists to external factors. So a body is always equilibrated by systems of external and internal forces. If the whole body is equilibrated it means that every elementary volume is equilibrated.

**Equilibrium Equations of the Whole Body.** To compose equilibrium equations of the whole body we consider it to be undeformable solid body and review it from point of view of the theoretical mechanics. There are six equilibrium equations for solid body we can write:

$$\left.\begin{array}{ll}
\int\limits_F p_{\nu x}\, dF + \int\limits_V X\, dV = 0, & \int\limits_F \left(p_{\nu z} y - p_{\nu y} z\right) dF + \int\limits_V \left(Zy - Yz\right) dV = 0, \\
\int\limits_F p_{\nu y}\, dF + \int\limits_V Y\, dV = 0, & \int\limits_F \left(p_{\nu x} z - p_{\nu z} x\right) dF + \int\limits_V \left(Xz - Zx\right) dV = 0, \\
\int\limits_F p_{\nu z}\, dF + \int\limits_V Z\, dV = 0, & \int\limits_F \left(p_{\nu y} x - p_{\nu x} y\right) dF + \int\limits_V \left(Yx - Xy\right) dV = 0,
\end{array}\right\} \quad (3.1)$$

where $F$ – is the body surface field for the surface integral; $V$ – is the body volume field for the volume integral.

**Equilibrium Equations of Elementary Tetrahedron.** Let's define in locality of some point $D$ elementary tetrahedron (see fig. 3.1). There are stresses and projections of external surface forces shown. Neglecting with volume forces it is necessary to compose three equilibrium equations of forces.

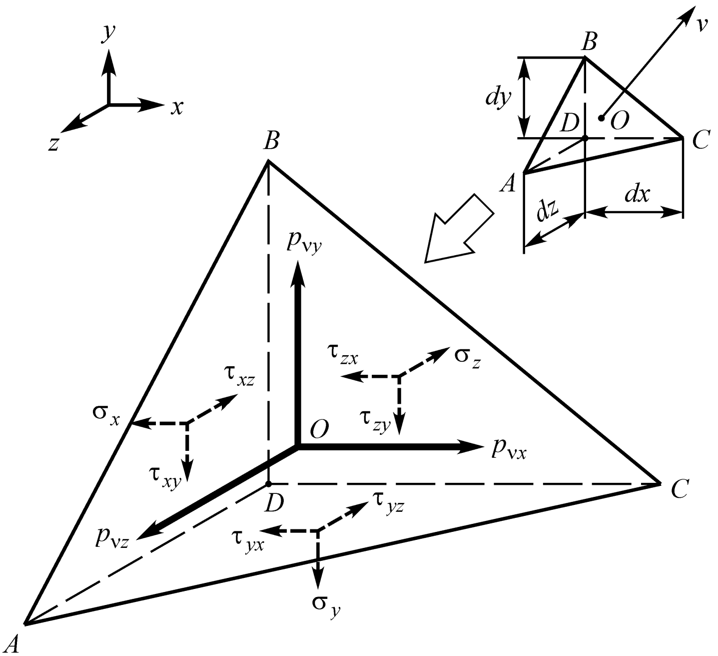



Fig. 3.1 – Elementary tetrahedron

For example we shall review one of these equations

$$\Sigma \text{ Forces}_x =$$

$$= 0: \quad -\sigma_x \cdot F_{ADB} - \tau_{yx} \cdot F_{ADC} - \tau_{zx} \cdot F_{CDB} + p_{\nu x} \cdot F_{ABC} = 0 \quad (3.2)$$

where $F_{ADB}$ – is the area of plane $ADB$, etc.

Denotation Σ $Forces_x = 0$ means sum of forces in projection on axis $x$ equals zero.

Because $ADB$, $ADC$, $CDB$ are projections of surface (plane) $ABC$ into coordinate axes, accepting $F_{ADB} = F_x$, $F_{ADC} = F_y$, $F_{CDB} = F_z$ and $F_{ABC} = F_\nu$ we shall have next

$$l = \frac{F_x}{F_\nu}; \quad m = \frac{F_y}{F_\nu}; \quad n = \frac{F_z}{F_\nu}.$$

Dividing (3.2) by $F_{ABC}$ and taking into account above presented expressions for $l$, $m$, $n$ we shall have next

$$-\sigma_x \cdot l - \tau_{yx} \cdot m - \tau_{zx} \cdot n + p_{\nu x} =$$

$$= 0 \Rightarrow p_{\nu x} = \sigma_x \cdot l + \tau_{yx} \cdot m + \tau_{zx} \cdot n . \tag{3.3}$$

Composing the rest equations like (3.2) for axes $y$ and $z$ we shall have two lack equations. Finally we have three equilibrium equations of elementary tetrahedron or so called the first group of equilibrium equations of linear mechanics of deformable body

$$\left.\begin{aligned} \sigma_x l + \tau_{yx} m + \tau_{zx} n = p_{\nu x}, \\ \tau_{xy} l + \sigma_y m + \tau_{zy} n = p_{\nu y}, \\ \tau_{xz} l + \tau_{yz} m + \sigma_z n = p_{\nu z}. \end{aligned}\right\} \tag{3.4}$$

**Equilibrium Equations of Elementary Parallelepiped.** In locality of some point $A$ we define elementary parallelepiped (see fig. 3.2). As it was told above stresses are functions of coordinates of points. These functions of stresses can be expanded in Taylor series in locality of some point. We have to expand the function like $f(x + dx, y + dy, z + dz)$. Within linear mechanics we neglect with nonlinear members of this series and take into account only linear members. At opposite faces of parallelepiped stresses distinguish for a very small magnitude. It is necessary to compose three equations of forces and three equations of moments.

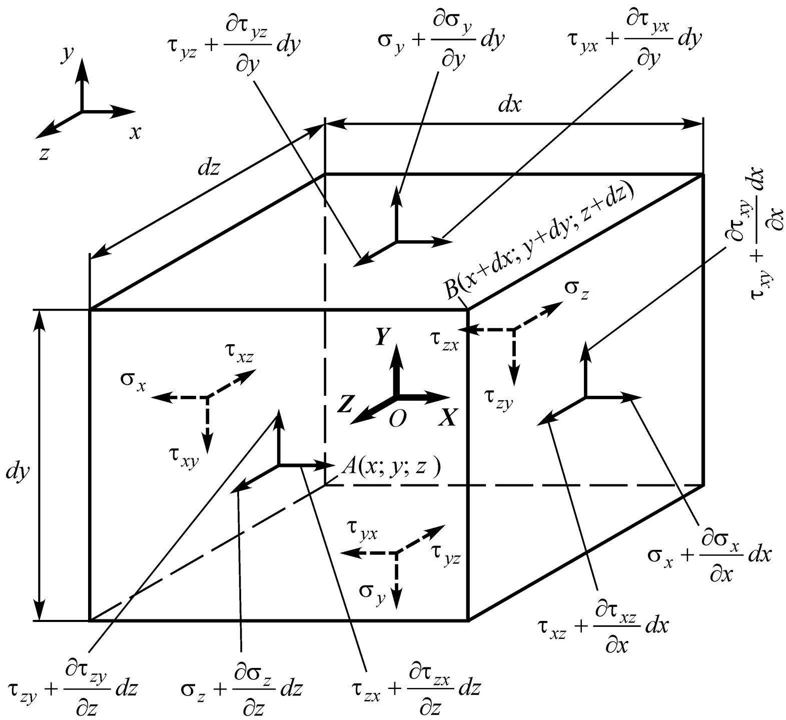



Fig. 3.2 – Elementary parallelepiped

One of equilibrium equations of forces can be written:

$$\left.\begin{aligned} &\Sigma\,\text{Forces}_x = 0: \quad -\sigma_x \cdot dy\,dz - \tau_{yx} \cdot dx\,dz - \tau_{zx} \cdot dx\,dy + \\ &+\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx\,dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\,dz + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy\,dz + \\ &+X \cdot dx\,dy\,dz - \rho \frac{\partial^2 u}{\partial t^2} \cdot dx\,dy\,dz = 0. \end{aligned}\right\} \quad (3.5)$$

Opening brackets, gathering similar variables and dividing by the multiplier $dx{\cdot}dy{\cdot}dz$ in (3.5), we shall have the first equilibrium equation for forces

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}-\rho\frac{\partial^2 u}{\partial t^2}=-X\,. \tag{3.6}$$

Composing the rest equations like (3.5) for axes $y$ and $z$ we shall have two lack equations. Finally we have three equilibrium equations of elementary parallelepiped or so called the second group of equilibrium equations of linear mechanics of deformable body

$$\left.\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}-\rho\frac{\partial^2 u}{\partial t^2}=-X,\\
&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{y}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}-\rho\frac{\partial^2 v}{\partial t^2}=-Y,\\
&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{z}}{\partial z}-\rho\frac{\partial^2 w}{\partial t^2}=-Z.
\end{aligned}\right\} \tag{3.7}$$

Equilibrium equations of moments can be composed relatively mass centre of the parallelepiped situated in point $O$. One of the equilibrium equations of moments can be written:

$$\left.\begin{aligned}
&\Sigma\ \text{Moments}_{xO}=0:\quad -\tau_{zy}\cdot dx\,dy\frac{1}{2}dz+\tau_{yz}dx\,dz\frac{1}{2}dy-\\
&-\left(\tau_{zy}+\frac{\partial\tau_{zy}}{\partial z}dz\right)dx\,dy\frac{1}{2}dz+\left(\tau_{yz}+\frac{\partial\tau_{yz}}{\partial y}dy\right)dx\,dz\frac{1}{2}dy=0.
\end{aligned}\right\} \tag{3.8}$$

Record $\Sigma$ Moments$_{xO}$ = 0 means the sum of moments in projection on axis $x$ relatively point $O$ equals zero.

Opening brackets, gathering similar values, dividing by multiplier $dx{\cdot}dy{\cdot}dz$ and eliminating smidgens of higher order in (3.8), we shall have

$$\tau_{zy} = \tau_{yz}\,. \tag{3.9}$$

Composing equilibrium equations of moments in projections on axes $y$ and $z$ relatively point $O$ similar to (3.8) we shall have the rest two equations.

As the result after composing of equilibrium equations of moments in projections on axes $x$, $y$, $z$ relatively point $O$ we shall have the next system of equations

$$\tau_{zy} = \tau_{yz}; \quad \tau_{zx} = \tau_{xz}; \quad \tau_{xy} = \tau_{yx}. \tag{3.10}$$

The system of equations (3.10) is mathematical formulation of the *reciprocity law for shearing stresses* (*law of twinning shear stresses*): for two decussate planes shear stresses acting on those planes and orthogonal to the line of intersection of decussate planes are directed to each other or from each other.

## LECTURE 4. PHYSICAL AND GEOMETRICAL RELATIONS

**Hook's Law for Isotropic Bodies.** Natural tests show that there is a linear dependency between components of stresses and strains when we have small deformation of elastic bodies for physically linear bodies

$$\left.\begin{aligned}
\varepsilon_x &= c_{11}\sigma_x + c_{12}\sigma_y + c_{13}\sigma_z + c_{14}\tau_{xy} + c_{15}\tau_{yz} + c_{16}\tau_{zx} \\
\varepsilon_y &= c_{21}\sigma_x + \ldots \qquad\qquad\qquad\qquad\qquad\qquad \ldots + c_{26}\tau_{zx} \\
\varepsilon_z &= c_{31}\sigma_x + \ldots \qquad\qquad\qquad\qquad\qquad\qquad \ldots + c_{36}\tau_{zx} \\
\gamma_{xy} &= c_{41}\sigma_x + \ldots \qquad\qquad\qquad\qquad\qquad\qquad \ldots + c_{46}\tau_{zx} \\
\gamma_{yz} &= c_{51}\sigma_x + \ldots \qquad\qquad\qquad\qquad\qquad\qquad \ldots + c_{56}\tau_{zx} \\
\gamma_{zx} &= c_{61}\sigma_x + \ldots \qquad\qquad\qquad\qquad\qquad\qquad \ldots + c_{66}\tau_{zx}
\end{aligned}\right\} \qquad (4.1)$$

The system (4.1) is called the common Hook's law, but factors $c_{ij}$ are elastic constants of material.

Due to linear dependency we have that for a given point of a body and for a given coordinate axes factors $c_{ij}$ have to be constant irrespective of kind of strain-stress state, i.e. of relations between magnitudes of stress components. Strains depend only of magnitudes of stresses acting in present instant time, and completely don't depend on the stresses acted in a body earlier. If said immutability of factors $c_{ij}$ is not provided, which takes place when stresses exceed some magnitude level defined individually for each material, the dependency between stresses and strains stop to be linear.

The consequence of linearity of equations (4.1) is so called the principle of independency of action of stresses: components of strains caused by some totality of stresses equal to the sum of components of strains caused by action of every stress component separately.

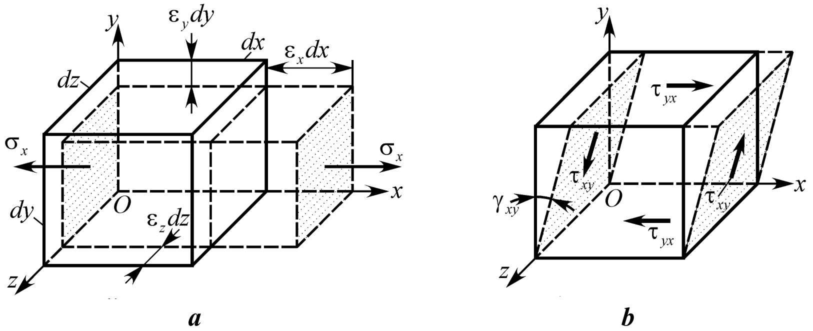



Fig. 4.1 – Elementary parallelepiped
to get relation between strains and stresses

Dependencies (4.1) contain 36 elastic constants, but only 24 of them are independent owing to the energy conservation law, because $c_{ij} = c_{ji}$.

For an ideal elastic isotropic body dependencies (4.1) can be greatly simplified. These dependencies can be obtained if to use known experimental data of the strength of materials related to simple kinds of deformation: to tension-compression and to pure shear, and taking into account immutability of factors $c_{ij}$.

For example let's define factors $c_{i1}$ at action of only stresses $\sigma_x$, when there is $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$. An elementary parallelepiped defined in the selected point of an ideal elastic body is under usual tension-compression in the direction of axis $x$ (see fig. 4.1,***a***).

This parallelepiped is in the same conditions as a specimen material in known tests of the strength of materials for definition of mechanic properties of materials for tension-compression on test machines.

Experiments show that linear strain $\varepsilon_x$ is

$$\varepsilon_x = \frac{\sigma_x}{E},$$

where $E$ – modulus of elasticity in tension, [Pa].

Simultaneously there is change of transversal sizes $dy$ and $dz$ of parallelepiped with contrary sign. Due to isotropy linear strains of edges $dy$ and $dz$ (and other elements in the plane $Oyz$) have to be identical and to present μ-th part of magnitude of longitudinal strain

$$\varepsilon_y = \varepsilon_z = -\mu\varepsilon_x = -\mu\frac{\sigma_x}{E},$$

where μ – is the factor of transversal strain or Poisson's factor.

It is obvious that at action of only stresses $\sigma_x$ owing to isotropy and homogeneity of material there is no distortion of orthogonal angles between facects/planes and edges of elementary parallelepiped. That's why there is $\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$.

Comparing obtained expressions for strains for the occasion, when $\sigma_x \neq 0$ and $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$ with dependencies (4.1) it is possible to determine

$$c_{11} = \frac{1}{E}, \quad c_{21} = c_{31} = -\frac{\mu}{E}, \quad c_{41} = c_{51} = c_{61} = 0\,.$$

Similar reasoning but applied for $\sigma_y$, $\sigma_z$ and comparison of obtained results with (4.1) bring us next conclusions

$$c_{22} = c_{33} = \frac{1}{E}, \quad c_{32} = c_{23} = c_{12} = c_{13} = -\frac{\mu}{E},$$
$$c_{42} = c_{52} = c_{62} = c_{43} = c_{53} = c_{63} = 0,$$

where $E$ and $\mu$ have the same meaning and magnitudes due to isotropy that in the first occasion for $\sigma_x$.

Let's review the occasion where there is acting only one of components of shear stresses, for example $\tau_{xy}$ (see fig. 4.1,***b***). The elementary parallelepiped in this case is under pure shear in plane $Oxy$, which cause as tests show shear strain in this plane

$$\gamma_{xy} = \frac{\tau_{xy}}{G},$$

where $G$ – is shear modulus or modulus of elasticity of the second class, [Pa].

Owing to isotropy there are no distortions of angles in two other coordinate planes and also there is next change of linear sizes of the parallelepiped: $\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$.

Comparing strains obtained for the occasion $\tau_{xy} \neq 0$, $\sigma_x = \sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = 0$ with (4.1) it can be concluded

$$c_{44} = \frac{1}{G}, \quad c_{41} = c_{42} = c_{43} = c_{45} = c_{46} = 0\,.$$

Reviewing in analogue way action of only one component $\tau_{yz}$ and after $\tau_{zx}$ we shall get

$$c_{55} = c_{66} = \frac{1}{G}, \quad c_{4j} = c_{5j} = c_{6j} = 0.$$

Shear modulus $G$ due to isotropy in all three occasions have the same magnitude.

Under the change of temperature of elementary parallelepiped its form don't distort but in all directions there are temperature linear strains are appeared the magnitudes of which is proportional to temperature and (to) factor of thermal linear extension α, $K^{-1}$:

$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0, \quad \varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha T .$$

Consequently calculating full strains appearing in the result of joint actions of stresses and temperature in right parts of (4.1) it is necessary to add members $+\alpha T$.

Considering obtained results presented above the common Hook's law for an ideal elastic isotropic homogenous body, where strains expressed via stresses can be written

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] + \alpha T; &\qquad \gamma_{xy} &= \frac{\tau_{xy}}{G};\\
\varepsilon_y &= \frac{1}{E}[\sigma_y - \mu(\sigma_z + \sigma_x)] + \alpha T; &\qquad \gamma_{yz} &= \frac{\tau_{yz}}{G};\\
\varepsilon_z &= \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)] + \alpha T; &\qquad \gamma_{zx} &= \frac{\tau_{zx}}{G}.
\end{aligned}\right\} \tag{4.2}$$

There is another form of Hook's law, where stresses expressed via strains

$$
\left.\begin{aligned}
&\sigma_x = 2G\left(\varepsilon_x + \frac{\mu\theta}{1-2\mu} - \frac{1+\mu}{1-2\mu}\alpha T\right), \qquad \tau_{xy} = G\gamma_{xy},\\
&\sigma_y = 2G\left(\varepsilon_y + \frac{\mu\theta}{1-2\mu} - \frac{1+\mu}{1-2\mu}\alpha T\right), \qquad \tau_{yz} = G\gamma_{yz},\\
&\sigma_z = 2G\left(\varepsilon_z + \frac{\mu\theta}{1-2\mu} - \frac{1+\mu}{1-2\mu}\alpha T\right), \qquad \tau_{zx} = G\gamma_{zx},\\
&\qquad\qquad G = \frac{E}{2(1+\mu)}, \quad \theta = \varepsilon_x + \varepsilon_y + \varepsilon_{z,}
\end{aligned}\right\} \tag{4.3}
$$

where $E$ – modulus of elasticity in tension, Pa; $G$ – shear modulus, Pa; $\mu$ – Poisson's ratio; $\alpha$ – factor of linear thermal extension, $K^{-1}$.

Factors $E$, $G$, $\mu$ are called technical elastic constants.

Hook's law is the third group of basic equations of the theory of elasticity. These equations are called physical equations.

**Cauchy Geometrical Relations Between Strains and Displacements.** Between displacements and strains there are relations. To get these relations we shall take in locality of some point $A$ (see fig. 4.2) three linear elements which are parallel to coordinate axes and mutually orthogonal to each other. There are coordinates of points $A$, $B$, $C$, $D$: $A(x; y; z)$, $B(x + dx; y; z)$, $C(x; y + dy; z)$, $D(x; y; z + dz)$. After deformation points $A$, $B$, $C$, $D$ will take new positions appropriately $A_1$, $B_1$, $C_1$, $D_1$. There are new coordinates of points: $A_1(x + u_A; y + v_A; z + w_A)$, $B_1(x + dx + u_B; y + v_B; z + w_B)$, $C_1(x + u_C; y + dy + v_C; z + w_C)$, $D_1(x + u_D; y + v_D; z + dz + w_D)$. Displacements in body are set by functions: $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$. Extending in Taylor series functions for displacements $u$, $v$, $w$ for points $A$, $B$, $C$, $D$ and taking into consideration only linear members we can get values of components of displacements in those points. After extending of displacements for points $A$, $B$, $C$, $D$ in Taylor series we shall have:

$$\left.\begin{aligned}
&u_A = u, \quad u_B = u + \frac{\partial u}{\partial x}dx, \quad u_C = u + \frac{\partial u}{\partial y}dy, \quad u_D = u + \frac{\partial u}{\partial z}dz,\\
&v_A = v, \quad v_B = v + \frac{\partial v}{\partial x}dx, \quad v_C = v + \frac{\partial v}{\partial y}dy, \quad v_D = v + \frac{\partial v}{\partial z}dz,\\
&w_A = w, \quad w_B = w + \frac{\partial w}{\partial x}dx, \quad w_B = w + \frac{\partial w}{\partial y}dy, \quad w_B = w + \frac{\partial w}{\partial z}dz.
\end{aligned}\right\} \quad (4.4)$$

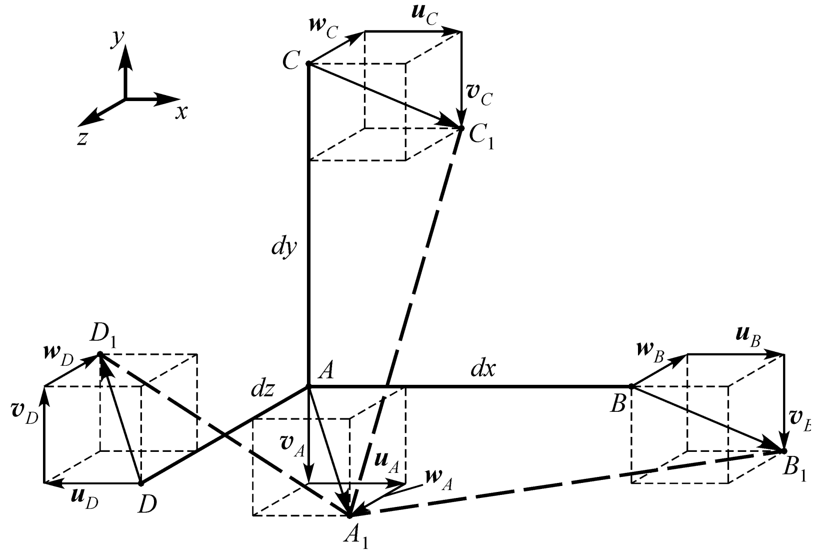



Fig. 4.2 – For getting of Cauchy relations

First we shall review linear element $dx$ between points $A$ and $B$. Coordinates of points $A$ and $B$ are presented above. Coordinates of points $A_1$ and $B_1$ considering (4.4) are:

$$\left.\begin{array}{l} A_1(x+u;\ \ y+v;\ \ z+w) \\ B_1\left(x+dx+u+\dfrac{\partial u}{\partial x}dx;\ \ y+v+\dfrac{\partial v}{\partial x}dx;\ \ z+w+\dfrac{\partial w}{\partial x}dx\right). \end{array}\right\}$$

Distance between points $A_1$ and $B_1$ is

$$A_1B_1 = dx\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\ . \tag{4.5}$$

But distance between points $A$ and $B$: $AB = dx$.

Linear relative strain in direction $AB$ (in direction of axis $x$) can be found using expression

$$\varepsilon_{AB} = \varepsilon_x = \frac{A_1B_1 - AB}{AB}\ . \tag{4.6}$$

Substituting (4.5) into (4.6) we shall have expression for $\varepsilon_x$

$$\varepsilon_x = \sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1\,. \tag{4.7}$$

Analogically reviewing linear elements $AC$, $AD$ we can get expressions for $\varepsilon_x$, $\varepsilon_z$. Expressions for $\varepsilon_x$, $\varepsilon_z$ can be got also using circular reordering of characters. Circular reordering of characters means to put instead given character next neighboring. Neighboring characters for example are: $x \to y \to z \to x$; $u \to v \to w \to u$, etc.

Further we shall define angular strain between $AB$ and $AC$ (between linear elements $dx$ and $dy$, i.e. in directions which are parallel appropriately to axes $x$ and $y$). Required angular strain between directions $AB$ and $AC$ can be found using expression

$$\sin\gamma_{xy} = \cos\left(\frac{\pi}{2} - \gamma_{xy}\right) = \cos\left(\overrightarrow{A_1B_1},^\wedge \overrightarrow{A_1C_1}\right). \tag{4.8}$$

Cosine of angle between vectors $\boldsymbol{A_1B_1}$ and $\boldsymbol{A_1C_1}$ can be found using known formula

$$\left.\begin{aligned}\cos\left(\overrightarrow{A_1B_1},\wedge\overrightarrow{A_1C_1}\right) &= \cos\left(\overrightarrow{A_1B_1},\wedge\vec{x}\right)\cdot\cos\left(\overrightarrow{A_1C_1},\wedge\vec{x}\right)+\\ &+\cos\left(\overrightarrow{A_1B_1},\wedge\vec{y}\right)\cdot\cos\left(\overrightarrow{A_1C_1},\wedge\vec{y}\right)+\\ &+\cos\left(\overrightarrow{A_1B_1},\wedge\vec{z}\right)\cdot\cos\left(\overrightarrow{A_1C_1},\wedge\vec{z}\right).\end{aligned}\right\} \qquad (4.9)$$

In compact notice (4.9) can be presented

$$\cos\left(\overrightarrow{A_1B_1},\wedge\overrightarrow{A_1C_1}\right) = l_{A_1B_1}\cdot l_{A_1C_1} + m_{A_1B_1}\cdot m_{A_1C_1} + n_{A_1B_1}\cdot n_{A_1C_1}, \quad (4.9')$$

where $l$, $m$, $n$ are direction cosines of vectors $\boldsymbol{A_1B_1}$ and $\boldsymbol{A_1C_1}$. Direction cosines of vectors $\boldsymbol{A_1B_1}$ and $\boldsymbol{A_1C_1}$ can be found in the way presented below.

Using coordinates of points $A_1$ and $B_1$ and (4.4) we can find that vector $\boldsymbol{A_1B_1}$ have next projections for coordinate axes:

$$\overrightarrow{A_1B_1}\left(\left(1+\frac{\partial u}{\partial x}\right)dx;\ \left(\frac{\partial v}{\partial x}\right)dx;\ \left(\frac{\partial w}{\partial x}\right)dx\right),$$

but the length of $\boldsymbol{A_1B_1}$ using (4.6) is $A_1B_1 = (1+\varepsilon_x)dx$. Hence direction cosines of vector $\boldsymbol{A_1B_1}$ are

$$\left.\begin{aligned}&\cos\left(\overrightarrow{A_1B_1},\wedge\vec{x}\right) = l_{A_1B_1} = \frac{1+\dfrac{\partial u}{\partial x}}{1+\varepsilon_x};\quad \cos\left(\overrightarrow{A_1B_1},\wedge\vec{y}\right) = m_{A_1B_1} = \frac{\dfrac{\partial v}{\partial x}}{1+\varepsilon_x};\\ &\qquad\qquad\cos\left(\overrightarrow{A_1B_1},\wedge\vec{z}\right) = n_{A_1B_1} = \frac{\dfrac{\partial v}{\partial x}}{1+\varepsilon_x}.\end{aligned}\right\} \quad (4.10)$$

Analogically direction cosines of vector $\boldsymbol{A_1C_1}$ can be found

$$\left.\begin{array}{l} \cos\left(\overrightarrow{A_1C_1}, \wedge \vec{x}\right) = l_{A_1C_1} = \dfrac{\dfrac{\partial u}{\partial y}}{1+\varepsilon_y}; \quad \cos\left(\overrightarrow{A_1C_1}, \wedge \vec{y}\right) = m_{A_1C_1} = \dfrac{1+\dfrac{\partial v}{\partial y}}{1+\varepsilon_y}; \\ \cos\left(\overrightarrow{A_1C_1}, \wedge \vec{z}\right) = n_{A_1C_1} = \dfrac{\dfrac{\partial w}{\partial y}}{1+\varepsilon_y}. \end{array}\right\} \quad (4.11)$$

Substituting (4.10) and (4.11) into right part of (4.9) we shall have

$$\sin\gamma_{xy} = \cos\left(\overrightarrow{A_1B_1}, \wedge \overrightarrow{A_1C_1}\right) =$$

$$= \frac{\left(1+\dfrac{\partial u}{\partial x}\right)\dfrac{\partial u}{\partial y} + \left(1+\dfrac{\partial v}{\partial y}\right)\dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial w}{\partial y}}{(1+\varepsilon_x)(1+\varepsilon_y)}. \quad (4.12)$$

Analogically $\gamma_{yz}$, $\gamma_{zx}$ can be found. Expressions of $\gamma_{yz}$, $\gamma_{zx}$ can also be written using circular reordering of characters, basing on the expression (4.12).

Extending (4.7) and (4.12) it is possible to write six accurate expressions which connect strains with displacements. Simplifying these accurate expressions in several approaches we shall have simple dependencies for the classic theory of elasticity

$$\left.\begin{array}{ll} \varepsilon_x = \dfrac{\partial u}{\partial x}, & \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}, \\ \varepsilon_y = \dfrac{\partial v}{\partial y}, & \gamma_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}, \\ \varepsilon_z = \dfrac{\partial w}{\partial z}, & \gamma_{zx} = \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}. \end{array}\right\} \quad (4.13)$$

Relations (4.13) are called Cauchy geometrical relations for linear theory of elasticity or simply Cauchy relations.

There are next geometrical meaning of partial derivatives of Cauchy relations (4.13) explained below:

$\frac{\partial v}{\partial x} \approx \beta_{xy}$ – rotation angle of element $dx$ in direction of axis $y$ (in plane $xy$);

$\frac{\partial w}{\partial x} \approx \beta_{xz}$ – rotation angle of element $dx$ in direction of axis $z$ (in plane $zx$);

$\frac{\partial u}{\partial y} \approx \beta_{yx}$ – rotation angle of element $dy$ in direction of axis $x$ (in plane $xy$);

$\frac{\partial w}{\partial y} \approx \beta_{yz}$ – rotation angle of element $dy$ in direction of axis $z$ (in plane $yz$);

$\frac{\partial u}{\partial z} \approx \beta_{zx}$ – rotation angle of element $dz$ in direction of axis $x$ (in plane $zx$);

$\frac{\partial v}{\partial z} \approx \beta_{zy}$ – rotation angle of element $dz$ in direction of axis $y$ (in plane $yz$).

These angles are positive if a rotation takes place in a shortest way between appropriate axes.

Because of assumption of relative rigidity of body the dependency between displacements and strains (4.13) turned out to be linear. It means that when displacements change in $n$ times as much strains change and vice versa. Moreover displacements matching for some of total displacements of body $u = \sum u_i$, $v = \sum v_i$, $w = \sum w_i$ can be defined as a sum of strains according to partial displacements $u_i$, $v_i$, $w_i$. On this basis bodies possessing the property of relative rigidity are called geometrically linear.

Investigations show that approximate relations (4.13) at determination of linear strains ε give absolute error not exceeding 1–2 % of quantity $\varepsilon_{max}$, if maximum angles of rotations of linear elements β satisfy to the condition $|\beta|_{max} \le 0.1\sqrt{|\varepsilon|_{max}}$, where $|\varepsilon|_{max}$ – maximum value of linear strain arising in body. To save the same precision defining angular strains γ by (4.13) the next conditions have to be satisfied: $|\varepsilon|_{max} \le 0.01$, $|\gamma|_{max} \le 0.02$.

Said ultimate values of strains ε and γ practically don't limit applying of Cauchy relations to steel and to other metal bodies in the whole range of elastic strains.

Six equations of Cauchy (4.13) set dependencies between nine functions: three displacements and six strains. Hence there are only three quantities which can be set independently. It is impossible to set six strains arbitrarily, because after substituting it into (4.13) there are three required displacements are connected by six expressions which in general case may be incompatible.

**Compatibility Conditions Expressed in Terms of Strains.** Incompatibility indicates the impossibility of appearance of given strains in body. Therefore strains can't be set arbitrarily. They have to satisfy to some relations, which can be got by the way of excluding of displacements from (4.13). Excluding displacements from (4.13) we shall have next relations

$$\left.\begin{aligned}
&\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}, &&\frac{\partial}{\partial x}\left(\frac{\partial\gamma_{xy}}{\partial z}-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}\right)=2\frac{\partial^2\varepsilon_x}{\partial y\partial z},\\
&\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}, &&\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)=2\frac{\partial^2\varepsilon_y}{\partial z\partial x},\\
&\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}, &&\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{yz}}{\partial x}\right)=2\frac{\partial^2\varepsilon_z}{\partial x\partial y}.
\end{aligned}\right\}\quad(4.14)$$

Equations (4.14) are called the Saint-Venant equations or equations of compatibility of strains or equations of continuity. Physical meaning of equations of compatibility (4.14) can be explained in the next way. For example we have body cut for numerous of elementary parallelepipeds before deformation as it's shown on the fig. 4.3.

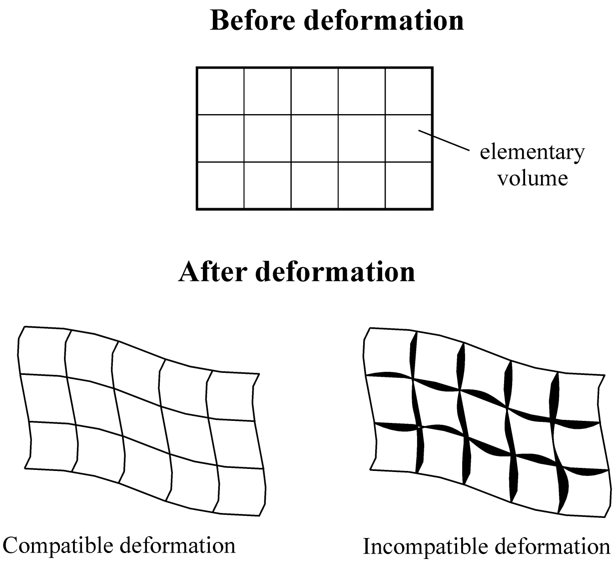



Fig. 4.3 – For explanation of compatibility

Then every parallelepiped was deformed in accordance with strains laws, which were set for the whole body. Assembling all parallelepipeds together after deformation we shall have the whole body after deformation. If given strains of parallelepipeds satisfy to compatibility conditions (4.14) we shall have that all parallelepipeds are closely adjacent to each other without voids between them after assembling. In opposite case there will be a lot of cracks and exfoliations in body. Therefore in a body which has property of continuity it is impossible appearance of strains which don't satisfy (4.14). Strains defined by given displacements using Cauchy relations satisfy to compatibility equations of strains automatically.

**Definition of Displacements for Known Strains.** Using Cauchy relations it is easy to define strain for known displacements. Let's define inverse task: to define displacement for known strains. Solution of this task is connected with integration of geometrical relations, which connect components of strains and displacements. To get formulas for displacements for known strains it is necessary to use known formula for definition of function knowing its full differential. After geometrical transformations we shall have next formulas for displacements

$$\left.\begin{aligned}
&u = u_0 + (y - y_0)\left(\frac{\partial u}{\partial y}\right)_0 + (z - z_0)\left(\frac{\partial u}{\partial z}\right)_0 + \int_{x_0}^{x} \varepsilon_x \, dx + \\
&\int_{y_0}^{y}\int_{y_0}^{y}\left(\frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \varepsilon_y}{\partial x}\right)_{x=x_0} dy^2 + \int_{z_0}^{z}\int_{z_0}^{z}\left(\frac{\partial \gamma_{zx}}{\partial z} - \frac{\partial \varepsilon_z}{\partial x}\right)_{x=x_0,\; y=y_0} dz^2 + \\
&+\frac{1}{2}(y - y_0)\int_{z_0}^{z}\left(\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right)_{x=x_0,\; y=y_0} dz.
\end{aligned}\right\} \quad (4.15)$$

$$\left.\begin{aligned}
v = v_0 &+ (z - z_0)\left(\frac{\partial v}{\partial z}\right)_0 + (x - x_0)\left(\frac{\partial v}{\partial x}\right)_0 + \int_{y_0}^{y} \varepsilon_y \, dy + \\
&+ \int_{z_0}^{z}\int_{z_0}^{z} \left(\frac{\partial \gamma_{yz}}{\partial z} - \frac{\partial \varepsilon_z}{\partial y}\right)_{y=y_0} dz^2 + \int_{x_0}^{x}\int_{x_0}^{x} \left(\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y}\right)_{y=y_0,\ z=z_0} dx^2 + \\
&+ \frac{1}{2}(z - z_0)\int_{x_0}^{x} \left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)_{y=y_0,\ z=z_0} dx.
\end{aligned}\right\} \qquad (4.16)$$

$$\left.\begin{aligned}
w = w_0 &+ (x - x_0)\left(\frac{\partial w}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial w}{\partial y}\right)_0 + \int_{z_0}^{z} \varepsilon_z \, dz + \\
&+ \int_{x_0}^{x}\int_{x_0}^{x} \left(\frac{\partial \gamma_{zx}}{\partial x} - \frac{\partial \varepsilon_x}{\partial z}\right)_{z=z_0} dx^2 + \int_{y_0}^{y}\int_{y_0}^{y} \left(\frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial \varepsilon_y}{\partial z}\right)_{z=z_0,\ x=x_0} dy^2 + \\
&+ \frac{1}{2}(x - x_0)\int_{y_0}^{y} \left(\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right)_{z=z_0,\ x=x_0} dy,
\end{aligned}\right\} \qquad (4.17)$$

where $x_0$, $y_0$, $z_0$ – are initial point for integration; $u_0$, $v_0$, $w_0$ – are initial displacements considering the given body to be completely rigid; $(\partial u/\partial y)_0$ – is the rotation angle of the element $dy$ in direction of axis $x$ (in plane $xOy$) considering given body to be completely rigid, etc. Initial rotation angles are connected with angular displacements by the next relations

$$\left.\begin{array}{c}\left(\dfrac{\partial u}{\partial y}\right)_0+\left(\dfrac{\partial v}{\partial x}\right)_0=(\gamma_{xy})_0,\quad \left(\dfrac{\partial v}{\partial z}\right)_0+\left(\dfrac{\partial w}{\partial y}\right)_0=(\gamma_{yz})_0,\\ \left(\dfrac{\partial w}{\partial x}\right)_0+\left(\dfrac{\partial u}{\partial z}\right)_0=(\gamma_{zx})_0.\end{array}\right\}. \qquad (4.18)$$

Sometimes they use next relations

$$\left.\begin{array}{c}\omega_x=\dfrac{1}{2}(\beta_{yz}-\beta_{zy})=\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}\right),\quad \omega_y=\dfrac{1}{2}(\beta_{zx}-\beta_{xz})=\dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}\right),\\ \omega_z=\dfrac{1}{2}(\beta_{xy}-\beta_{yx})=\dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right).\end{array}\right\} \qquad (4.19)$$

Parameters $\omega_x$, $\omega_y$, $\omega_z$ are rotation angles of elements around axes accordingly $x$, $y$, $z$. These elements bisect directions between axes $y$ and $z$, $z$ and $x$, $x$ and $y$ accordingly. These quantities characterize in average rotation of material in locality of given point around proper axis.

If finding displacements by (4.15–4.17) there is chosen isolated coordinate system which is not connected with other bodies, foundations, restrictions etc. for constants of integration it is possible to set arbitrary values. In particular because displacements of a body as total solid is no of interest it is possible to put all six values to zero.

To exclude average rotations in locality of initial point it can be possible to put to zero components of rotation $(\omega_x)_0=(\omega_y)_0=(\omega_z)_0=0$, from which using (4.19) we shall have

$$\left.\begin{array}{c}\left(\dfrac{\partial u}{\partial y}\right)_0=\left(\dfrac{\partial v}{\partial x}\right)_0=\dfrac{1}{2}(\gamma_{xy})_0,\quad \left(\dfrac{\partial v}{\partial z}\right)_0=\left(\dfrac{\partial w}{\partial y}\right)_0=\dfrac{1}{2}(\gamma_{yz})_0,\\ \left(\dfrac{\partial w}{\partial x}\right)_0=\left(\dfrac{\partial u}{\partial z}\right)_0=\dfrac{1}{2}(\gamma_{zx})_0.\end{array}\right\} \qquad (4.20)$$

Also it is possibly to attach coordinate axes to a body in any other way but providing elimination of displacements of solid whole not inhibiting to any possible deformation. Depending on ways of attachment of coordinate axes there will be different laws of displacements which have to be taken into account analyzing results.

**Additional Kinematic Relations for Multiply Connected Bodies.** As more complete investigations show for simply connected body the Saint-Venant compatibility conditions (4.14) are necessary and sufficient for existence of such strains in body and consequently possibility to define appropriate displacements using (4.15)–(4.17).

But for multiply connected bodies the mentioned condition being necessary stop to be sufficient. In this case displacements have to be uninterrupted single-valued functions on a body point coordinates at any closed contour *s* (or closed surface), enveloping each inside cavity not crossing body boundaries. This condition can be expressed using next relations

$$\left.\begin{aligned}
&\oint_s \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \right) = 0, \\
&\oint_s \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz \right) = 0, \\
&\oint_s \left( \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz \right) = 0.
\end{aligned}\right\} \qquad (4.21)$$

For simply connected bodies the relation (4.21) is satisfied automatically if strains satisfy to Saint-Venant compatibility conditions (4.14).

Thus for calculation of multiply connected bodies Cauchy relations (4.13) have to be supplemented by equations (4.21).

**Common structure of general equations of the theory of elasticity.** There are next groups of strain-stress state presented in the table 4.1.

Table 4.1 – Groups of strain-stress state

| Name of group | Denotation |
|---|---|
| Displacements | $u, v, w$ |
| Stresses | $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ |
| Strains | $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ |

In the table 4.2 there are presented groups of general equations of the theory of elasticity.

Table 4.2 – Groups of general equations of the theory of elasticity

| # of group | Name of equation | Reference |
|---|---|---|
| 1 | Equilibrium equations of elementary tetrahedron | (3.4) |
| 2 | Equilibrium equations of elementary parallelepiped | (3.7) |
| 3 | Hook's law, *or* physical relations | (4.2) for $\varepsilon = f(\sigma)$<br>(4.3) for $\sigma = f(\varepsilon)$ |
| 4 | Cauchy equations, *or* geometrical relations | (4.13) |

**General Equations of Geometrically Nonlinear Theory of Elasticity.** Thereby extending (4.7) and (4.12) we can write accurate expressions which connect strains with displacements within nonlinear theory of elasticity

$$\left.\begin{aligned}
\varepsilon_x &= \sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2}-1,\\
\varepsilon_y &= \sqrt{\left(1+\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}-1,\\
\varepsilon_z &= \sqrt{\left(1+\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2}-1,\\
\gamma_{xy} &= \arcsin\frac{\left(1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y}+\left(1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}}{\left(1+\varepsilon_x\right)\left(1+\varepsilon_y\right)},\\
\gamma_{yz} &= \arcsin\frac{\left(1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial z}+\left(1+\frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial y}+\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z}}{\left(1+\varepsilon_y\right)\left(1+\varepsilon_z\right)},\\
\gamma_{zx} &= \arcsin\frac{\left(1+\frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial x}+\left(1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial z}+\frac{\partial v}{\partial z}\cdot\frac{\partial v}{\partial x}}{\left(1+\varepsilon_z\right)\left(1+\varepsilon_x\right)}.
\end{aligned}\right\} \quad (4.22)$$

Expressions (4.22) are also called geometrical relations.

Replacing in (4.22) radicals by sums of two first members of Newton binomial and values of arcsines by its arguments it is possibly to get the next geometrical relations of nonlinear theory of elasticity

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right],\\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right],\\
\varepsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right],\\
\gamma_{xy} &= \left(1 + \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + \left(1 + \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},\\
\gamma_{yz} &= \left(1 + \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial z} + \left(1 + \frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z},\\
\gamma_{zx} &= \left(1 + \frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial x} + \left(1 + \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x}.
\end{aligned}\right\} \qquad (4.23)$$

Expressions (4.23) are true for maximum linear strains $|\,\varepsilon\,|_{\max} \le 0.005$ and maximum shear stresses $|\,\gamma\,|_{\max} \le 0.2$ within less than 1 % in comparison with exact solution (4.22).

Further simplifications can be made in case when one of displacements and its derivatives exceeds in an order of magnitude the rest ones. For example if displacement $w$ and its derivatives exceed in an order of magnitude the rest displacements $u$, $v$ and its derivatives the relations (4.23) can be written

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},\\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z},\\
\varepsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x}.
\end{aligned}\right\} \tag{4.24}$$

Expressions presented above are used in the theory of bending of thin plates and shells. At last it is possibly to show if linear strains $\varepsilon \le 0.005$, and rotation angles $\partial u/\partial y$, $\partial u/\partial z$, $\partial v/\partial x$, $\partial v/\partial z$, $\partial w/\partial x$, $\partial w/\partial y \le 0.007$ with error not more than 2–3 % from formulae (4.23) it is possible to get Cauchy relations (4.13).

## LECTURE 5. DIRECT AND INVERSE PROBLEMS OF THE THEORY OF ELASTICITY

**Direct and Inverse Problems of the Theory of Elasticity.** There are two general problems in the theory of elasticity: direct and inverse problems. On the fig. 5.1 there is scheme where the problems of the theory of elasticity are shown.

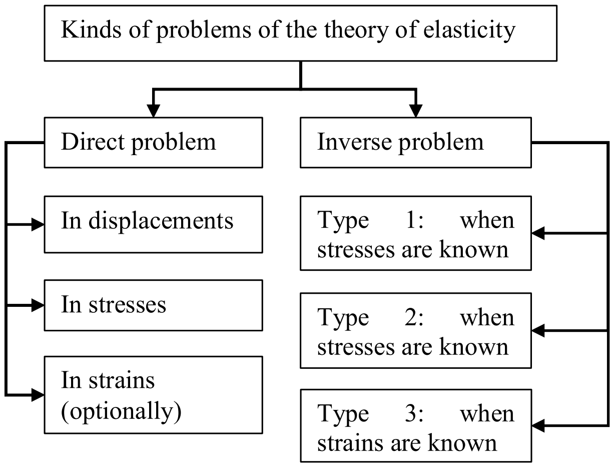



Fig. 5.1 – Kinds of problems of the theory of elasticity

For the direct problem it is necessary for a given body, set constraints, external loadings and temperature field to define displacements, strains and stresses. For the inverse problem it is necessary for a given body with its known constraints and displacements or strains or stresses to define the rest two lacking groups of components of strain-stress state and external loadings with temperature field.

**Direct Problem.** There are two kinds of the direct problem of the theory of elasticity: direct problem in displacements and direct problem in stresses. *Direct problem in displacements* means that displacements are unknown values for the first step of solution of this problem. To solve the direct problem in displacements it is necessary to solve so called Lame equations which can be written as

$$\left.\begin{aligned}
&\nabla^2 u+\frac{1}{1-2\mu}\frac{\partial\theta}{\partial x}=-\frac{X}{G}+\frac{2(1+\mu)}{1-2\mu}\frac{\partial T}{\partial x}\alpha,\\
&\nabla^2 v+\frac{1}{1-2\mu}\frac{\partial\theta}{\partial y}=-\frac{Y}{G}+\frac{2(1+\mu)}{1-2\mu}\frac{\partial T}{\partial y}\alpha,\\
&\nabla^2 w+\frac{1}{1-2\mu}\frac{\partial\theta}{\partial z}=-\frac{Z}{G}+\frac{2(1+\mu)}{1-2\mu}\frac{\partial T}{\partial z}\alpha.
\end{aligned}\right\} \tag{5.1}$$

Boundary conditions for Lame equations are

$$\left.\begin{aligned}
&p_{\nu x}=G\left(\frac{\partial u}{\partial\nu}+\frac{\partial\rho}{\partial x}+\frac{2l}{1-2\mu}[\mu\theta-(1+\mu)\alpha T]\right),\\
&p_{\nu y}=G\left(\frac{\partial v}{\partial\nu}+\frac{\partial\rho}{\partial y}+\frac{2m}{1-2\mu}[\mu\theta-(1+\mu)\alpha T]\right),\\
&p_{\nu z}=G\left(\frac{\partial w}{\partial\nu}+\frac{\partial\rho}{\partial z}+\frac{2n}{1-2\mu}[\mu\theta-(1+\mu)\alpha T]\right),
\end{aligned}\right\} \tag{5.2}$$

where

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z};$$

$\dfrac{\partial(\,)}{\partial \nu} = l\dfrac{\partial(\,)}{\partial x} + m\dfrac{\partial(\,)}{\partial y} + n\dfrac{\partial(\,)}{\partial z}$ – derivative along normal vector to surface;

$\rho = lu + mv + nw$ – projection of displacement into normal vector to surface;

$\nabla^2(\,) = \dfrac{\partial^2(\,)}{\partial x^2} + \dfrac{\partial^2(\,)}{\partial y^2} + \dfrac{\partial^2(\,)}{\partial z^2}$ – Laplace operator;

$\alpha$ – factor of linear thermal extension.

There are also kinematic boundary conditions for the parts of surface, i.e. some laws of displacements are set at the parts $S_i$ of a body surface

$$\left.\begin{aligned} u_i &= u_i(\bar{x}, \bar{y}, \bar{z}); \\ v_i &= v_i(\bar{x}, \bar{y}, \bar{z}); \\ w_i &= w_i(\bar{x}, \bar{y}, \bar{z}), \end{aligned}\right\} \tag{5.3}$$

where $i = 1, 2, \dots n$ – index of given part of body surface; $n$ – total number of parts.

Equations (5.1) and (5.2) are equilibrium equations of elementary parallelepiped and elementary tetrahedron appropriately but expressed via displacements using sequentially Hook's law for getting strains and Cauchy geometrical relations. Expressions (5.2) are force boundary conditions (BCs) and (5.3) are kinematical BCs.

Thus to solve the direct problem in displacements it is necessary to solve Lame equations (5.1) together with boundary conditions (5.2) and (5.3) considering (4.21) for multiply connected bodies.

On the fig. 5.2 there are major steps in solution of the direct problem of the theory of elasticity in displacements are presented.

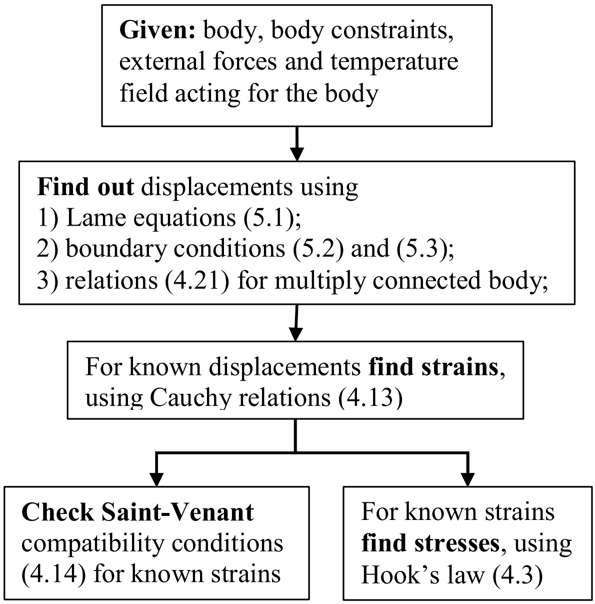



Fig. 5.2 – Sequence in solution of the direct problem of the theory of elasticity in displacements

*Direct problem in stresses* means that stresses are unknown values for the first step of solution of this problem. There are six components of stresses. So we have to have at least six equations to find these stresses: three equations of elementary parallelepiped and three equations of elementary tetrahedron as boundary conditions. If there are displacements set on parts of body surface like (5.3) it is necessary to express these displacements via stresses using sequentially Cauchy relations to get strains and Hook's law to get stresses. But these six stresses aren't independent (analogously as strains are). So it is necessary to have additional relations which connect these six unknown stresses. These additional relations are compatibility conditions for stresses, which are called Beltrami-Mitchell compatibility conditions and can be written as (without considering of temperature)

$$\left.\begin{aligned}
&\nabla^2\sigma_x+\frac{1}{1+\mu}\frac{\partial^2\Theta}{\partial x^2}=-2\frac{\partial X}{\partial x}-\frac{\mu}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right);\\
&\nabla^2\sigma_y+\frac{1}{1+\mu}\frac{\partial^2\Theta}{\partial y^2}=-2\frac{\partial Y}{\partial y}-\frac{\mu}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right);\\
&\nabla^2\sigma_z+\frac{1}{1+\mu}\frac{\partial^2\Theta}{\partial z^2}=-2\frac{\partial Z}{\partial z}-\frac{\mu}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right);\\
&\nabla^2\tau_{xy}+\frac{1}{1+\mu}\frac{\partial^2\Theta}{\partial x\,\partial y}=-\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right);\\
&\nabla^2\tau_{yz}+\frac{1}{1+\mu}\frac{\partial^2\Theta}{\partial y\,\partial z}=-\left(\frac{\partial Y}{\partial z}+\frac{\partial Z}{\partial y}\right);\\
&\nabla^2\tau_{zx}+\frac{1}{1+\mu}\frac{\partial^2\Theta}{\partial z\,\partial x}=-\left(\frac{\partial Z}{\partial x}+\frac{\partial X}{\partial z}\right).
\end{aligned}\right\} \quad (5.4)$$

where $\Theta=\sigma_x+\sigma_y+\sigma_z$;

$$\nabla^2(\,)=\frac{\partial^2(\,)}{\partial x^2}+\frac{\partial^2(\,)}{\partial y^2}+\frac{\partial^2(\,)}{\partial z^2}$$ – the Laplace operator.

Expressions (5.4) are compatibility conditions in terms of stresses. They can be got using Saint-Venant compatibility conditions applying Hook's law and equilibrium equations of elementary parallelepiped.

Thus to solve the direct problem in stresses it is necessary to solve equilibrium equations of elementary parallelepiped (3.7) together with equilibrium equations of elementary tetrahedron (3.4) as boundary conditions and with boundary conditions (5.3) (previously transformed via stresses as it was explained above) taking into account (4.21) for multiply connected bodies. In (4.21) displacements also have to be expressed via stresses.

On the fig. 5.3 there are major steps in solution of the direct problem of the theory of elasticity in stresses are presented.

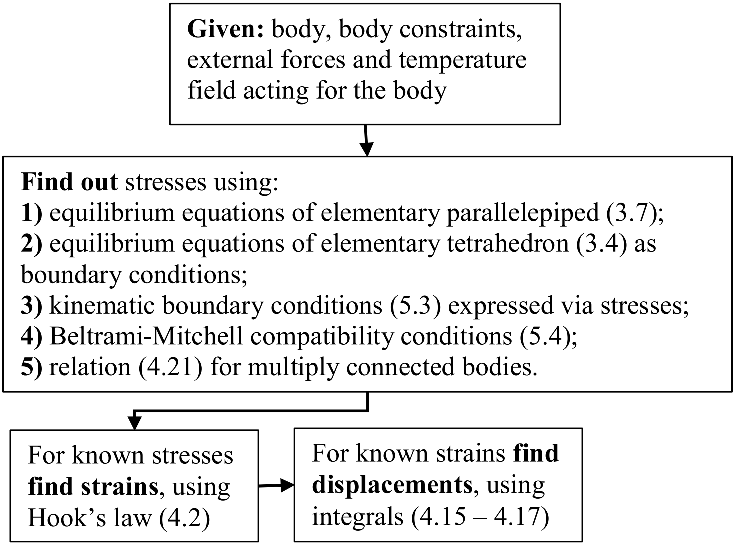



Fig. 5.3 – Sequence in solution of the direct problem of the theory of elasticity in stresses

Direct problem in strains (see fig. 5.1) can be solved in the same way as direct problem in stresses. But first it is necessary to express stains through stresses using Hook's law.

**Inverse Problem of the Theory of Elasticity.** As it is shown on the fig. 5.1 there are three kinds of the inverse problem. To obtain single-valued solution of the inverse problem there must be set temperature field or volume forces. In opposite case there can be numerous combinations of temperature fields and volume forces causing the same strain-stress state. That's why below it is suggested that temperature field $T = T(x, y, z)$ is set.

For the *first* kind of the inverse problem there are laws of displacements are known. It is necessary to find stresses and strains. Using Cauchy relations we can find the laws for strains and after applying Hook's law for known strains we can find stresses. Using equilibrium equations of elementary volumes for known stresses or displacements (Lame equations) we can find external volume and surface forces.

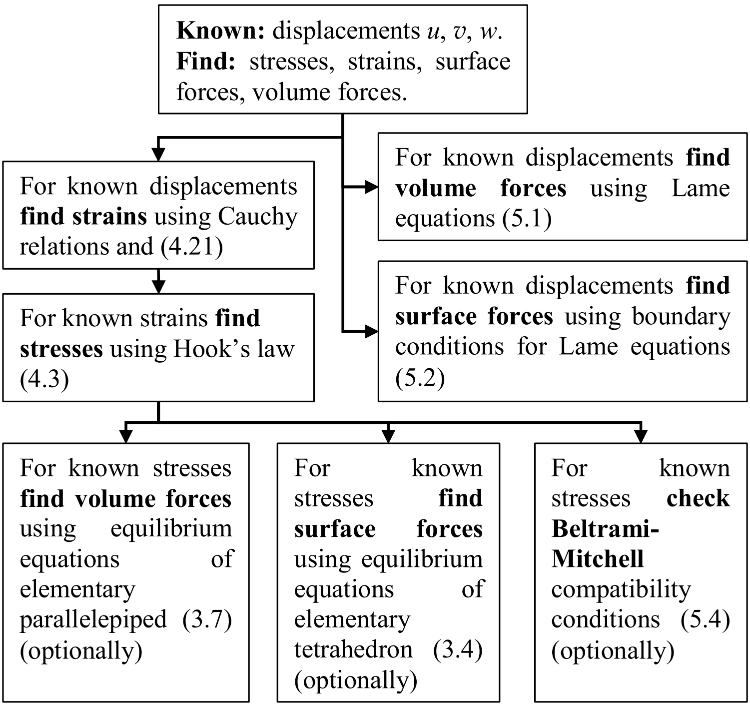



Fig. 5.4 – First kind of the inverse problem of the theory of elasticity

For the *second* kind of the inverse problem there are laws of stresses are known. It is necessary to find strains and displacements. Using Hook's law we can find strains. For known strains we can find displacements using relations (4.15 ÷ 4.17). Using equilibrium equations of elementary volumes for known stresses or displacements (Lame equations) we can find external volume and surface forces.

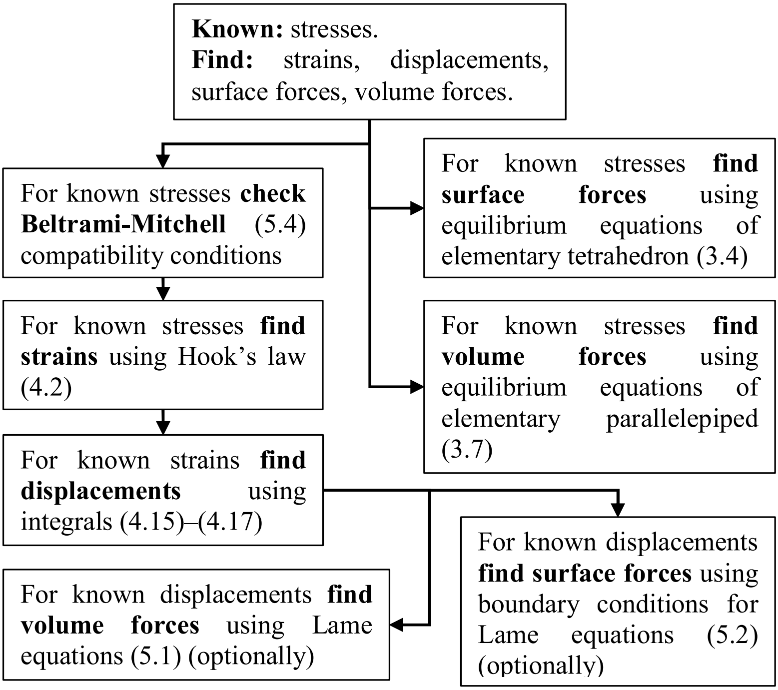



Fig. 5.5 – Second kind of the inverse problem
of the theory of elasticity

For the *third* kind of the inverse problem there are laws of strains are known. It is necessary to find the laws for displacements and stresses. Using integrals (4.15 ÷ 4.17) we can find displacements.

Also using Hook's law we can find stresses. Applying equilibrium equations for known stresses or displacements (Lame equations) we can find external volume and surface forces.

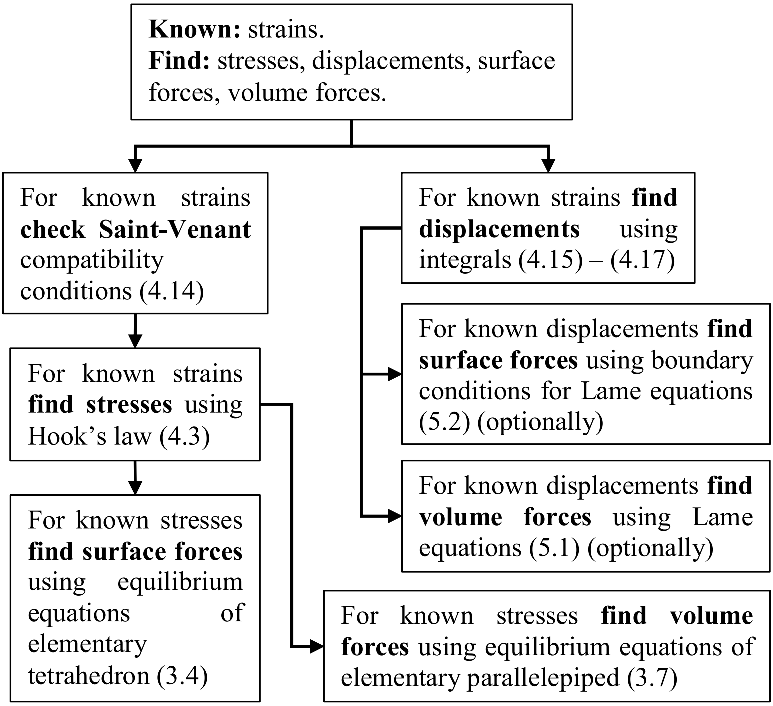



Fig. 5.6 – Third kind of the inverse problem of the theory of elasticity

**Saint-Venant Principle.** In practical applications the Saint-Venant (1855) principle is very important and widely used. There are two identical formulations of it.

1) A system of external forces acting in frames of a very small field of surface or volume of a body and which (i.e. system) is statically equivalent to zero (principal vector and principal moment equal to

zero) cause stresses and strains rapidly descending when moving off mentioned field.

2) Strains-stress state caused by a system of external forces applied to a small field of surface or volume of a body at some distance it (i.e. strain-stress state) depends only on principle vector and principle moment of external forces and don't depend on particulars of its distribution in the limits of a said field.

From the Saint-Venant principle there is the next conclusion: change of character of distribution of external loading within small field under retention of the quantity and the direction of principal vector and principle moment cause change of strain-stress state only within this field or close by it. At distances from centers of said fields which exceed about one-one and a half the largest its sizes the influence of redistribution of external loading practically disappears.

It is necessary to take into account that the field of a body which is at issue above in the Saint-Venant principle has to be considered to be a piece of a body hitting into limits of the volume, sizes of which (i.e. of volume) equal to the largest distance between points of application of the reviewed system of external loadings. Moreover, the said part of a body have to be simply closed and monolithic, limited by convex surface. At inobservance of these conditions the Saint-Venant principle is of no use.

The Saint-Venant principle has no accurate quantitative mathematic proving, but all existing solutions and practical examples completely confirm its validity. If some problem for the given law of loads can't be solved exactly it can be replaced by more simple problem, which is equivalent to the first one in the sense of the Saint-Venant principle, and the obtained solution can be expanded into the first problem.

**Half-Inverted Saint-Venant Method.** The solution of many problems of the theory of elasticity significantly can be relieved owing to applying of so called the half-inverted Saint-Venant method. The entity of this method is narrowed down to the next.

As it is known the inverse problem of the theory of elasticity can be easily solved. It allows us to use the scheme of solution of the inverse problem to get the solution of the direct problem. Presetting one of the groups of the strain-stress state such as stresses, strains or displacements and solving the inverse problem they define appropriate external forces which response to the given strain-stress state. If external forces appear to be exact as they are specified the solution we have got is accurate due to the unique theorem. In an opposite occasion it is necessary to check other variants up till we achieve the correct solution. But in this way there are little chances to get the solution which would be a little bit interest.

That's why Saint-Venant proposed so called ***half-inverted method*** (1855). In this method the mechanism of solution of the inverse problem is also used as it was told above. But we preset not all the group of some strain-stress state but only part of it relatively which we have some knowledge. This knowledge can be got on basis of qualitative analysis, experimental data, and existing exact or approximate solutions of other close problems. Substitution of assumed part of the solution into the general system of equations of the theory of elasticity may significantly simplify the problem. If the simplified system of equations we have got (after substitution of assumed part) can be solved it means it is possible to find the unknown part of the solution. But if the mentioned system of equations appears to be incompatible it tells us about fallaciousness of the suggested part of the solution and it is necessary to accept other hypothesizes.

Half-inverted method is widely used in the theory of elasticity. The classic example of applying of this method is developing by Saint-Venant the theory of torsion of prismatic bars.

**Superposition Principle** is true only for linear mechanics. It can be formulated in the next way. If there is system of external loadings, which consists of several forces acting for the body simultaneously, the displacement (stresses, strain) in some point, caused by all external forces equals to the sum of displacements (stresses, strains) caused by every external force acting separately.

Mathematically this principle can be formulated

$$u_A(P_1, P_2, \ldots, P_n) = u_A(P_1) + u_A(P_2) + \ldots + u_A(P_n) \qquad (5.5)$$

where $P_1$, $P_2$, … , $P_n$ are external forces both concentrated and distributed ones; $u_A$ – is some displacement (stresses, strain) in some point $A$.

## LECTURE 6. SIMPLE PROBLEMS OF THE THEORY OF ELASTICITY

In the strength of materials there are simple relations connecting stresses, strains or displacements with external forces. Within this lecture we shall try to estimate accuracy of those expressions got in frames of the strength of materials from positions of the theory of elasticity.

**Pure Bending of Prismatic Beam.** Let's review pure bending of prismatic console beam (see fig. 6.1,***a***), the left end of which is rigidly constrained but the right end is loaded by concentrated bending moment $M_A$. Throwing off connections and replacing the system by statically equivalent we have the scheme as it is shown on the fig. 6.1,***b***. At end surfaces there are surface forces $p_{xx}$ are applied which are statically equivalent in the plane $xzO$ mutually equilibrated by bending moments $M_A$. There is no volume forces and temperature field.

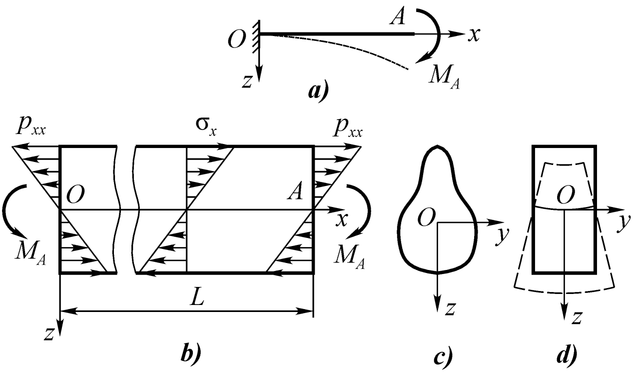



Fig. 6.1 – Pure bending of beam

Axes $y$ and $z$ are central principal axes but $x$ is longitudinal one coming through mass centers of cross sections. Given external loadings (by only bending moments $M_A$) cause next stress state

$$\sigma_x = -\frac{M \cdot z}{I}, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0\,, \qquad (6.1)$$

where $M = M(x) = M_A$ – is internal bending moment (the moment of internal forces); $I$ – is the principal central inertia moment relatively axis $y$.

The solution (6.1) satisfy identically to compatibility conditions of Beltrami-Mitchell and to equilibrium equations of elementary tetrahedron. Equilibrium equations of elementary tetrahedron on prismatic surface $\nu(l = 0,\ m^2 + n^2 = 1)$ are also satisfied in the absence of surface forces on it.

At end surfaces $x = 0$, $\nu(-1, 0, 0)$ and $x = L$, $\nu(1, 0, 0)$ two last equilibrium equations of elementary tetrahedron are satisfied identically, but from the 1-st one it is possibly to get

$$\left.\begin{aligned} p_{xx} &= \sigma_x l = \frac{M \cdot z}{I} \text{ for left end,} \\ p_{xx} &= \sigma_x l = -\frac{M \cdot z}{I} \text{ for right end.} \end{aligned}\right\} \qquad (6.2)$$

Surface loading $p_{xx}$ is really statically equivalent to bending moments $M$. The principle vector $\boldsymbol{R}$ equals

$$R = \int\limits_F p_{xx} dF = \frac{M}{I} \int\limits_F z dF = 0\,, \qquad (6.3)$$

because $y$ – is the central axis.

Bending moment relatively axis $y$ can be found

$$\left.\begin{aligned}
&\int_F p_{xx} z dF = \frac{M}{I}\int_F z^2\,dF = M \quad \text{for left end,}\\
&\int_F z^2\,dF = I,\\
&\int_F p_{xx} z dF = -\frac{M}{I}\int_F z^2\,dF = -M \quad \text{for right end.}
\end{aligned}\right\} \tag{6.4}$$

Thus the solution (6.1) is accurate in case if normal loads at ends of beam causing bending moment are distributed in the same way as the normal stresses in arbitrary cross section of beam, i.e. linearly on axis $z$. If mentioned condition is not provided, then in accordance with Saint-Venant principle this solution have to be reviewed as an approximate. At that in regions nearby to ends of beam the law of $\sigma_x$ gradually transform from stated at ends $\sigma_x = \pm p_{xx}$ to defined by (6.1).

For nonprismatic beam $I(x)$ the solution (6.1) is precise when there are some additional volume and surface loads act. But if the change of the cross section of a beam is going on smoothly with little gradient then the solution (6.1) have to be reviewed as an approximate.

Strains can be define using Hook's law

$$\varepsilon_x = -\frac{Mz}{EI}, \quad \varepsilon_y = \varepsilon_z = \mu\frac{Mz}{EI}, \quad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0. \tag{6.5}$$

Displacements can be found using integrals (4.15)–(4.17)

$$u = -\frac{M}{EI}xz, \quad v = \mu\frac{M}{EI}yz, \quad w = \frac{M}{2EI}[x^2 + \mu(z^2 - y^2)]. \tag{6.6}$$

To find (6.6) there were next relations accepted

$$\left.\begin{aligned} &x = y = z = 0, \quad u_0 = v_0 = w_0 = 0, \\ &\left(\frac{\partial u}{\partial y}\right)_0 = \left(\frac{\partial u}{\partial z}\right)_0 = \left(\frac{\partial v}{\partial z}\right)_0 = 0. \end{aligned}\right\} \tag{6.7}$$

For the neutral axis when $z = y = 0$ using $w$ in (6.6) it is possibly to get the well known relation for console beam for deflections

$$w = \frac{Mx^2}{2EI}. \tag{6.8}$$

Orthogonal planes in orthogonal cross sections before deformation change after deformation as it is shown on the fig. 6.1,***d***. At that horizontal fibers hog for concentric circles, but vertical ones keep radius positions.

**Torsion of Round Prismatic Bar.** Let's review torsion of prismatic round bar (see fig. 6.2). At ends of a bar there are tangent forces $p_{zx}$, $p_{zy}$ which are statically equivalent for two torques $M_z$ which are equal for magnitudes but opposite for directions. There is no volume forces and temperature field.

Using flat cross-section hypothesis in the strength of materials there is next formula for full shear stresses acting perpendicularly to radiuses can be got

$$\tau_z = \frac{M_z \cdot r}{I_p}, \tag{6.9}$$

where $M_z$ – is the internal torque which equals to external torque; $r$ – is the distance from the mass centre of the cross section to arbitrary point on the same section; $I_p$ – polar inertia moment of the cross section.

Projecting $\tau_z$ into axes $x$ and $y$ considering the rest components to be zero we have

$$\left.\begin{aligned} &\tau_{zx} = -\tau_z \frac{y}{r} = -\frac{M_z}{I_p} y, \quad \tau_{zy} = \tau_z \frac{x}{r} = \frac{M_z}{I_p} x, \\ &\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0. \end{aligned}\right\} \qquad (6.10)$$

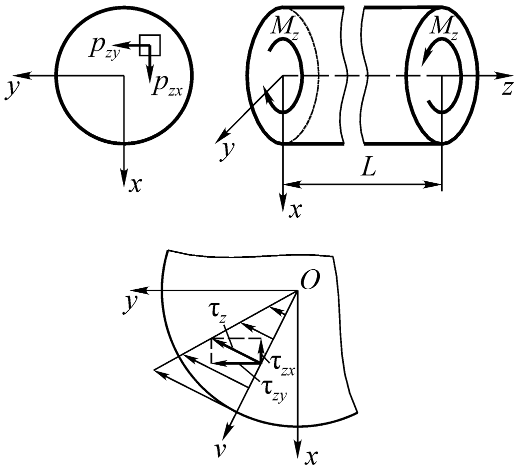



Fig. 6.2 – Torsion of round bar

The solution (6.10) satisfy to Beltrami-Mitchell compatibility conditions and to equilibrium equations of elementary parallelepiped.

Equilibrium equations of elementary tetrahedron on cylinder surface $x^2 + y^2 = R^2$, $\nu(l = x/R,\ m = y/R,\ n = 0)$ on which for the reason of the problem statement $p_{\nu x} = p_{\nu y} = p_{\nu z} = 0$ are also satisfied.

On the butt surfaces the third equilibrium equation of elementary tetrahedron is satisfied ($p_{zz} = 0$) but the first and the second ones give next dependencies

$$\left.\begin{array}{l}\text{for left butt } z=0,\ \nu(0,0,n=-1)\\ p_{zx}=\tau_{zx}n=\dfrac{M_z\cdot y}{I_p},\quad p_{zy}=\tau_{zy}n=-\dfrac{M_z\cdot x}{I_p},\\ \text{for right butt } z=L,\ \nu(0,0,n=1)\\ p_{zx}=\tau_{zx}n=\dfrac{M_z\cdot y}{I_p},\quad p_{zy}=\tau_{zy}n=-\dfrac{M_z\cdot x}{I_p}.\end{array}\right\} \qquad (6.11)$$

The system of surface tangent loads $p_{zx}=p_{zy}$ is statically equivalent to torque $M_z$ at each butt.

Thus the solution (6.10) is really the accurate solution of reviewed problem, but only in case if surface forces on butts are really distributed as laws of shear forces in inside cross sections. In other cases the solution (6.10) can be used as an approximate applying the Saint-Venant principle. But if the change of the cross section of a bar is smooth with a little gradient then the solution (6.10) have to be reviewed as approximate. If the form of a bar is not round the solution (6.10) loss its truth.

Using Hook's law it is possibly to get strains

$$\varepsilon_x=\varepsilon_y=\varepsilon_z=\gamma_{xy}=0,\quad \gamma_{zx}=\frac{M_z}{GI_p}y,\quad \gamma_{zy}=-\frac{M_z}{GI_p}x\,. \qquad (6.12)$$

Accepting (6.7) and (4.15)–(4.17) we shall have the next relations for displacements

$$u=\frac{M_z}{GI_p}yz,\quad v=\frac{M_z}{GI_p}xz,\quad w=0\,. \qquad (6.13)$$

Orthogonal cross sections still to be flat ($w\equiv 0$) and undeformed in its plane ($\varepsilon_x=\varepsilon_y=\varepsilon_z=\gamma_{xy}$), rotating for angle $\theta=\alpha z$ (where $\alpha=M_z/(GI_p)$ is a linear rotation angle) relatively cross section $z=0$.

# LECTURE 7. RELATIONS FOR STRESS STATE IN POINT

**Stress State at Rotation of Axes. Stress State in Inclined Elemental Areas.** The stress state in some point is set by the stresses $\sigma_x$, $\tau_{xy}$, … , $\sigma_z$ in coordinate system *xyzO*. It is necessary to define components of this stress state $\sigma_{x'}$, $\tau_{x'y'}$, … , $\sigma_{z'}$ in new coordinate system $x'y'z'O'$ direction cosines of which are set by nine direction cosines (see table 7.1).

Table 7.1 – Direction cosines

| Old axes | New axes | | |
|---|---|---|---|
| | $x'$ | $y'$ | $z'$ |
| $x$ | $l_1$ | $l_2$ | $l_3$ |
| $y$ | $m_1$ | $m_2$ | $m_3$ |
| $z$ | $n_1$ | $n_2$ | $n_3$ |

Coordinate systems *xyzO* and are mutually orthogonal to each other. Let's review how to compose expressions for the stress state for one elemental area, normal vector to which is $\nu = x'(l_1, m_1, n_1)$ but tangent vectors are $y'$, $z'$, relatively old axes *xyzO*. Components of full stresses in plane $\nu = x'$ can be defined using equilibrium equations of elementary tetrahedron

$$\left.\begin{aligned}\sigma_x l_1 + \tau_{yx} m_1 + \tau_{zx} n_1 &= p_{x'x},\\ \tau_{xy} l_1 + \sigma_y m_1 + \tau_{zy} n_1 &= p_{x'y},\\ \tau_{xz} l_1 + \tau_{yz} m_1 + \sigma_z n_1 &= p_{x'z}.\end{aligned}\right\} \tag{7.1}$$

Composing the sum of projections (7.1) into vectors/axes $x'$, $y'$, $z'$ to get normal and shear components of full stresses for elemental area $\nu = x'$ we shall have

$$\left.\begin{aligned}\sigma_{x'} &= p_{x'x} l_1 + p_{x'y} m_1 + p_{x'y} n_1,\\ \tau_{x'y'} &= p_{x'x} l_2 + p_{x'y} m_2 + p_{x'y} n_2,\\ \tau_{x'z'} &= p_{x'x} l_3 + p_{x'y} m_3 + p_{x'y} n_3.\end{aligned}\right\} \tag{7.2}$$

Substituting in (7.2) expressions for $p_{x'x}$, $p_{x'y}$, $p_{x'z}$ from (7.1) we shall have final expressions for $\sigma_{x'}$, $\tau_{x'y'}$, $\tau_{x'z'}$. Doing the same operations mentioned above for elemental areas $\nu = y'(l_2, m_2, n_2)$, $\nu = z'(l_3, m_3, n_3)$ we shall have the rest components of stress state. Finally transformation formulas for components of stress state specified in coordinate axes $xyzO$ to new coordinate axes $x'y'z'O'$ (at rotation of coordinate axes from position $xyzO$ to position $x'y'z'O'$) have next presentation

$$\left.\begin{aligned}\sigma_{x'} &= \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2\tau_{xy} l_1 m_1 + 2\tau_{yz} m_1 n_1 + 2\tau_{zx} n_1 l_1,\\ \sigma_{y'} &= \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2\tau_{xy} l_2 m_2 + 2\tau_{yz} m_2 n_2 + 2\tau_{zx} n_2 l_2,\\ \sigma_{z'} &= \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2\tau_{xy} l_3 m_3 + 2\tau_{yz} m_3 n_3 + 2\tau_{zx} n_3 l_3,\\ \tau_{x'y'} &= \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}\left(l_1 m_2 + m_1 l_2\right) +\\ &\qquad + \tau_{yz}\left(m_1 n_2 + n_1 m_2\right) + \tau_{zx}\left(l_1 n_2 + n_1 l_2\right),\\ \tau_{y'z'} &= \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}\left(l_2 m_3 + m_2 l_3\right) +\\ &\qquad + \tau_{yz}\left(m_2 n_3 + n_2 m_3\right) + \tau_{zx}\left(l_2 n_3 + n_2 l_3\right),\\ \tau_{z'x'} &= \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}\left(l_3 m_1 + m_3 l_1\right) +\\ &\qquad + \tau_{yz}\left(m_3 n_1 + n_3 m_1\right) + \tau_{zx}\left(l_3 n_1 + n_3 l_1\right).\end{aligned}\right\} \tag{7.3}$$

**Principal Planes and Principal Stresses.** Let's review elementary parallelepiped defined in locality of some point. As it was mentioned above faces of elementary parallelepiped are always parallel to coordinate planes. Rotating coordinate axes we shall have different elementary parallelepipeds appropriate stresses and strains acting on its faces. At last we can rotate our coordinate axes in a such way that shear stresses on all six faces on three mutual orthogonal coordinate elemental planes disappear, but there will be only normal stresses on those elemental planes. These planes are called principal. So *principal planes* are planes on which there are no shear stresses. *Principal normal stresses* are stresses acting in principal planes.

Let's explain how to obtain magnitudes and directions of principal stresses. Let's assume there is some point *A* with known stress tensor in it defined in coordinate system *xyzO* (see fig. 7.1).

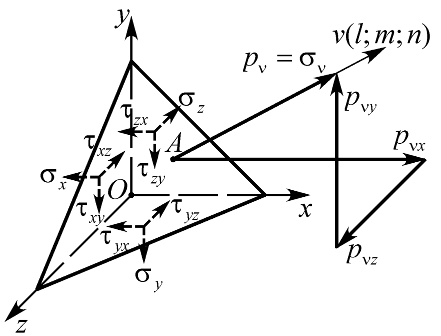



Fig. 7.1 – For explanation of principle stresses

Let's guess that the inclined plane with direction cosines $\nu(l; m; n)$ coming trough this point is principle. As for the principle plane full shear stresses equal to zero i.e. $\tau_\nu = 0$, full stresses $p_\nu$ are full normal stresses $\sigma_\nu$ simultaneously. Direction cosines of full stresses $p_\nu = \sigma_\nu$ are the same as for the normal vector to the principle plane, i.e. $\nu(l; m; n)$.

That's why it is possibly to write

$$p_{\nu x} = p_{\nu} l = \sigma_{\nu} l, \quad p_{\nu y} = p_{\nu} m = \sigma_{\nu} m, \quad p_{\nu z} = p_{\nu} n = \sigma_{\nu} n. \tag{7.4}$$

Substituting components $p_{\nu x}, p_{\nu y}, p_{\nu z}$ defined above by (7.4) into equilibrium equations of elementary tetrahedron (3.4) we can get next

$$\left.\begin{aligned} \sigma_x l + \tau_{yx} m + \tau_{zx} n = \sigma_{\nu} l, \\ \tau_{xy} l + \sigma_y m + \tau_{zy} n = \sigma_{\nu} m, \\ \tau_{xz} l + \tau_{yz} m + \sigma_z n = \sigma_{\nu} n. \end{aligned}\right\} \Rightarrow \left.\begin{aligned} (\sigma_x - \sigma_{\nu}) l + \tau_{yx} m + \tau_{zx} n = 0, \\ \tau_{xy} l + (\sigma_y - \sigma_{\nu}) m + \tau_{zy} n = 0, \\ \tau_{xz} l + \tau_{yz} m + (\sigma_z - \sigma_{\nu}) n = 0. \end{aligned}\right\} \tag{7.5}$$

As zero solution for direction cosines is impossible due to known dependency

$$l^2 + m^2 + n^2 = 1, \tag{7.6}$$

the determinant of homogenous equations (7.5) has to equal zero. In the opposite case the system (7.5)–(7.6) is incompatible. Expanding the determinant (7.5) we shall get the nonlinear or so called cubical equation which allows us to find principle stresses

$$\sigma_{\nu}^3 - I_1 \sigma_{\nu}^2 + I_2 \sigma_{\nu} - I_3 = 0, \tag{7.7}$$

where

$$\left.\begin{aligned} & I_1(T_{\sigma}) = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3, \\ & I_2(T_{\sigma}) = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1, \\ & I_3(T_{\sigma}) = \sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{zx}^2 - \sigma_z \tau_{xy}^2 = \sigma_1 \sigma_2 \sigma_3. \end{aligned}\right\} \tag{7.8}$$

Quantities $I_1, I_2, I_3$ are called stress invariants. Stress invariants don't depend on orientation of coordinate system in space relatively body. It means that reviewing strain-stress states in some point relatively different coordinate axes we shall have that stress and strain tensors will be different, but stress invariants will always be the same.

The equation (7.7) is obtained from the symmetry determinant and as it is proved in the higher algebra it has three real roots $\sigma_1$, $\sigma_2$, $\sigma_3$ which are principal stresses arising on the three principle planes. Principal normal stresses have to be ordered applying the next rule: $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Direction cosines of principal stresses can be found using (7.5).

Let's review how to get direction cosines for $\sigma_1$. As the determinant of (7.5) equals zero there are only two independent equations in (7.5) from which two direction cosines can be expressed via the third one. For example

$$m_1 = a_1 l_1, \quad n_1 = b_1 l_1, \tag{7.9}$$

where $a_1$, $b_1$ are some unknown factors which have to be found.

Substituting (7.9) into (7.6) we shall have next relations

$$l_1 = \pm \frac{1}{\sqrt{1 + a_1^2 + b_1^2}},$$

$$m_1 = \pm \frac{a_1}{\sqrt{1 + a_1^2 + b_1^2}}, \quad n_1 = \pm \frac{b_1}{\sqrt{1 + a_1^2 + b_1^2}}. \tag{7.10}$$

Also substituting (7.9) into (7.5) we shall have next equations

$$\left.\begin{aligned} &\tau_{yx} a_1 + \tau_{zx} b_1 = \sigma_1 - \sigma_x, \\ &(\sigma_y - \sigma_1) a_1 + \tau_{zy} b_1 = -\tau_{xy}, \\ &\tau_{yz} a_1 + (\sigma_z - \sigma_1) b_1 = -\tau_{xz}. \end{aligned}\right\} \tag{7.11}$$

Solving mutually two any equations of three ones given in (7.11) we have to find $a_1$ and $b_1$. After using (7.10) for known $a_1$ and $b_1$ we can find direction cosines $l_1$, $m_1$, $n_1$ of normal vector of principal plane where $\sigma_1$ takes place. Analogously direction cosines for $\sigma_2$ and $\sigma_3$ can be found substituting sequentially $\sigma_\nu = \sigma_2$, $\sigma_\nu = \sigma_3$, repeating procedure described above.

Found direction cosines for three principle planes have to satisfy for the next relations

$$\left.\begin{array}{l} \sum\limits_{i=1}^{i=3} l_i^2 = 1, \quad \sum\limits_{i=1}^{i=3} m_i^2 = 1, \quad \sum\limits_{i=1}^{i=3} n_i^2 = 1, \quad l_i^2 + m_i^2 + n_i^2 = 1, \quad i = 1,\ 2,\ 3; \\ \\ \sum\limits_{i=1}^{i=3} l_i m_i = 0, \quad \sum\limits_{i=1}^{i=3} m_i n_i = 0, \quad \sum\limits_{i=1}^{i=3} n_i l_i = 0, \quad l_i l_j + m_i m_j + n_i n_j = 0, \\ \text{where} \quad ij = 12,\ 23,\ 31. \end{array}\right\} \qquad (7.12)$$

Taking into consideration absence of shear stresses on principle planes stress tensor in principle coordinate axes *1*, *2*, *3* can be written as

$$T_\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \qquad (7.13)$$

**Stress Ellipsoid.** Let's review some stress state in arbitrary point defined by the stress tensor (7.13) in principle axes *1*, *2*, *3* and some plane in the same point, direction cosines *l*, *m*, *n* of which are defined relatively principle axes *1*, *2*, *3*. Using equilibrium equations of elementary tetrahedron and changing designations *x*, *y*, *z* there into *1*, *2*, *3* accordingly also applying tensor components (7.13) we shall have projections of full stresses $p_\nu$ into coordinate axes in the given plane

$$p_{\nu 1} = \sigma_1 l, \quad p_{\nu 2} = \sigma_2 m, \quad p_{\nu 3} = \sigma_3 n. \qquad (7.14)$$

From (7.14) we have that

$$l = \frac{p_{\nu 1}}{\sigma_1}, \quad m = \frac{p_{\nu 2}}{\sigma_2}, \quad n = \frac{p_{\nu 3}}{\sigma_3}. \qquad (7.15)$$

Taking into account that $l^2 + m^2 + n^2 = 1$ we shall have next relation

$$\left(\frac{p_{\nu 1}}{\sigma_1}\right)^2 + \left(\frac{p_{\nu 2}}{\sigma_2}\right)^2 + \left(\frac{p_{\nu 3}}{\sigma_3}\right)^2 = 1. \qquad (7.16)$$

As components $p_{\nu 1}, p_{\nu 2}, p_{\nu 3}$ can be considered as coordinates of end of vector of full stresses $\boldsymbol{p}_{\boldsymbol{\nu}}$, there is conclusion based on the (7.16) that this end of a vector of full stresses defines closed surface which is called ellipsoid. Axes of this ellipsoid match principle axes and equal to according principle strains $\sigma_1$, $\sigma_2$, and $\sigma_3$. This ellipsoid is called the Lame ellipsoid (see fig. 7.2).

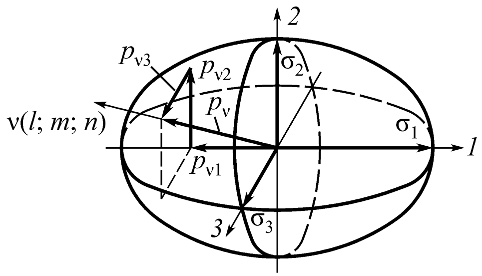



Fig. 7.2 – Stress ellipsoid of Lame

**Maximum Shear Stresses and Its Planes.** Among all planes, which can be defined in some point there is the one in which shear stresses have the largest value on modulus. It is convenient to find the solution in principle axes *1*, *2*, *3*, considering stress tensor (7.13) to be known. For some plane defined in some point there is normal vector **ν** which have direction cosines $l$, $m$, $n$ relatively principle axes. In that plane there are components of full stresses $p_{\nu 1}, p_{\nu 2}, p_{\nu 3}$ defined by (7.14).

Squares of full stresses, full normal stresses and full shear stresses can be found using obvious expressions

$$\left.\begin{aligned} p_\nu^2 &= p_{\nu1}^2 + p_{\nu2}^2 + p_{\nu3}^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2, \\ \sigma_\nu^2 &= (p_{\nu1} l + p_{\nu2} m + p_{\nu3} n)^2 = (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2, \\ \tau_\nu^2 &= p_\nu^2 - \sigma_\nu^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2. \end{aligned}\right\} \quad (7.17)$$

There is problem to find optimal values of direction cosines $l$, $m$, $n$ relatively principle axes which provide maximum to $\tau_\nu^2$. Reviewing the most common occasion, when $\sigma_1$, $\sigma_2$, $\sigma_3$ are different we shall have next optimal values of direction cosines $l$, $m$, $n$ which provide only maximums to $\tau_\nu^2$ in (7.17):

$$\left.\begin{aligned} &l = 0, \quad m = \pm\frac{1}{\sqrt{2}}, \quad n = \pm\frac{1}{\sqrt{2}}, \\ &l = \pm\frac{1}{\sqrt{2}}, \quad m = 0, \quad n = \pm\frac{1}{\sqrt{2}}, \\ &l = \pm\frac{1}{\sqrt{2}}, \quad m = \pm\frac{1}{\sqrt{2}}, \quad n = 0. \end{aligned}\right\} \quad (7.18)$$

Substituting direction cosines (7.18) into the third equation of (7.17) we can get next values of maximum shear stresses acting in according planes defined by (7.18)

$$\tau_1 = \frac{\sigma_2 - \sigma_3}{2}, \quad \tau_2 = \tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}, \quad \tau_3 = \frac{\sigma_1 - \sigma_2}{2}, \quad (7.19)$$

where $\tau_{\max}$ – are the largest shear stresses of maximum shear stresses.

So $\tau_1$, $\tau_2$, $\tau_3$ are maximum shear stresses, which arise in planes parallel to according principal axes *1*, *2*, *3*. These planes divide orthogonal angle in two between the rest principle directions (see fig. 7.3).

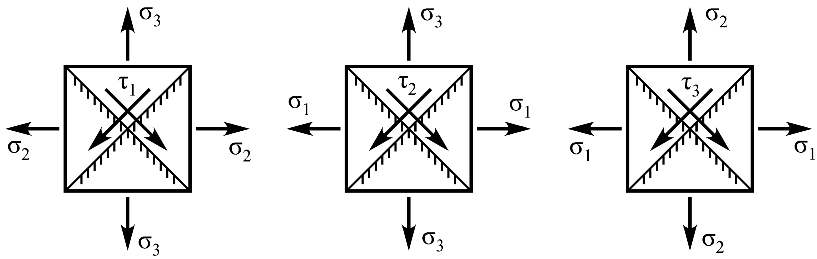



Fig. 7.3 – Planes of the maximum shear stresses

**Octahedron Stresses and Intensity of Stresses.** With the exception on reviewed above principal normal stresses and maximum shear stresses there is interest of stresses acting in planes equally inclined to principle axes *1*, *2*, *3*. These planes are called *octahedron planes* (see fig. 7.4).

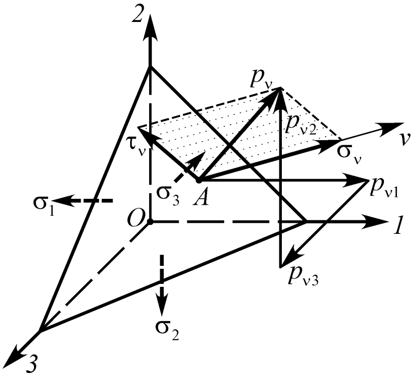



Fig. 7.4 – Octahedron stresses

Directions cosines of normal vectors of octahedron planes are next

$$l = l_{oct} = \pm\frac{1}{\sqrt{3}}, \quad m = m_{oct} = \pm\frac{1}{\sqrt{3}}, \quad n = n_{oct} = \pm\frac{1}{\sqrt{3}}. \quad (7.20)$$

Normal octahedron $\sigma_{oct}$ and shear octahedron $\tau_{oct}$ stresses are defined using expressions

$$\left.\begin{aligned}
&\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \sigma_{av},\\
&\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{2}{3}\sqrt{\tau_1^2 + \tau_2^2 + \tau_3^2},\\
&\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)},
\end{aligned}\right\} \quad (7.21)$$

where $\sigma_{av}$ – are average normal stresses in point.

In the theory of plasticity they operate with quantity proportional to $\tau_{oct}$ and which is called the *intensity of stresses*

$$\sigma_i = \frac{3}{\sqrt{2}}\tau_{oct}. \quad (7.22)$$

## LECTURE 8. RELATIONS FOR STRAIN STATE IN POINT

**Transformation of Displacements and Strains at Rotation of Coordinate Axes.** There is displacement $\rho = (u, v, w)$ of some point in a body stated in the coordinate system $xyzO$, but it is necessary to define displacement $\rho' = (u', v', w')$ of the same point but in the new coordinate system $x'y'z'O'$, axes of which have the same direction cosines as it is presented in the table 7.1. It can be done using next expressions

$$\left.\begin{aligned} u' &= ul_1 + vm_1 + wn_1, \\ v' &= ul_2 + vm_2 + wn_2, \\ w' &= ul_3 + vm_3 + wn_3. \end{aligned}\right\} \tag{8.1}$$

Components of strains $\varepsilon_{x'}$, $\gamma_{x'y'}$, … , $\varepsilon_{z'}$ in the new coordinate system $x'y'z'O'$ for assumptions accepted in frames of the linear theory of elasticity can be found using Cauchy relations if to replace components $(u, v, w)$ for $(u', v', w')$ in it, but coordinates $x$, $y$, $z$ for coordinates $x'$, $y'$, $z'$.

The linear strain $\varepsilon_{x'}$ in the direction of axis $x'$ is this

$$\varepsilon_{x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial x'}. \tag{8.2}$$

Direction cosines $l_1, m_1, n_1$ can be found using expressions

$$l_1 = \frac{\partial x}{\partial x'}, \quad m_1 = \frac{\partial y}{\partial y'}, \quad n_1 = \frac{\partial z}{\partial z'}. \tag{8.3}$$

Substituting (8.1) into (8.2) considering (8.3) we shall have next

$$\left.\begin{aligned} \varepsilon_{x'} = \left( \frac{\partial u}{\partial x} l_1 + \frac{\partial v}{\partial x} m_1 + \frac{\partial w}{\partial x} n_1 \right) l_1 + \left( \frac{\partial u}{\partial y} l_1 + \frac{\partial v}{\partial y} m_1 + \frac{\partial w}{\partial y} n_1 \right) m_1 + \\ + \left( \frac{\partial u}{\partial z} l_1 + \frac{\partial v}{\partial z} m_1 + \frac{\partial w}{\partial z} n_1 \right) n_1 \end{aligned}\right\} \tag{8.4}$$

taking into account Cauchy relations instead derivatives in (8.4) we have next

$$\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_y m_1^2 + \varepsilon_z n_1^2 + 2\varepsilon_{xy} l_1 m_1 + 2\varepsilon_{yz} m_1 n_1 + 2\varepsilon_{zx} n_1 l_1 . \tag{8.5}$$

In the same way it is possibly to get formulae for the rest components of strain in axis $x', y', z'$. Finally we shall have

$$\left.\begin{aligned} &\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_y m_1^2 + \varepsilon_z n_1^2 + 2\varepsilon_{xy} l_1 m_1 + 2\varepsilon_{yz} m_1 n_1 + 2\varepsilon_{zx} n_1 l_1, \\ &\varepsilon_{y'} = \varepsilon_x l_2^2 + \varepsilon_y m_2^2 + \varepsilon_z n_2^2 + 2\varepsilon_{xy} l_2 m_2 + 2\varepsilon_{yz} m_2 n_2 + 2\varepsilon_{zx} n_2 l_2, \\ &\varepsilon_{z'} = \varepsilon_x l_3^2 + \varepsilon_y m_3^2 + \varepsilon_z n_3^2 + 2\varepsilon_{xy} l_3 m_3 + 2\varepsilon_{yz} m_3 n_3 + 2\varepsilon_{zx} n_3 l_3, \\ &\varepsilon_{x'y'} = \varepsilon_x l_1 l_2 + \varepsilon_y m_1 m_2 + \varepsilon_z n_1 n_2 + \varepsilon_{xy} \left( l_1 m_2 + m_1 l_2 \right) + \\ &\qquad + \varepsilon_{yz} \left( m_1 n_2 + n_1 m_2 \right) + \varepsilon_{zx} \left( l_1 n_2 + n_1 l_2 \right), \\ &\varepsilon_{y'z'} = \varepsilon_x l_2 l_3 + \varepsilon_y m_2 m_3 + \varepsilon_z n_2 n_3 + \varepsilon_{xy} \left( l_2 m_3 + m_2 l_3 \right) + \\ &\qquad + \varepsilon_{yz} \left( m_2 n_3 + n_2 m_3 \right) + \varepsilon_{zx} \left( l_2 n_3 + n_2 l_3 \right), \\ &\varepsilon_{z'x'} = \varepsilon_x l_3 l_1 + \varepsilon_y m_3 m_1 + \varepsilon_z n_3 n_1 + \varepsilon_{xy} \left( l_3 m_1 + m_3 l_1 \right) + \\ &\qquad + \varepsilon_{yz} \left( m_3 n_1 + n_3 m_1 \right) + \varepsilon_{zx} \left( l_3 n_1 + n_3 l_1 \right), \end{aligned}\right\} \tag{8.6}$$

where $\varepsilon_{xy} = \gamma_{xy}/2$, $\varepsilon_{yz} = \gamma_{yz}/2$, $\varepsilon_{zx} = \gamma_{zx}/2$.

**Principal Linear and Maximum Shear Strains.** There are several ways to find principal linear strains. One of the ways is to calculate principal linear strains $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ using known values of principal stresses using next expressions

$$\left.\begin{aligned}
\varepsilon_1 &= \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)];\\
\varepsilon_2 &= \frac{1}{E}[\sigma_2 - \mu(\sigma_3 + \sigma_1)];\\
\varepsilon_3 &= \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)];\\
\gamma_{12} &= \gamma_{23} = \gamma_{31} = 0,
\end{aligned}\right\} \tag{8.7}$$

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ – principle linear strains; $\gamma_{12}$, $\gamma_{23}$, $\gamma_{31}$ – are principle shear strains.

But if there are only stains are known it is possibly to find principle strains solving next equation

$$\varepsilon_\nu^3 - I_1\varepsilon_\nu^2 + I_2\varepsilon_\nu - I_3 = 0\,, \tag{8.8}$$

where

$$\left.\begin{aligned}
I_1(T_\varepsilon) &= \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,\\
I_2(T_\varepsilon) &= \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{zx}^2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1,\\
I_3(T_\varepsilon) &= \varepsilon_x\varepsilon_y\varepsilon_z + 2\varepsilon_{xy}\varepsilon_{yz}\varepsilon_{zx} - \varepsilon_x\varepsilon_{yz}^2 - \varepsilon_y\varepsilon_{zx}^2 - \varepsilon_z\varepsilon_{xy}^2 = \varepsilon_1\varepsilon_2\varepsilon_3.
\end{aligned}\right\} \tag{8.9}$$

Quantities $I_1$, $I_2$, $I_3$ are called strain invariants. Strain invariants don’t depend on orientation of coordinate system in space relatively body. There are always only three real roots of (8.8) which are principal linear strains. Directions of principal linear strains are the same as directions of principal normal stresses.

Maximum shear strains $\gamma_1, \gamma_2, \gamma_3$ can be found using expressions

$$\left.\begin{aligned}
\gamma_1 &= \frac{\tau_1}{G} = \frac{(\sigma_2 - \sigma_3)}{2G} = \varepsilon_2 - \varepsilon_3, \\
\gamma_2 &= \gamma_{\max} = \frac{\tau_2}{G} = \frac{(\sigma_1 - \sigma_3)}{2G} = \varepsilon_1 - \varepsilon_3, \\
\gamma_3 &= \frac{\tau_3}{G} = \frac{(\sigma_1 - \sigma_2)}{2G} = \varepsilon_1 - \varepsilon_2.
\end{aligned}\right\} \tag{8.10}$$

The system of equations from which we define directions cosines of principal strains can be got analogously as for stresses.

**Octahedron Strain and Intensity of Strains.** The linear strain of element the direction of which coincides with the normal vector to octahedron plane equals to the average linear strain in point.

Linear octahedron $\varepsilon_{oct}$ and shear octahedron $\gamma_{oct}$ strains are defined using expressions

$$\left.\begin{aligned}
\varepsilon_{oct} &= \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z) = \varepsilon_{av}, \\
\gamma_{oct} &= \frac{\tau_{oct}}{G} = \frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2} = \frac{2}{3}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}, \\
\gamma_{oct} &= \frac{2}{3}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)},
\end{aligned}\right\} \tag{8.11}$$

where $\varepsilon_{av}$ – is average normal strain in point.

In the theory of plasticity they operate with quantity proportional to $\gamma_{oct}$ which is called the *intensity of strains*

$$\varepsilon_i = \frac{\gamma_{oct}}{\sqrt{2}}. \tag{8.12}$$

**Deformation of Volume and Form.** Let's define relative change of volume of material in locality of some point as a result caused by deformation. Before deformation sizes of elementary tetrahedron are *dx*, *dy*, *dz*, and the volume is $V = dx{\cdot}dy{\cdot}dz$. After the deformation its sizes are $(1 + \varepsilon_x)dx$, $(1 + \varepsilon_y)dy$, $(1 + \varepsilon_z)dz$, and the volume $V_1 = (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)\, dx{\cdot}dy{\cdot}dz$. Relative change of the volume or volume deformation can be defined in the next way

$$\left.\begin{aligned}\theta = \frac{V_1 - V}{V} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1 = \\ = \varepsilon_x + \varepsilon_y + \varepsilon_z = 3\varepsilon_{av}.\end{aligned}\right\} \qquad (8.13)$$

There is no multiplications of linear strains in (8.13) which evidently are infinitesimal of higher order in comparison with strains itself. Thus the volume deformation equal to the first strain invariant or tripled value of average linear strains.

Arbitrary strain state in a point always can be presented as a sum of two deformed states: tensor of one of it $T_\varepsilon^0$ include all volume deformation of initial state without change of form of elements, and the tensor of the second state $D_\varepsilon$ is additional to the first one. Mentioned tensors can be written

$$T_\varepsilon^0 = \begin{bmatrix} \varepsilon_{av} & 0 & 0 \\ 0 & \varepsilon_{av} & 0 \\ 0 & 0 & \varepsilon_{av} \end{bmatrix}, \qquad (8.14)$$

$$D_\varepsilon = \begin{bmatrix} \varepsilon_x - \varepsilon_{av} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_{av} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_{av} \end{bmatrix}, \qquad (8.15)$$

$$T_{\varepsilon} = T_{\varepsilon}^{0} + D_{\varepsilon}, \tag{8.16}$$

where $T_{\varepsilon}^{0}$ – is *tensor of volume deformation* or *spherical strain tensor*; $D_{\varepsilon}$ – is *tensor of deformation of a form* or *strain deviator*; $T_{\varepsilon}$ – full strain tensor, which is the sum of spherical strain tensor and strain deviator.

Spherical tensor $T_{\varepsilon}^{0}$ describe change of volume of a body only, but the strain deviator $D_{\varepsilon}$ describe change of body form only.

Stress tensor can be presented as the sum of two tensors analogously as we showed for strain tensor. One of these stress tensors fits to spherical strain tensor and denoted as . The second stress tensor match strain deviator $D_{\varepsilon}$ and denoted as $D_{\sigma}$. These stress tensors can be written as

$$T_{\sigma}^{0} = \begin{bmatrix} \sigma_{av} & 0 & 0 \\ 0 & \sigma_{av} & 0 \\ 0 & 0 & \sigma_{av} \end{bmatrix}, \tag{8.17}$$

$$D_{\sigma} = \begin{bmatrix} \sigma_{x} - \sigma_{av} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} - \sigma_{av} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} - \sigma_{av} \end{bmatrix} \tag{8.18}$$

$$T_{\sigma} = T_{\sigma}^{0} + D_{\sigma}, \tag{8.19}$$

where $T_{\sigma}^{0}$ – spherical stress tensor; $D_{\sigma}$ – stress deviator; $T_{\sigma}$ – full stress tensor, which is the sum of spherical stress tensor and stress deviator.

Using Hook's laws it is possibly to get the next relation between components of spherical tensors

$$\left.\begin{aligned} \sigma_{av} = 3E_0\varepsilon_{av}, \quad \varepsilon_{av} = \frac{\sigma_{av}}{3E_0}, \quad E_0 = \frac{E}{3(1-2\mu)}, \\ T_\sigma^0 = E_0 T_\varepsilon^0, \quad T_\varepsilon^0 = \frac{1}{E_0} T_\sigma^0, \end{aligned}\right\} \tag{8.20}$$

where $E_0$ – is the volume modulus of elasticity.

Analogously there are relations between components of deviators of stresses and strains

$$\left.\begin{aligned} \sigma_x - \sigma_{av} = 2G(\varepsilon_x - \varepsilon_{av}), \quad \tau_{xy} = 2G\left(\frac{1}{2}\gamma_{xy}\right), \\ \sigma_y - \sigma_{av} = 2G(\varepsilon_y - \varepsilon_{av}), \quad \tau_{yz} = 2G\left(\frac{1}{2}\gamma_{yz}\right), \\ \sigma_z - \sigma_{av} = 2G(\varepsilon_z - \varepsilon_{av}), \quad \tau_{zx} = 2G\left(\frac{1}{2}\gamma_{zx}\right), \\ D_\sigma = 2GD_\varepsilon, \quad D_\varepsilon = \frac{1}{2G} D_\sigma. \end{aligned}\right\} \tag{8.21}$$

The relation (8.20) is the law of body volume change but the (8.21) is the law of body form change.

**Specific Elastic Energy of a Body.** In a process of static loading of elastic body on account of inevitable motion of points caused by applied forces these forces produce work. If a body is completely elastic then for static unloading it will renew its initial form and sizes, and internal forces disappearing little by little will do the same work as external forces in the process of loading.

Let's review elastic body the strain-stress state of which is set by stress tensor $T_\sigma$ and strain tensor $T_\varepsilon$. These tensors are connected with each other by Hook's law. Let's separate in locality of some point elementary parallelepiped with sizes *dx*, *dy*, *dz*. If to review this parallelepiped as a separate body then stresses acting on its facets are conservative external surface forces, the work of which for static action equals to potential energy in the volume of this parallelepiped.

To simplify reasoning we consider that all components of stresses and appropriate strains change simultaneously proportionally to one parameter. In this case relation between two components of the same name of strain and stress tensors is linear during all the process of loading (unloading).

For example if to review only action of normal stresses $\sigma_x$ for facets orthogonal to axis $x$, the force acting on these facets is $\sigma_x \cdot dy \cdot dz$, but the distance between facets change for the magnitude $\varepsilon_x \cdot dx$. The dependency between $\sigma_x$ and $\varepsilon_x$ is linear (the proportional factor has no matter). It allows to write the expression for the work of the force $\sigma_x \cdot dy \cdot dz$ along axis $x$

$$A_{\sigma_x} = \frac{1}{2}(\sigma_x dydz)(\varepsilon_x dx). \qquad (8.22)$$

If on the same facets there are two shear forces (equal for absolute magnitudes but opposite for direction) parallel to axis $z$, $\tau_{xz} dydz$, and the relative deposition of facets in the direction of acting forces due to shear equals $\gamma_{xz} dx$, the work of these forces considering linear dependency between $\tau_{xz}$ and $\gamma_{xz}$ is

$$A_{\tau_{xz}} = \frac{1}{2}(\tau_{xz} dydz)(\gamma_{xz} dx). \qquad (8.23)$$

Expressions analogue to (8.22), (8.23) can be obtained for other components of strain-stress state. Total work $dA$ of all forces or equal in magnitude potential energy $dW$ accommodated in the volume of elementary parallelepiped $dx \cdot dy \cdot dz$ is

$$dA = dV = W\, dxdydz, \qquad (8.24)$$

where

$$\left.\begin{aligned}
&W(x,y,z)=\frac{1}{2}\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)=\\
&=\frac{1}{2E}[\sigma_x^2+\sigma_y^2+\sigma_z^2-2\mu(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x)+2(1+\mu)(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)]=\\
&=G\left[\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2+\frac{\mu}{1-2\mu}\theta^2+\frac{1}{2}(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2)\right]=\\
&=G\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+\frac{\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2+\right.\\
&\qquad\left.+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]\right\},\\
&\theta=\varepsilon_x+\varepsilon_y+\varepsilon_z,
\end{aligned}\right\} \quad (8.25)$$

the value $W$ – is the *specific potential energy of deformation*, joule/m$^3$.

Total potential energy $\ddot{I}$ in the whole body volume can be found using expression

$$\Pi=\iiint\limits_V W(x,y,z)dxdydz\ . \quad (8.26)$$

The specific potential energy of deformation $W$ splits into specific potential energy of volume $W_0$ and form $W_f$. So $W$ can be presented

$$W=W_0+W_f\ . \quad (8.27)$$

Specific potential energy of volume $W_0$ can be found using expressions

$$W_0=\frac{3(1-2\mu)}{2E}\sigma_{av}^2=\frac{3}{2E_0}\sigma_{av}^2=\frac{3E_0}{2}\varepsilon_{av}^2=\frac{\sigma_{oct}^2}{2E_0}. \quad (8.28)$$

Specific potential energy of form $W_f$ can be found using expressions

$$W_f=\frac{3(1+\mu)}{2E}\tau_{oct}^2=\frac{1}{2}\sigma_i\varepsilon_i=\frac{3E}{4(1+\mu)}\varepsilon_i^2=\frac{(1+\mu)}{3E}\sigma_i^2\ . \quad (8.29)$$

**Basic Theories of Strength and Plasticity.** Solution of the direct problem of the theory of elasticity, i.e. definition of stresses, strains and displacements is not end in itself. We solve direct or contrary problem in frames of design of some construction. In the strength of materials we compare real stresses in construction with allowable ones. Thus we have to find the most tensed point in a body and to compare it with some admissible stresses. So there is the problem to define the most tensed i.e. the most dangerous point in a body.

Sometimes it is necessary to compare several points in loaded body for their stress state. It is necessary to know which point is more tensed but which is less tensed. There is conclusion that it is necessary to have some unique parameter which would be able to estimate stress state in point uniquely. This parameter is usually called *equivalent stresses*.

For every point of a body there is complex stress state in general case because there are six independent components of stresses there. The problem of definition of unique parameter or equivalent stresses mentioned above can be simplified if to set the criteria of comparability of different complex stress states with simple ones such as tension-compression or shear. Definition of mentioned criteria is the part of the problem of so called theories/hypothesis of strength and plasticity. There are a lot of such theories. But we shall review only several of it.

There are four classic or basic theories of strength.

The **1-st** strength theory is called the theory of maximum normal stresses. If we have ordered principle stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ equivalent stresses can be found using next simple relations

$$\left.\begin{aligned} \sigma_{eq} &= \sigma_1, \quad \text{if } \sigma_1 > 0; \\ \sigma'_{eq} &= \sigma_3, \quad \text{if } \sigma_3 < 0. \end{aligned}\right\}. \tag{8.30}$$

First expression in (8.30) defines equivalent stresses for simple tension but the second one defines equivalent stresses for simple compression.

The **2-nd** strength theory is called the theory of maximum linear strains. There are two expressions for equivalent stresses

$$\left.\begin{aligned} \sigma_{eq} &= \sigma_1 - \mu(\sigma_2 + \sigma_3) > 0, \\ \sigma'_{eq} &= \sigma_3 - \mu(\sigma_1 + \sigma_2) < 0. \end{aligned}\right\} \qquad (8.31)$$

First expression of (8.31) defines equivalent stresses for simple tension but the second one defines equivalent stresses for simple compression. Equivalent stresses defined by (8.31) have to be reviewed together in estimation of strength.

The **3-rd** strength theory is called the theory of maximum shear stresses. Equivalent stresses for this theory can be found using expression

$$\sigma_{eq} = \sigma_1 - \sigma_3 \,. \qquad (8.32)$$

The **4-th** strength theory is called the theory of specific potential energy of deformation of the body form (after *Richard Edler von Mises* and *Maksymilian Tytus Huber*). Equivalent stresses for this theory can be found using expression

$$\left.\begin{aligned} \sigma_{eq} = \sigma_i &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \\ &= \frac{2}{\sqrt{2}}\sqrt{\tau_1^2 + \tau_2^2 + \tau_3^2} = \\ &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}, \end{aligned}\right\} \qquad (8.33)$$

where $\sigma_i$ – is intensity of stresses, which is used in the theory of plasticity.

The third and the fourth theories are widely used because these theories give satisfactory results and confirmed by natural tests. The third theory fits well for brittle materials but the fourth one for plastic materials.

Equivalent stresses have to be compared with allowable stresses. Allowable stresses are stresses which are considered to be dangerous. It is forbidden for the construction to have equivalent stresses more than allowable ones. In opposite case we consider that our construction don't carry its functional possibilities.

The condition of sufficient strength can be written

$$\sigma_{eq} \leq [\sigma], \quad [\sigma] = \frac{\sigma_0}{n}, \tag{8.34}$$

where $[\sigma]$ – are allowable stresses; $\sigma_0$ – are some dangerous stresses for given construction for given conditions of exploitation (for example yield limit or ultimate strength, etc.); $n$ – is accepted safety factor for given dangerous state.

# LECTURE 9. PURE TORSION OF PRISMATIC BARS. BASICS

**Initial Explanations.** The theory of pure torsion of prismatic bar is applied to prismatic bars, the cross section of which may be simply connected region or multiply connected regions. The position of comparison axis *xOy* is free. Outer cylindrical surface of a bar is defined by equation $f(\overline{x},\overline{y}) = 0$, but internal cylindrical surfaces are defined by equations $f_i(\overline{x_i},\overline{y_i}) = 0$, where $i$ – is the number of internal channel (see fig. 9.1).

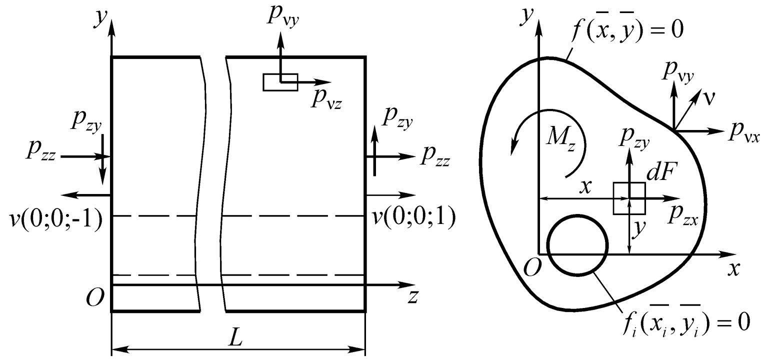



Fig. 9.1 – For explanation of pure torsion

**Principal Equations of the Theory of Pure Torsion.** External forces have to satisfy to the next conditions:

**a)** on a cylinder surface $f(\overline{x},\overline{y})=0$ and $f_i(\overline{x_i},\overline{y_i})=0$ there are no surface external forces

$$p_{\nu x}=p_{\nu y}=p_{\nu z}=0, \tag{9.1}$$

**b)** on the right end $z=L$, $\nu(0, 0, 1)$ there are only external shear/tangent forces take place

$$\left.\begin{aligned} p_{\nu x}&=p_{zx}(x,y)\neq 0,\\ p_{\nu y}&=p_{zy}(x,y)\neq 0,\\ p_{\nu z}&=p_{zz}=0,\end{aligned}\right\} \tag{9.2}$$

which create only torque

$$\int_F p_{zx}dF=0,\quad \int_F p_{zy}dF=0,\quad \int_F (p_{zy}x-p_{zx}y)dF=M_z, \tag{9.3}$$

**c)** on the left end $z=0$, $\nu(0, 0, -1)$ – there are the same conditions, but before $M_z$ in (9.3) it's necessary to put sign minus.

**d)** There is no volume forces and temperature field acting to a bar.

Integrals in (9.3) are taken in the field occupied by material of the cross section of the bar. Reviewing the strain-stress state we have that only two shear stresses are no zero but the rest stresses are zero

$$\left.\begin{aligned} &\sigma_x=\sigma_y=\sigma_z=\tau_{xy}=0,\\ &\tau_{zx}=\tau_{zx}(x,y),\quad \tau_{zy}=\tau_{zy}(x,y).\end{aligned}\right\} \tag{9.4}$$

Substituting (9.4) into equilibrium equations of the elementary parallelepiped, without considering volume forces $X=Y=Z=0$ we shall have next

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}=0. \tag{9.5}$$

Substituting (9.4) into Beltrami-Mitchell compatibility conditions gives next

$$\nabla^2\tau_{zy} = 0, \quad \nabla^2\tau_{zx} = 0, \quad \nabla^2() = \frac{\partial^2()}{\partial x^2} + \frac{\partial^2()}{\partial y^2}. \qquad (9.6)$$

So that $\tau_{zx}(x, y)$ and $\tau_{zy}(x, y)$ have to be defined from equations (9.5), (9.6).

To simplify the problem it's assumed that there is some function $\psi = \psi(x, y)$ which satisfy to the next relations

$$\tau_{zx} = \alpha G\frac{\partial\psi}{\partial y}, \quad \tau_{zy} = -\alpha G\frac{\partial\psi}{\partial x}, \qquad (9.7)$$

where $G$ – is shear modulus; $\alpha$ – some free constant, which will be explained later.

Substituting (9.7) into (9.5) and (9.6) we shall have that (9.5) are satisfied identically, but compatibility equations (9.6) are transformed to the next expressions

$$\left.\begin{aligned} \nabla^2\left(\alpha G\frac{\partial\psi}{\partial y}\right) &= \frac{\partial}{\partial y}\left(\alpha G\nabla^2\psi\right) = 0, \\ \nabla^2\left(\alpha G\frac{\partial\psi}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\alpha G\nabla^2\psi\right) = 0, \end{aligned}\right\} \qquad (9.8)$$

from which we have that

$$\alpha G\nabla^2\psi = K = \text{const}, \quad \nabla^2\psi = \frac{K}{\alpha G}. \qquad (9.9)$$

Let's put the right part of the (9.9) to be always –2. Finally we have next

$$\nabla^2\psi(x, y) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2. \qquad (9.10)$$

Hence the solution of the problem of pure torsion of prismatic bar was narrowed down to definition of the stress function (9.10) which is of Poisson's type differential equation, but for physical meaning is the compatibility equation.

**Boundary Conditions.**

To solve (9.10) it's necessary to have boundary conditions for outer contour and for inside contours. Let's review force boundary conditions which are equilibrium equations of elementary tetrahedron. On outer prismatic surface $f(\bar{x},\bar{y})=0$ directions cosines of outer normal vector ν (see fig. 9.1) are

$$l=\cos\beta=\frac{dy}{ds},\quad m=\sin\beta=-\frac{dx}{ds},\quad n=0\,, \qquad (9.11)$$

where $ds$ – is an arc differential of outer contour; $s$ – is tangent vector.

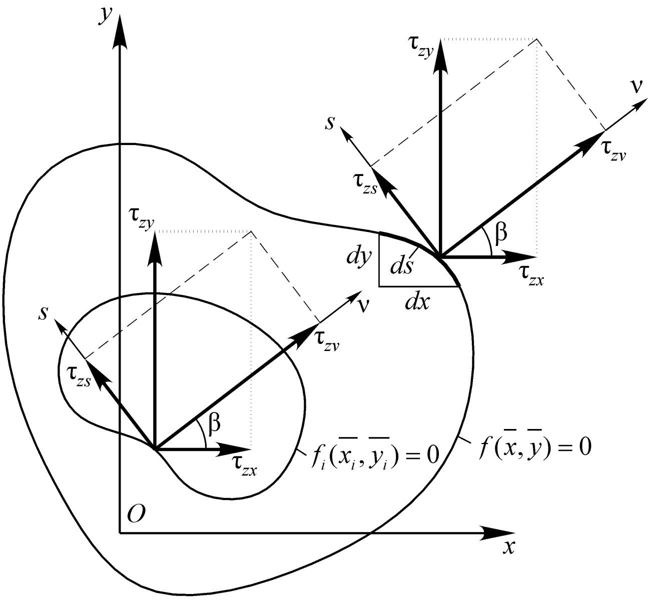



Fig. 9.2 – To explain boundary conditions

Substituting (9.11) and (9.4) into equilibrium equations of elementary tetrahedron we shall have that first two equations are satisfied identically, but the third one gives next

$$\tau_{zx}\cos\beta+\tau_{zy}\sin\beta=\tau_{zx}\frac{dy}{ds}-\tau_{zy}\frac{dx}{ds}=\tau_{z\nu}=0, \qquad (9.12)$$

substituting into (9.12) right parts of (9.7) instead of shear stresses there is

$$\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial s}=\frac{\partial\psi}{\partial s}=0. \qquad (9.13)$$

The relation (9.13) is *the condition of equality to zero of the projection of full shear stresses into normal vector into outer contour in all contour points*. It means that full shear stresses on boundaries of the cross section are always directed along tangent vector of the contour.

On basis of (9.13) the outer contour derivative of the stress function equals zero, thus for this contour it have keep constant value

$$\psi(\bar{x},\bar{y})=c_0=\mathrm{const}. \qquad (9.14)$$

In case of bar with multiply connected regions (of the cross section) we have next

$$\tau_{z\nu_i}=0, \quad \frac{\partial\psi(\bar{x}_i,\bar{y}_i)}{\partial s_i}=0,$$

$$\psi(\bar{x}_i,\bar{y}_i)=c_i=\mathrm{const}, \quad i=1,2,\ldots,n, \qquad (9.15)$$

where $\bar{x}_i, \bar{y}_i$ – are coordinates of closed contour encircling $i$-th internal channel; $c_i$ – is the constant value of the stress function on contour of the $i$-th internal channel; $n$ – number of internal channels.

Variation of ordinates of $\psi(x, y)$ for arbitrary constant value $A$ don't influence for the commonality of a solution, it means that value $A$ can always be selected in a such way to provide the stress function on the outer contour to be zero

$$\psi(\bar{x}, \bar{y}) = c_0 = 0, \tag{9.16}$$

which will be suggested further instead (9.14).

For the bar end $z = L$, $\nu(0,0,1)$ the third equation of equilibrium equations of the elementary tetrahedron is satisfied identically, but from the first and the second ones we have next expressions for external loadings

$$p_{zx} = \tau_{zx} = \alpha G \frac{\partial \psi}{\partial y}, \quad p_{zy} = \tau_{zy} = -\alpha G \frac{\partial \psi}{\partial x}. \tag{9.17}$$

These loads (9.17) have to satisfy to integral relations (9.3). Two of (9.3) are satisfied identically, but the third gives for a simply connected bar next

$$M_z = C\alpha, \quad C = 2G \int_F \psi dF, \quad \alpha = \frac{M_z}{C}. \tag{9.18}$$

For multiply connected bar considering that on internal channels there are conditions (9.15) are provided we have next

$$M_z = C\alpha, \quad C = 2G\left[\int_F \psi dF + \sum_{i=1}^{n} c_i F_i\right], \quad \alpha = \frac{M_z}{C}, \tag{9.19}$$

where $M_z$ – torque; $C$ – is total bar rigidity; $\alpha$ – is the linear angle of torsion of the bar, rad/m; $G$ – shear modulus; $c_i$ – is the stress function value on the contour of $i$-th channel; $F_i$ – is the area circumscribed by contour of $i$-th channel; $F$ – is the cross section area occupied by bar material.

**Torsion of Thin-Walled Simply Connected Compound Bars.** In building it is often necessary to calculate thin-walled simply connected compound bars (so called open profile bars) for pure torsion (see fig. 9.3).

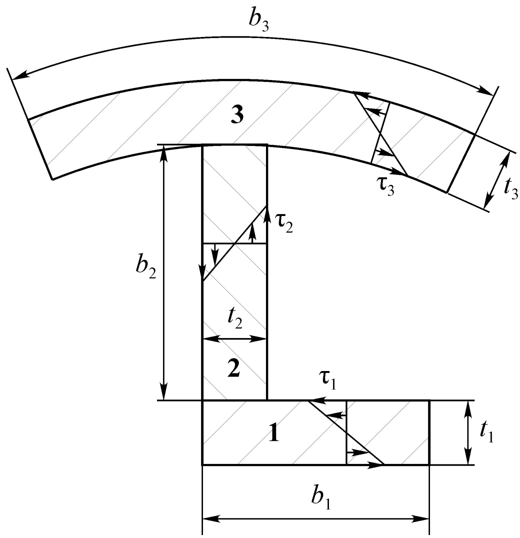



Fig. 9.3 – Thin-walled simply connected compound bar

To have simple relations we consider that thickness $t$ within each stripe is constant and the relation of thickness $t$ to length $b$ of each stripe is 1/15 and less. It allows for us to write the formula for rigidity and shear stresses for compound bar consisting with thin stripes

$$C_i = \frac{1}{3} G t_i^3 b_i, \quad C = \frac{1}{3} G \sum_{i=1}^{m} t_i^3 b_i, \quad \alpha = \frac{M_z}{C}, \quad \tau_i = \alpha G t_i, \quad (9.20)$$

where $C_i$ – is the rigidity of each stripe; $C$ – total rigidity of compound bar; $t_i$ – thickness of each stripe; $b_i$ – length of each stripe; $\tau_i$ – maximum shear stresses, as it is shown on the fig. 9.3.

## LECTURE 10. ADDITIONAL QUESTIONS OF PURE TORSION

**Strains and Displacements.** Using Hook's law, considering (9.4), (9.7) we have next

$$\left.\begin{aligned} &\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0, \\ &\tau_{zx} = \alpha \frac{\partial \psi}{\partial y}, \quad \tau_{zy} = -\alpha \frac{\partial \psi}{\partial x}. \end{aligned}\right\} \tag{10.1}$$

Assuming the conditions of attaching of coordinate axes to a body in point $x = y = z = 0$ here below

$$\left.\begin{aligned} &u_0 = v_0 = w_0 = 0, \\ &\left(\frac{\partial u}{\partial y}\right)_0 = \left(\frac{\partial u}{\partial z}\right)_0 = \left(\frac{\partial v}{\partial z}\right)_0 = 0. \end{aligned}\right\} \tag{10.2}$$

Substituting strains (10.1) into expressions for displacements (see the topic: definition of displacements for known strains) after integration we shall have next

$$u = -\alpha z y, \quad v = \alpha z x. \tag{10.3}$$

The displacement $w$ can be obtained using (10.1), (10.3) and formulae for finding displacements for known strains

$$\left.\begin{aligned} &\frac{\partial w}{\partial x} = \gamma_{zx} - \frac{\partial u}{\partial z} = \alpha\left(\frac{\partial \psi}{\partial y} + y\right), \\ &\frac{\partial w}{\partial y} = \gamma_{zy} - \frac{\partial v}{\partial z} = -\alpha\left(\frac{\partial \psi}{\partial x} + x\right), \\ &w(x, y) = \int_0^x \frac{\partial w}{\partial x} dx + \int_0^y \frac{\partial w}{\partial y} dy = \alpha w_0(x, y), \end{aligned}\right\} \tag{10.4}$$

where

$$w_0(x, y) = \int_0^x \frac{\partial \psi}{\partial y} dx + \int_0^y \frac{\partial \psi}{\partial x} dy + xy. \tag{10.5}$$

Analyzing results (9.18)–(9.19), (10.1)–(10.5) it's possibly to define next particulars of a bar deformation. Cross sections of a bar don't deform in its plane ($\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$) twisting as a rigid disk for the angle $\alpha z$ relatively immovable section $z = 0$. The function $w_0(x, y)$ defining the character of distortion is called the *Saint-Venant torsion function* or *deplanation*. All cross sections have the same deplanation, that's why all points laying on one longitudinal fiber move along axis $z$ equally, without longitudinal strain ($\varepsilon_z = 0$).

**Circulation of Shear Stresses for Multiply Connected Bars.** For multiply connected bars there are stress function values on channel contours $c_i$ are unknown. It's necessary to have conditions to define it. Below it is shown how to do that.

From (10.3) it's implied that displacements $u(x, y)$ and $v(x, y)$ are single-valued functions. To the condition of single-valuedness it is necessary to submit only displacement $w(x, y)$ (10.4) on closed contours $s_i$ encircling each of internal channels. This condition can be written

$$\left.\begin{aligned} &\oint_s \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy\right) = \alpha \oint_s \left[\left(\frac{\partial \psi}{\partial y} + y\right) dx - \left(\frac{\partial \psi}{\partial x} + x\right) dy\right] = \\ &= \alpha\left[\frac{1}{\alpha G} \oint_s (\tau_{zx} dx + \tau_{zy} dy) = \oint_s (y dx - x dy)\right] = 0. \end{aligned}\right\} \tag{10.6}$$

Considering relations (9.11) it is possibly to write

$$\left.\begin{aligned}\tau_{zx}dx+\tau_{zy}dy=\left(\tau_{zx}\frac{dx}{ds}+\tau_{zy}\frac{dy}{ds}\right)ds=\\(-\tau_{zx}\sin\beta+\tau_{zy}\cos\beta)ds=\tau_{zs}ds.\end{aligned}\right\}\qquad(10.7)$$

As the direction of comparison axes $xOy$ can be chosen in a free way, identifying $x$ with $\nu$, but $y$ with $s$ it's possibly to write

$$\tau_{z\nu}=\alpha G\frac{\partial\psi}{\partial s},\quad \tau_{zs}=-\alpha G\frac{\partial\psi}{\partial\nu}.\qquad(10.8)$$

Substituting (10.8) into (10.6) and after path tracing the $i$-th contour $s_i$ using (10.6) we shall have next

$$\oint\limits_{s_i}\tau_{zs}ds=2\alpha GF_{s_i},\qquad(10.9)$$

$$\oint\limits_{s_i}\frac{\partial\psi}{\partial\nu}ds=2F_{s_i}.\qquad(10.10)$$

The integral (10.9) is called the *circulation of shear stresses* on the closed contour $s_i$, but the relations (10.9) and (10.10) are the mathematical formulation of the theorem of circulation of shear stresses.

Composing (10.10) for closed contours $s_i$ encircling internal channels, it is possibly to get the necessary number of equations for definition of constants $c_i$.

Contours $s_i$ may coincide with appropriate contours of internal channels or encircle them (contours of internal channels) in any way, but $s_i$ don't have to exceed the bounds of material occupied by a bar.

**Common Algorithm for Calculation of Pure Torsion.** On the fig. 10.1 the common algorithm of calculation of prismatic multiply connected bars for pure torsion is presented.

**1)** Define stress function $\psi(x, y, c_i)$ as the solution of (9.10), satisfying (9.16) on outer contour and (9.15) on inside contours $s_i$.

**2)** For inside closed contours $s_i$ of internal channels we compose equations of circulation of shear stresses (10.10) to define $c_i$.

**3)** Find bar rigidity $C$ and the linear torsion angle $\alpha$ using (9.19).

**4)** Find stresses (9.7), strains (10.1), displacements (10.3), (10.4).

For simply connected bars there is no p.2 in the algorithm presented on the fig. 10.1.

| |
|---|
| **1)** Define stress function $\psi(x, y, c_i)$ as the solution of (9.10), satisfying (9.16) on outer contour and (9.15) on inside contours $s_i$. |
| ↓ |
| **2)** For inside closed contours $s_i$ of internal channels we compose equations of circulation of shear stresses (10.10) to define $c_i$. |
| ↓ |
| **3)** Find bar rigidity $C$ and the linear torsion angle $\alpha$ using (9.19) |
| ↓ |
| **4)** Find stresses (9.7), strains (10.1), displacements (10.3), (10.4). |

Fig. 10.1 – Common algorithm of calculation of bar for pure torsion

**Torsion of Thin-Walled Multiply Connected Bars.** Torsion of thin-walled bars presents an interest, because many constructions are thin-walled. For example ship hull can be reviewed as a thin-walled bar. There is the thin-walled bar presented on the fig. 10.2.

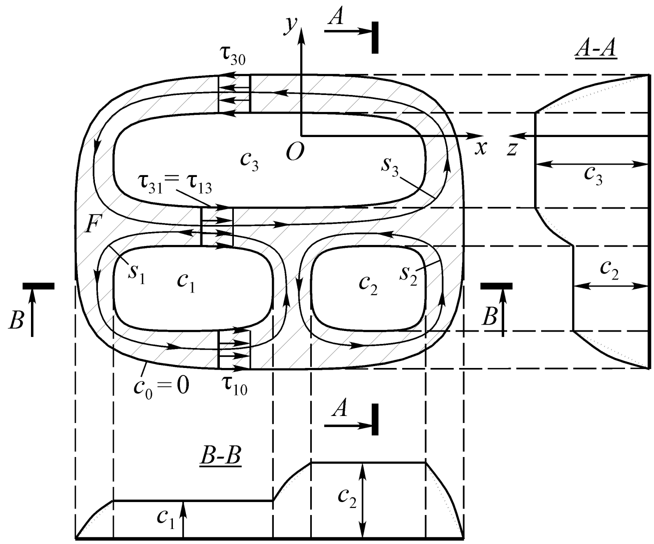



Fig. 10.2 – Thin-walled multiply connected bar

There are simplifications in the torsion theory which allow to get simple solutions with no loss of accuracy:

**a)** the area of material occupied by material of a bar is quite small thus in (9.19) the value of integral $\int_F \psi dF \to 0$ is quite small and we may not to consider it there. That why (9.19) for thin-walled bars can be written

$$M_z = C\alpha, \quad C = 2G\sum_{i=1}^{n} c_i F_i, \quad \alpha = \frac{M_z}{C}; \qquad (10.11)$$

**b)** The contour $s_i$ is situated too close to the contour of $i$-th internal channels. These contours are almost identical. As the stress function value on the contour is constant $\psi(\overline{x_i}, \overline{y_i}) = c_i = \text{const}$ (see. (9.15)), the derivative for stress function $\psi(\overline{x_i}, \overline{y_i})$ along tangent direction $s$ for the contour $s_i$ is zero (almost zero), that's why on basis of (10.8) we have that

$$\tau_{z\nu} = 0; \tag{10.12}$$

**c)** the thickness of thin-walled bars is too small that the change of stress function $\psi(x, y)$ is not significant and it is possibly to consider that this change is linear within transverse direction of contour $c_i$ (see fig. 10.2). It allows us to write next

$$\frac{\partial\psi}{\partial\nu} = \frac{c_j - c_i}{t(s_i)}, \quad \tau_{zs} = \alpha G \frac{c_i - c_j}{t(s_i)}, \tag{10.13}$$

where $t(s_i)$ – is the thickness of a wall of thin-walled profile in the point of the contour $s_i$ for which stresses are defined; $c_i$, $c_j$ – values of stress function $\psi(x, y)$ on contours of $i$-th and $j$-th channels, which are divided by thickness $t(s_i)$; $c_0 = 0$ – is the value of the stress function for outer contour.

Substituting (10.13) into equation of circulation of shear stresses (10.9) we shall have next

$$\left.\begin{aligned} &\oint_{s_i} \tau_{zs} ds = \alpha G \int_{s_i} \frac{c_i - c_j}{t(s_i)} ds = 2\alpha G F_{s_i}, \\ &c_i \oint_{s_i} \frac{ds}{t(s_i)} - \sum_j c_j \int_{s_{ij}} \frac{ds}{t(s_i)} = 2F_{s_i}, \quad i = 1, 2, \dots, n, \end{aligned}\right\} \tag{10.14}$$

where $i$ – are numbers of $i$-th internal channels; $j$ – numbers of closed channels, adjoining to $i$-th channel; $s_{ij}$ – is the segment of contour $s_i$ within length of internal wall dividing $i$-th and $j$-th channels; $F_{si}$ – area of inside contour $s_i$; $n$ – total count of internal channels in composition of the bar cross section.

The algorithm of calculation of pure torsion of thin-walled bar for this occasion is also reduced and can be presented as it is shown on the fig. 10.3.

**1)** For inside closed contours $s_i$ of internal channels we compose equations of circulation of shear stresses (10.14) to define $c_i$

↓

**2)** Find bar rigidity $C$ and the linear torsion angle α using (10.11)

↓

**3)** Find stresses (10.13), (10.12), strains (10.1), displacements (10.3), (10.4)

Fig. 10.3 – Algorithm of calculation
of thin-walled multiply connected bar for pure torsion

**1)** For inside closed contours $s_i$ of internal channels we compose equations of circulation of shear stresses (10.14) to define $c_i$

**2)** Find bar rigidity $C$ and the linear torsion angle α using (10.11)

**3)** Find stresses (10.13), (10.12), strains (10.1), displacements (10.3), (10.4)

# LECTURE 11. TWO-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY IN RECTANGULAR COORDINATES

**Basic Concepts of Plane Strain-Stress State. Major Equations and Relations.**

Plane strain-stress state (or two-dimensional state, 2D state) is reviewed for thin plates the contour of which satisfies to the next equation $f(\bar{x}, \bar{y}) = 0$ (see fig. 11.1).

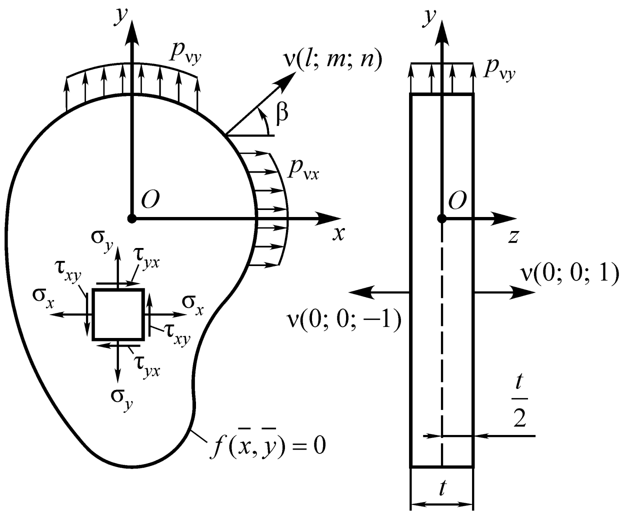



Fig. 11.1 – For explanation of plane strain-stress state

Contour of that plane is considered to be boundary surface which have normal direction vector **v**($l = \cos\beta$, $m = \sin\beta$, $n = 0$). Plane strain-stress state can be applied for sheeting of ships for different pipes and volumes with thin walls.

Plane strain-stress state takes place in thin plates when there are next conditions provided:

1. There are next surface loadings on boundary surface:

$$p_{\nu x} = p_{\nu x}(\bar{x}, \bar{y}), \quad p_{\nu y} = p_{\nu y}(\bar{x}, \bar{y}), \quad p_{\nu z} = 0. \qquad (11.1)$$

2. There are no external surface loadings at foundation planes of the plate, i.e. side planes are free

$$p_{\nu x} = 0, \quad p_{\nu y} = 0, \quad p_{\nu z} = 0. \qquad (11.2)$$

3. Volume external forces and temperature field distributed uniformly for thickness, functions of which don’t depends on coordinate $z$

$$X = X(x, y), \quad Y = Y(x, y), \quad Z = Z(x, y), \quad T = T(x, y). \qquad (11.3)$$

4. There are only $\sigma_x$, $\sigma_y$, $\tau_{xy}$ are nonzero stresses. The rest components of stresses are zero $\sigma_z = 0$, $\tau_{yz} = 0$, $\tau_{zx} = 0$. So stress tensor can be written

$$T_\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}. \qquad (11.4)$$

Equations and relations for massive body can be simplified for plane strain-stress state.

Equilibrium equations of elementary tetrahedron on side planes $z = \pm t/2$ are satisfied identically by stress tensor (11.4), but on the contour of a plate $f(\bar{x}, \bar{y}) = 0$ it can be written

$$\left.\begin{aligned} \sigma_x l + \tau_{yx} m &= p_{\nu x}, \\ \tau_{xy} l + \sigma_y m &= p_{\nu y}. \end{aligned}\right\} \qquad (11.5)$$

Equilibrium equations of elementary parallelepiped can be written

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} - \rho \frac{\partial^2 u}{\partial t^2} = -X, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} - \rho \frac{\partial^2 v}{\partial t^2} = -Y. \end{aligned}\right\} \tag{11.6}$$

Hook's law for plane strain-stress state can be written (without considering of temperature)

$$\left.\begin{aligned} &\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y); \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x); \\ &\varepsilon_z = -\frac{\mu}{E}(\sigma_x + \sigma_y); \quad \gamma_{xy} = \frac{\tau_{xy}}{G}; \quad \gamma_{yz} = 0; \quad \gamma_{zx} = 0. \end{aligned}\right\} \tag{11.7}$$

From (11.7) there is conclusion can be made

$$\varepsilon_z = -\frac{\mu}{1-\mu}(\varepsilon_x + \varepsilon_y). \tag{11.8}$$

In another form Hook's law can be presented (without considering of temperature)

$$\left.\begin{aligned} &\sigma_x = \frac{E}{1-\mu^2}(\varepsilon_x + \mu\varepsilon_y); \quad \sigma_y = \frac{E}{1-\mu^2}(\varepsilon_y + \mu\varepsilon_x); \\ &\sigma_z = 0; \quad \tau_{xy} = G\gamma_{xy}; \quad \tau_{yz} = 0; \quad \tau_{zx} = 0. \end{aligned}\right\} \tag{11.9}$$

Compatibility conditions expressed via strains and stresses can be written

$$\left.\begin{aligned} &\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \\ &\nabla^2(\sigma_x + \sigma_y) = 0. \end{aligned}\right\} \tag{11.10}$$

Expressions (11.10) are simplifications of compatibility conditions of Saint-Venant's and Beltrami-Mitchell's.

Lame equations can also be simplified. Without considering of temperature and volume external forces Lame equations can be written

$$\left.\begin{aligned} &\nabla^2 u + \frac{1+\mu}{1-\mu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = 0, \\ &\nabla^2 v + \frac{1+\mu}{1-\mu}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) = 0. \end{aligned}\right\} \qquad (11.11)$$

Cubic nonlinear equation for definition of principle stresses can be written

$$[\sigma_\nu^2 - \sigma_\nu(\sigma_x + \sigma_y) + (\sigma_x\sigma_y - \tau_{xy}^2)]\sigma_\nu = 0\,. \qquad (11.12)$$

Roots of the equation (11.12) are principal stresses defined by the next expressions

$$\left.\begin{aligned} &\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \\ &\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \\ &\sigma_3 = 0. \end{aligned}\right\} \qquad (11.13)$$

It's obvious that any orthogonal plane to axis $z$ in which $\sigma_z = \tau_{yz} = \tau_{zx} = 0$ is principle. The rest two principle planes are parallel to axis $z$. For definition of angles of inclination of principal planes to axis $x$ the next expression can be used

$$\operatorname{tg} 2\beta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\,. \qquad (11.14)$$

The maximum shear stresses can be found using expressions

$$\left.\begin{aligned}&\tau_1 = \pm\frac{\sigma_2}{2}, \quad \tau_2 = \pm\frac{\sigma_1}{2},\\ &\tau_3 = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}.\end{aligned}\right\} \qquad (11.15)$$

The largest on modulus value of maximum shear stresses can be defined by expression

$$\tau_{\max} = \max(|\tau_1|, |\tau_2|, |\tau_3|). \qquad (11.16)$$

Principle strains and maximum shear strains can be found using expressions

$$\left.\begin{aligned}&\varepsilon_1 = \frac{1}{E}(\sigma_1 - \mu\sigma_2), \quad \varepsilon_2 = \frac{1}{E}(\sigma_2 - \mu\sigma_1),\\ &\varepsilon_3 = -\frac{\mu}{E}(\sigma_1 + \sigma_2), \quad \gamma_{\max} = \frac{\tau_{\max}}{G}.\end{aligned}\right\} \qquad (11.17)$$

The expression for the 4-th strength theory can be written

$$\sigma_{eq} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}\,. \qquad (11.18)$$

Expressions for definition of displacements for known strains can be presented

$$\left.\begin{aligned}&u = u_0 + (y - y_0)\left(\frac{\partial u}{\partial y}\right)_0 + \int_{x_0}^{x}\varepsilon_x dx + \int_{y_0}^{y}\int_{y_0}^{y}\left(\frac{\partial\gamma_{xy}}{\partial y} - \frac{\partial\varepsilon_y}{\partial x}\right)_{x=x_0} dy^2,\\ &v = v_0 + (x - x_0)\left(\frac{\partial v}{\partial x}\right)_0 + \int_{y_0}^{y}\varepsilon_y dy + \int_{x_0}^{x}\int_{x_0}^{x}\left(\frac{\partial\gamma_{xy}}{\partial x} - \frac{\partial\varepsilon_x}{\partial y}\right)_{y=y_0} dx^2.\end{aligned}\right\} \qquad (11.19)$$

Formulas of transformation of stresses components on moving to the new coordinate system $x'y'zO$ (rotated relatively old system $xyzO$ for angle β) defined by direction cosines

$$\left.\begin{aligned} &l_1 = \cos\beta, \quad m_1 = \sin\beta, \quad n_1 = 0, \\ &l_2 = -\sin\beta, \quad m_2 = \cos\beta, \quad n_2 = 0, \\ &l_3 = 0, \quad m_3 = 0, \quad n_3 = 1, \end{aligned}\right\} \qquad (11.20)$$

can be written

$$\left.\begin{aligned} &\sigma_{x'} = \sigma_x \cos^2\beta + \tau_{xy} \sin 2\beta + \sigma_y \sin^2\beta, \\ &\sigma_{y'} = \sigma_x \sin^2\beta - \tau_{xy} \sin 2\beta + \sigma_y \cos^2\beta, \\ &\tau_{x'y'} = \frac{1}{2}(\sigma_y - \sigma_x)\sin 2\beta + \tau_{xy} \cos 2\beta, \end{aligned}\right\} \qquad (11.21)$$

where β – is the angle between positive directions of axes $x$ and $x'$, measured anticlockwise.

**Plane Strain.** When there is so called plane strain takes place then in addition to described above loads on lateral faces of plate $z = \pm\, t/2$ there are additional normal loads appear $p_{vz} = \pm\, p_{vz}(x, y)$ which completely eliminate deplanation of that surfaces. Such strain takes place in middle layers of long prismatic bodies loaded by uniformly distributed self-balancing loads along length perpendicular to axis $z$. But in this case there are

$$\varepsilon_z = a + bx + cy, \quad \sigma_z = E\varepsilon_z + \mu(\sigma_x + \sigma_y), \qquad (11.22)$$

where factors $a$, $b$, $c$ are defined via principle vector and principle moment of load $p_{vz}(\bar{x}, \bar{y}, \bar{z})$. Basic equations and solution methods for problems of plane strain-stress state and plane strain are completely analogous. That's why in future we shall limit to exposition of the first problem.

**Schemes of Solution of Direct Problem of Plane Strain-Stress State. Stress Function.** To solve the direct problem in stresses of the plane strain-stress state there is direct way to solve the system of equations (11.6) with second equation in (11.10). The resolution system for plane strain-stress state is presented below

$$\left.\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}-\rho\frac{\partial^2 u}{\partial t^2}=-X,\\&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}-\rho\frac{\partial^2 v}{\partial t^2}=-Y,\\&\nabla^2(\sigma_x+\sigma_y)=0.\end{aligned}\right\}\qquad(11.23)$$

Boundary conditions in addition to the system (11.23) are equilibrium equations of elementary tetrahedron (11.5) and also kinematic boundary conditions at parts $f_i$ of contour $f(\bar{x},\bar{y})=0$ of the body set on contour

$$\left.\begin{aligned}u_i&=u_i(\bar{x},\bar{y});\\v_i&=v_i(\bar{x},\bar{y}).\end{aligned}\right\}\qquad(11.24)$$

But the solution of the direct problem of the plane strain-stress state can be simplified if to use so called stress function $\psi(x, y)$. This function is connected with stresses by relations

$$\sigma_x=\frac{\partial^2\psi}{\partial y^2},\quad \sigma_y=\frac{\partial^2\psi}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2\psi}{\partial x\,\partial y}.\qquad(11.25)$$

Substituting (11.25) into the 3-rd of (11.23) we have the next biharmonic equation

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\,\partial y^4}+\frac{\partial^4\psi}{\partial y^4}=\nabla^2\nabla^2\psi=0.\qquad(11.26)$$

The function $\psi = \psi\ (x, y)$, which satisfy to the biharmonic equation (11.26) is called *stress function*.

Using equilibrium equations of elementary tetrahedron (11.5) and expressions for stresses (11.25) we can get boundary conditions for the stress function

$$\frac{\partial^2 \psi}{\partial y^2} l - \frac{\partial^2 \psi}{\partial x\, \partial y} m = p_{\nu x}, \quad -\frac{\partial^2 \psi}{\partial x\, \partial y} l + \frac{\partial^2 \psi}{\partial x^2} m = p_{\nu y}\,. \quad (11.27)$$

The consequence of solution of the direct problem of the plane strain-stress state of the theory of elasticity can be presented as it is shown on the fig. 11.2.

**Find stress function ψ** which:

**1)** have to satisfy to biharmonic equation (11.26)

**2)** have to satisfy to boundary conditions (11.27)

For known stress function ψ **find:**

**1)** stresses, using (11.25);

**2)** strains, using (11.7), (11.8);

**3)** displacements, using (11.19)

**Find stress function ψ** which:
**1)** have to satisfy to biharmonic equation (11.26)
**2)** have to satisfy to boundary conditions (11.27)

↓

For known stress function ψ **find:**
**1)** stresses, using (11.25);
**2)** strains, using (11.7), (11.8);
**3)** displacements, using (11.19)

Fig. 11.2 – The consequence of solution of the direct problem of the plane strain-stress state

**Plain Problem of the Theory of Elasticity in Polynomials.** There is no common solution for biharmonic equation (11.26) for arbitrary contour of plate, loadings and temperature field. But for some occasions for orthogonal, circular and elliptical plates there are partial analytic solutions, which can be used in practice.

For rectangular plates there are next boundary conditions at edges (see fig. 11.3)

$$\left.\begin{array}{ll} \text{when} \quad x=0, & \nu(l=-1;\, m=0):\quad \sigma_x=-p_{\nu x},\quad \tau_{xy}=-p_{\nu y}; \\ \text{when} \quad x=a, & \nu(1;0):\quad \sigma_x=p_{\nu x},\quad \tau_{xy}=p_{\nu y}; \\ \text{when} \quad y=0, & \nu(0;-1):\quad \sigma_y=-p_{\nu y},\quad \tau_{xy}=-p_{\nu x}; \\ \text{when} \quad y=b, & \nu(0;1):\quad \sigma_y=p_{\nu y},\quad \tau_{xy}=p_{\nu x}, \end{array}\right\} \qquad (11.28)$$

where $p_{\nu x}$, $p_{\nu y}$ are projections of external surface loadings into accordant axes.

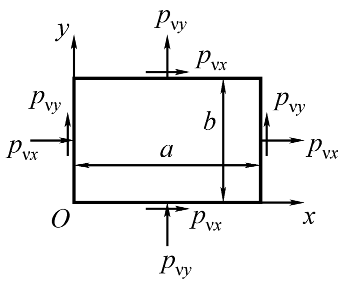



Fig. 11.3 – For boundary conditions of rectangular plate

There is so called solution *in polynomials* for rectangular plates. To get it is necessary to set the stress function in the next presentation

$$\left.\begin{array}{l} \psi(x,y)=a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+a_6x^3+a_7x^2y+ \\ \qquad +a_8xy^2+a_9y^3+a_{10}xy^3+a_{11}x^3y, \end{array}\right\} \qquad (11.29)$$

where $a_i$ $(i=0,1,2,\ldots,11)$ are constant factors.

Substituting stress function (11.29) into biharmonic equation (11.26) we can insure that stress function satisfy to biharmonic equation identically. Substituting (11.29) into expressions for stresses (11.25) we shall have next

$$\left.\begin{aligned} \sigma_x &= 2a_5 + 2a_8 x + 6a_9 y + 6a_{10} xy, \\ \sigma_y &= 2a_3 + 6a_6 x + 2a_7 y + 6a_{11} xy, \\ \tau_{xy} &= -(a_4 + 2a_7 x + 2a_8 y + 3a_{10} y^2 + 3a_{11} x^2). \end{aligned}\right\} \quad (11.30)$$

Let's review simple example of solution (11.30). Assuming $a_5 \neq 0$, but the rest factors $a_i = 0$ we shall have $\sigma_x = 2a_5$, $\sigma_y = \tau_{xy} = 0$, what fits to the occasion of simple tension-compression of rectangular stripe by the normal uniform pressure (according to (11.28)) $p_{vx} = -2a_5$ applied for the side $x = 0$, and $p_{vx} = 2a_5$ – for the side $x = a$.

Considering some factors $a_i$ to be zero but some $a_i$ to be nonzero we shall have different strain-stress states for plate: tension-compression, bending, shear etc.

Using polynomials of higher order than (11.29) we shall have sequence of other solutions. For this case polynomial factors have to be chosen for it to satisfy biharmonic equation (11.26) and given boundary conditions.

Using solutions in polynomials it's impossibly to consider interrupted loadings and concentrated forces.

# LECTURE 12. TWO-DIMENTIONAL PROBLEM IN TRIGONOMETRICAL SERIES. EFFECTIVE WIDTH OF WIDE BEAM FLANGES

**Solution of Two-Dimensional Problem for Rectangular Plates in Trigonometric Series.** A lot of problems can be solved if to accept the stress function $\psi$ of plane stress-strain state as the trigonometric series for the plate presented on the fig. 11.3

$$\psi(x,y)=\sum_{n=1}^{n=\infty} f_n(y)\sin\left(\frac{n\pi x}{a}\right), \qquad (12.1)$$

$$\psi(x,y)=\sum_{n=1}^{n=\infty} f_n(y)\cos\left(\frac{n\pi x}{a}\right), \qquad (12.2)$$

where $n = 1, 2, \ldots, +\infty$; $f_n(y)$ – are functions which have to be defined for it to satisfy biharmonic equation (11.26) $\nabla^2\nabla^2\psi = 0$.

Let's review the first solution of (12.1). Substituting (12.1) into (11.26) we shall have next

$$\sum_{n=1}^{n=\infty}\left[\frac{\partial^4 f_n(y)}{\partial y^4}-2\left(\frac{n\pi}{a}\right)^2\frac{\partial^2 f_n(y)}{\partial y^2}+\left(\frac{n\pi}{a}\right)^4 f_n(y)\right]\sin\frac{n\pi x}{a}=0. \qquad (12.3)$$

To provide zero for trigonometric series in the left side of (12.3) it is necessary for the factors of this series to be zero i.e.

$$\left.\begin{array}{l}\dfrac{\partial^4 f_n(y)}{\partial y^4}-2\left(\dfrac{n\pi}{a}\right)^2\dfrac{\partial^2 f_n(y)}{\partial y^2}+\left(\dfrac{n\pi}{a}\right)^4 f_n(y)=0\\ n=1,\ 2,\ \ldots,\ +\infty.\end{array}\right\}\quad(12.4)$$

Solving these homogenous linear differential equations we have partial solutions are

$$\left.\begin{array}{l}e^{\alpha_n\eta},\quad \eta e^{\alpha_n\eta},\quad e^{-\alpha_n\eta},\quad \eta e^{-\alpha_n\eta},\quad \text{or}\\ \operatorname{sh}\alpha_n\eta,\quad \operatorname{ch}\alpha_n\eta,\quad \eta\operatorname{sh}\alpha_n\eta,\quad \eta\operatorname{ch}\alpha_n\eta,\\ \text{where}\\ \alpha_n=n\pi\gamma,\quad \gamma=\dfrac{b}{a},\quad \eta=\dfrac{y}{b},\end{array}\right\}\quad(12.5)$$

but the general solutions can be written

$$f_n(y)=(A_n+B_n\eta)\operatorname{sh}\alpha_n\eta+(C_n+D_n\eta)\operatorname{ch}\alpha_n\eta,\quad(12.6)$$

where $\eta$ – is dimensionless coordinate; $A_n$, $B_n$, $C_n$, $D_n$ – are free constants.

Of course it is possibly to use (12.6) directly for further explanations, but we shall have complex expressions. That's why we shall look for convenient presentation of $f_n(y)$.

As is well known any linear independent combinations of partial solutions (12.5) are also partial solutions from which it is possibly to compose different forms of general solutions. For practical usage it is convenient to accept next form of $f_n(y)$

$$f_n(y)=$$

$$=\frac{b^2}{\alpha_n^2}\left[B_{1n}\chi_{2n}(1-\eta)+B_{2n}\chi_{2n}(\eta)+B_{3n}\chi_{0n}(1-\eta)+B_{4n}\chi_{0n}(\eta)\right],\ (12.7)$$

where $B_{jn}$ – are arbitrary constants;

$$\left.\begin{aligned}
\chi_{0n}(\eta) &= \frac{(2+\alpha_n \operatorname{cth}\alpha_n)\operatorname{sh}\alpha_n\eta - \alpha_n\eta\cdot\operatorname{ch}\alpha_n\eta}{2\operatorname{sh}\alpha_n},\\
\chi_{1n}(\eta) &= \frac{(1+\alpha_n \operatorname{cth}\alpha_n)\operatorname{ch}\alpha_n\eta - \alpha_n\eta\cdot\operatorname{sh}\alpha_n\eta}{2\operatorname{sh}\alpha_n}\\
\chi_{2n}(\eta) &= \chi_{0n}(\eta) - \frac{\operatorname{sh}\alpha_n\eta}{\operatorname{sh}\alpha_n},\\
\chi_{3n}(\eta) &= \chi_{1n}(\eta) - \frac{\operatorname{ch}\alpha_n\eta}{\operatorname{sh}\alpha_n},\\
\chi_{4n}(\eta) &= \chi_{2n}(\eta) - \frac{2\operatorname{sh}\alpha_n\eta}{\operatorname{sh}\alpha_n}.
\end{aligned}\right\} \quad (12.8)$$

There is the next rule for derivatives for the functions $\chi_{in}(\eta)$

$$\chi_{in}(\eta) = \frac{1}{\alpha_n}\frac{d\chi_{i-1,n}}{d\eta} = \frac{1}{\alpha_n^i}\frac{d^i\chi_{0n}}{d\eta^i}. \qquad (12.9)$$

There is the next relation for $\chi_{in}(\eta)$ which allows us to define $\chi_{in}(\eta)$ with any index $i$

$$\chi_{i+4,n} - 2\chi_{i+2,n} + \chi_{i,n} = 0, \qquad (12.10)$$

but functions $\chi_{0n}(1-\eta)$ and $\chi_{2n}(1-\eta)$ are defined by expressions (12.8) if to replace $\eta$ for $(1-\eta)$.

The stress function $\psi$ considering (12.7) can be written

$$\left.\begin{aligned}
&\varphi(x,y) = \sum_{n=1}^{+\infty}\frac{b^2}{\alpha_n^2}\left[B_{1n}\chi_{2n}(1-\eta) + B_{2n}\chi_{2n}(\eta) + B_{3n}\chi_{0n}(1-\eta) + B_{4n}\chi_{0n}(\eta)\right]\sin n\pi\xi,\\
&\xi = \frac{x}{a}
\end{aligned}\right\} \quad (12.11)$$

Substituting (12.11) into formulas for stresses (11.25) we shall have expressions for stresses

$$\left.\begin{aligned}
\sigma_x &= \sum_{n=1}^{+\infty}\left[B_{1n}\chi_{4n}(1-\eta)+B_{2n}\chi_{4n}(\eta)+B_{3n}\chi_{2n}(1-\eta)+B_{4n}\chi_{2n}(\eta)\right]\sin n\pi\xi,\\
\sigma_y &= -\sum_{n=1}^{+\infty}\left[B_{1n}\chi_{2n}(1-\eta)+B_{2n}\chi_{2n}(\eta)+B_{3n}\chi_{0n}(1-\eta)+B_{4n}\chi_{0n}(\eta)\right]\sin n\pi\xi,\\
\tau_{xy} &= -\sum_{n=1}^{+\infty}\left[B_{1n}\chi_{3n}(1-\eta)-B_{2n}\chi_{3n}(\eta)+B_{3n}\chi_{1n}(1-\eta)-B_{4n}\chi_{1n}(\eta)\right]\cos n\pi\xi.
\end{aligned}\right\} \quad (12.12)$$

After substituting expressions for stresses (12.12) into (11.28) we shall have relations for boundary conditions

$$\left.\begin{aligned}
&\text{at sides } \xi=0\ (x=0) \text{ and } \xi=1\ (x=a)\\
&p_{\nu x}=-\sigma_x(0,\eta)=0, \quad p_{\nu y}=-\tau_{xy}(0,\eta)\neq 0,\\
&p_{\nu x}=\sigma_x(1,\eta)=0, \qquad p_{\nu y}=\tau_{xy}(1,\eta)\neq 0,
\end{aligned}\right\} \quad (12.13)$$

$$\left.\begin{aligned}
&\text{at side } \eta=0\ (y=0)\\
&p_{\nu x}=-\tau_{xy}(\xi,0)=-\sum_{n=1}^{+\infty}\left[B_{1n}\chi_{3n}(1)-B_{2n}\chi_{3n}(0)+B_{3n}\chi_{1n}(1)-B_{4n}\chi_{1n}(0)\right]\cos n\pi\xi,\\
&p_{\nu y}=-\sigma_y(\xi,0)=\sum_{n=1}^{\infty}B_{3n}\sin n\pi\xi.
\end{aligned}\right\} \quad (12.14)$$

$$\left.\begin{aligned}
&\text{at side } \eta=1\ (y=b)\\
&p_{\nu x}=\tau_{xy}(\xi,1)=\sum_{n=1}^{+\infty}\left[B_{1n}\chi_{3n}(0)-B_{2n}\chi_{3n}(1)+B_{3n}\chi_{1n}(0)-B_{4n}\chi_{1n}(1)\right]\cos n\pi\xi,\\
&p_{\nu y}=\sigma_y(\xi,1)=-\sum_{n=1}^{\infty}B_{4n}\sin n\pi\xi.
\end{aligned}\right\} \quad (12.15)$$

There is the next sequence in solution of two-dimensional problem in trigonometric series (12.1):

1) There are factors of Fourier series for external loads have to be defined

$$\left.\begin{aligned}
&\text{at side } \eta = 0\ (y = 0)\\
&p_{\nu y} = \sum_{n=1}^{\infty} p_{1n} \sin n\pi\xi, \quad p_{\nu x} = \sum_{n=1}^{\infty} \tau_{1n} \cos n\pi\xi,\\
&\text{at side } \eta = 1\ (y = b)\\
&p_{\nu y} = \sum_{n=1}^{\infty} p_{2n} \sin n\pi\xi, \quad p_{\nu x} = \sum_{n=1}^{\infty} \tau_{2n} \cos n\pi\xi.
\end{aligned}\right\} \qquad (12.16)$$

2) Substituting series (12.16) for external loads into conditions (12.14), (12.15) and equating accordant series factors we shall get equations for definition of factors $B_{jn}$.

3) After $B_{jn}$ factors are found it is possibly to define stresses using (12.12).

The solution (12.1) was found by L. Failon (1903) and independently by S.I. Belzetskiy (1905), that's why it is called Failon-Belzetskiy solution.

Let's review another particular form (12.2) of plane strain-stress state which was obtained by M. Reebjer. Substituting (12.2) into biharmonic equation (11.26) $\nabla^2\nabla^2\psi = 0$, we shall have for $f_n(y)$ the same equations (12.4) and hence the same general solutions (12.7) that is for the solution of Failon-Belzetskiy.

For this occasion stresses are defined by expressions

$$\left.\begin{aligned}
\sigma_x &= \frac{\partial^2\psi}{\partial y^2} = \sum_{n=1}^{\infty} \frac{\partial^2 f_n(y)}{\partial y^2} \cos n\pi\xi,\\
\sigma_y &= \frac{\partial^2\psi}{\partial x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{a}\right)^2 f_n(y) \cos n\pi\xi,\\
\tau_{xy} &= -\frac{\partial^2\psi}{\partial x\,\partial y} = \sum_{n=1}^{\infty} \left(\frac{n\pi}{a}\right) \frac{\partial f_n(y)}{\partial y} \sin n\pi\xi.
\end{aligned}\right\} \qquad (12.17)$$

Analyzing solutions of (12.1) and (12.2) it is can be concluded that the solution of Failon-Belzetskiy fits approximately to conditions of constraints shown on the fig. 12.1,***a***, but the Reebjer solution fits to conditions of constraints shown on fig.12.1,***b***.

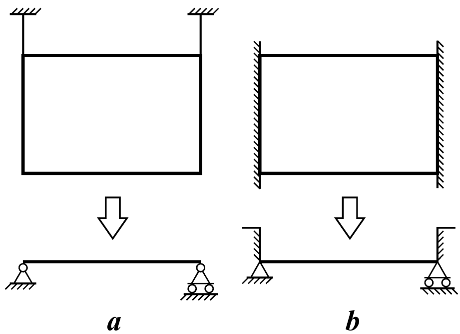
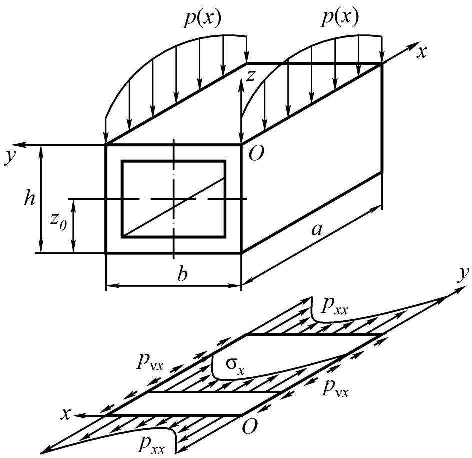



Fig. 12.1 – For explanation of solutions in trigonometric series

Using solutions of (12.1) and (12.2) there are numerous practical results were got. But solutions of (12.1) and (12.2) described here is not sufficient, that's why more complicated trigonometric series are used, but laboriousness of those solutions is huge.

**Effective Width of Wide Beam Flanges.** Analytical solutions of the problem of plane strain-stress state can be used for theoretical explorations of deformation (bending, tension-compression) of thin-walled beams with high walls and wide flanges. As it is known from the strength of materials the behavior of thin-walled beams is some distinguish from the behavior of usual beams. Solutions in trigonometric series can help to explore strain-stress state in those beams both in walls and in flanges. Because the distribution of normal and shear stresses here is complex and nonlinear and don't fit to distributions obtained in frames of the strength of materials. Here below we show only how to define the effective width of wide beam flanges.

Let's explain this problem. For example we have usual orthogonal bending for the beams presented on the fig. 12.2.



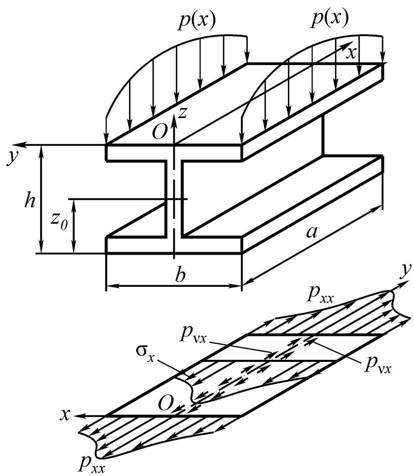




Figure 12.2 – For explanation of effective width of wide beam flanges

On the fig. 12.2 we can see the real distribution of normal stresses on flanges. It's obvious the distribution is nonuniform. So the formulas of the strength of materials can't be applied for normal stresses. So the problem for this occasion is to define the effective width of the flange to be able to apply expressions of the strength of materials to calculate stresses. This effective width of beam flange is reduced in comparison with real width due to shear lag. Within this reduced beam flange we consider that normal stresses are distributed uniformly. Hereafter we shall call this reduced flange the effective flange.

For effective flange there are next demands:

1) longitudinal linear strains and normal stresses have to be distributed uniformly within width of effective flange (the hypothesis of flat cross sections is true);

2) for equal longitudinal forces in appropriate orthogonal cross sections of effective and real flanges linear strains of effective flange have to equal to the linear strains of real flange in lines of connection of flanges with wall.

Because for the effective flange it's necessary to apply the same force as for real flange we shall have next

$$\sigma st = t\int_0^b \sigma_x dy\,, \tag{12.18}$$

where $\sigma$ – are normal stresses distributed uniformly within width of effective flange; $s$ – width of effective flange; $t$ – thickness of effective flange; $\sigma_x$ – is nonuniform distribution of normal stresses within width of real flange.

Using (12.18) the width of effective flange $s$ can be written

$$s = \frac{1}{E\varepsilon_{xk}}\int_0^b \sigma_x dy, \tag{12.19}$$

where $\varepsilon_{xk}$ – are linear strains in real flange in line of connection of girdle and web.

For convince there is the reductive factor φ used which is the relation of width of effective flange and width or real flange

$$\varphi = \frac{s}{b}, \quad s = \varphi b. \tag{12.20}$$

Quantities $\varepsilon_{xk}$, $\sigma_x$ have to be defined as the result of solutions of plane strain-stress state for real flange. This flange has to be reviewed as a plate with stated boundary conditions. The boundary conditions have to be defined for every particular occasion depending on constructive particulars of beam and conditions of its deformation. Many important results can be obtained using solutions in trigonometric series based on expressions (12.1), (12.2). Magnitudes of reductive factor φ for many important and widespread occasions can be found in special literature, reference books and manuals.

# LECTURE 13. TWO-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY IN POLAR COORDINATES

**Basic Relations of Plane Strain-Stress State in Polar Coordinates.** The formulas of the connection between orthogonal Cartesian and polar coordinate systems are these

$$\left.\begin{aligned} &x = r\cos\varphi, \quad y = r\sin\varphi, \quad \frac{\partial r}{\partial x} = \cos\varphi, \quad \frac{\partial r}{\partial y} = \sin\varphi, \\ &r = \sqrt{x^2 + y^2}, \quad \varphi = \operatorname{arctg}\frac{y}{x}, \quad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r}, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r}. \end{aligned}\right\} \tag{13.1}$$

Substituting expressions (13.1) into expressions of elements of plane strain-stress state written in previous lectures we shall have expressions for plane strain-stress state in polar coordinates.

The stresses, strains and displacements for plane strain-stress state in polar coordinate system are shown on the fig. 13.1.

There is the connection between displacements $u$ and $v$ in polar coordinate system and between displacements $u^*$ and $v^*$ in Cartesian orthogonal coordinate system

$$\left.\begin{aligned} u^* &= u\cos\varphi - v\sin\varphi, \\ v^* &= v\cos\varphi + u\sin\varphi. \end{aligned}\right\} \tag{13.2}$$

The linear strain of radial elements $dr$ is notified as $\varepsilon_r$ (radial linear strain) but the linear strain of tangential elements $r\,d\varphi$ is $\varepsilon_\varphi$ (tangential linear strain). The shear angle between radial element $dr$ and tangential one $r\,d\varphi$ is $\gamma_{r\varphi} = \gamma_{\varphi r} = \beta_1 + \beta_2$. Normal stresses $\sigma_r$ act in planes, which are perpendicular to radius $r$, but $\sigma_\varphi$ act on radial planes.

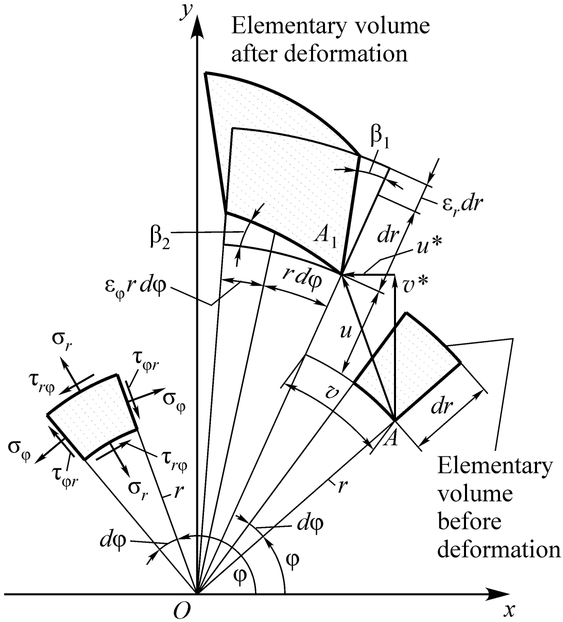



Fig. 13.1 – For explanation of elements for plane strain-stress state

Considering orthogonality of coordinate directions $r$ and $\varphi$, Hook's law for a plane strain-stress state in polar coordinate system can be written

$$\left.\begin{aligned} &\varepsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\varphi), \quad \varepsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \mu\sigma_r), \quad \gamma_{r\varphi} = \frac{\tau_{r\varphi}}{G}, \\ &\sigma_r = \frac{E}{1-\mu^2}(\varepsilon_r + \mu\varepsilon_\varphi), \quad \sigma_\varphi = \frac{E}{1-\mu^2}(\varepsilon_\varphi + \mu\varepsilon_r), \quad \tau_{r\varphi} = G\gamma_{r\varphi}. \end{aligned}\right\} \quad (13.3)$$

Dependencies between stresses and strains in polar and rectangular coordinates are next

$$\left.\begin{aligned}
\sigma_r &= \sigma_x \cos^2\varphi + \tau_{xy}\sin 2\varphi + \sigma_y \sin^2\varphi,\\
\sigma_\varphi &= \sigma_x \sin^2\varphi - \tau_{xy}\sin 2\varphi + \sigma_y \cos^2\varphi,\\
\tau_{r\varphi} &= \frac{1}{2}(\sigma_y - \sigma_x)\sin 2\varphi + \tau_{xy}\cos 2\varphi,
\end{aligned}\right\} \tag{13.4}$$

$$\left.\begin{aligned}
\varepsilon_r &= \varepsilon_x \cos^2\varphi + \gamma_{xy}\sin\varphi\cdot\cos\varphi + \varepsilon_y \sin^2\varphi,\\
\varepsilon_\varphi &= \varepsilon_x \sin^2\varphi - \gamma_{xy}\sin\varphi\cdot\cos\varphi + \varepsilon_y \cos^2\varphi,\\
\gamma_{r\varphi} &= (\varepsilon_y - \varepsilon_x)\sin 2\varphi + \gamma_{xy}\cos 2\varphi,
\end{aligned}\right\} \tag{13.5}$$

There are Cauchy relations, which connect displacements and strains in polar coordinates for a plane strain-stress state

$$\left.\begin{aligned}
&\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\varphi = \frac{1}{r}\left(u + \frac{\partial v}{\partial \varphi}\right),\\
&\gamma_{r\varphi} = \frac{1}{r}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} - \frac{v}{r}.
\end{aligned}\right\} \tag{13.6}$$

Biharmonic equation (11.26) in polar coordinate system can be written

$$\left.\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial \varphi^2}\right) = 0,\\
&\text{or in extended view}\\
&\frac{\partial^4\psi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\psi}{\partial r^3} + \frac{1}{r^2}\left(2\frac{\partial^4\psi}{\partial r^2\partial\varphi^2} - \frac{\partial^2\psi}{\partial r^2}\right) +\\
&+\frac{1}{r^3}\left(\frac{\partial\psi}{\partial r} - 2\frac{\partial^3\psi}{\partial r\partial\varphi^2}\right) + \frac{1}{r^4}\left(4\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^4\psi}{\partial\varphi^4}\right) = 0.
\end{aligned}\right\} \tag{13.7}$$

Stresses in polar coordinates for a plane strain-stress state can be written

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\left(\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2}\right), \quad \sigma_\varphi = \frac{\partial^2\psi}{\partial r^2}, \\ \tau_{r\varphi} &= \frac{1}{r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\varphi}-\frac{\partial^2\psi}{\partial r\,\partial\varphi}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\right). \end{aligned}\right\} \tag{13.8}$$

Equilibrium equations of elementary tetrahedron can be writte

$$\left.\begin{aligned} \sigma_r l + \tau_{\varphi r} m &= p_{\nu r}, \\ \tau_{r\varphi} l + \sigma_\varphi m &= p_{\nu\varphi}. \end{aligned}\right\} \tag{13.9}$$

where $l$, $m$ – are direction cosines of outer normal vector $\nu$ relatively radial direction $r$ and tangential direction $\varphi$; $p_{\nu r}$, $p_{\nu\varphi}$ – are projections of external forces into directions $r$ and $\varphi$.

**Stress Concentrations in Plates With Holes.** Using stress function (13.7) in polar coordinates and appropriate boundary conditions there are some useful analytic solutions can be obtained. These solutions have practical usage mostly for exploring of stress concentrations for plates with holes.

*Stress concentrators* are places of abrupt change of geometry in a solid body or in constructions.

*Stress concentration* is a local enlarging of stresses in stress concentrators in comparison with nominal stresses acting far enough from concentrators.

The level of stress concentration can be estimated by a stress concentration factor

$$k = \frac{\sigma_c}{\sigma_{\text{nom}}}, \tag{13.10}$$

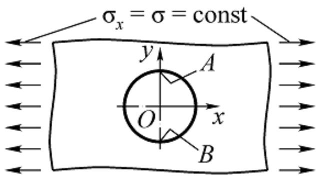
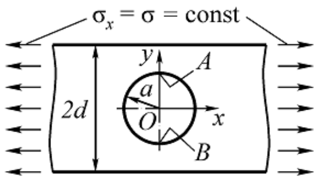
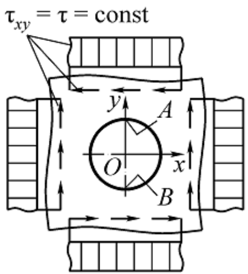
where $\sigma_c$ – are stresses in concentrator; $\sigma_{\text{nom}}$ – are nominal stresses defined far enough from concentration zone.

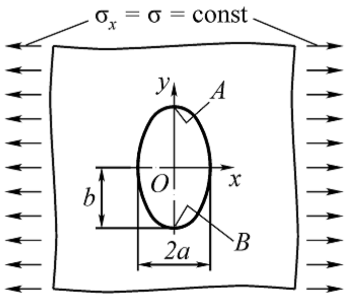
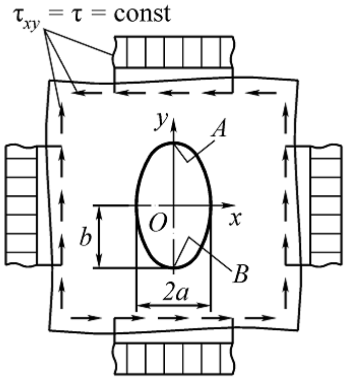
Stresses in (13.10) may be normal, shear, some equivalent, etc., but the nature in numerator and denominator in (13.10) has to be the same.

There are often constructions occur which have plates with holes of different forms, and these plates are in a plane strain-stress state. It's necessary to estimate the level of stress concentrations. For some occasions accurate analytic solutions of plane strain-stress state in polar coordinates can be used.

In the table 13.1 there are stress concentration factors are presented for different occasions of a plane strain-stress state for plates with holes.

Table 13.1 – Stress concentration factors for plates with holes

| # | The name of calculations scheme | The calculation scheme | Stress concentration factor in points $A$ and $B$. |
|---|---|---|---|
| 1 | Infinite plate with round hole under normal nominal stresses |   | $k = 3$ |
| 2 | Finite plate with round hole under normal nominal stresses (but the length along $x$ is infinite) |   | $k = \dfrac{6}{\left(1 - \dfrac{a^2}{d^2}\right)\left(2 + \dfrac{a^2}{d^2}\right)}$ |
| 3 | Infinite plate with round hole under shear nominal stresses |   | $k = \dfrac{4}{\sqrt{3}} = 2{,}31$ |

| # | The name of calculations scheme | The calculation scheme | Stress concentration factor in points *A* and *B*. |
|---|---|---|---|
| 4 | Infinite plate with elliptical hole under normal nominal stresses |  | $k = 1 + 2\dfrac{b}{a}$ |
| 5 | Infinite plate with elliptical hole under shear nominal stresses |  | $k = \dfrac{4}{\sqrt{3}(1 - m^2)}, \quad m = \dfrac{a - b}{a + b}$ |

# LECTURE 14. NUMERICAL METHODS IN THE THEORY OF ELASTICIY

**The General Description of Structures Strength Analysis Methods.** A mathematical model describes behavior of the physical model of a body that is affected by factors causing appearance of a strain-stress state in it. Selection or development of the efficient method of the mathematical model of investigation is one of the missions of the strength analysis.

Mathematical models for models in the theory of elasticity have been described here using mostly differential equations. But we should note that other forms of mathematical description of physical models exist, such as functional equations, integral equations, etc. In the theory of elasticity and structural mechanics variational principle is widely used. It is based on the energy theorems and principles. This form is usable for development of approximate methods of mathematical model of investigation. For now methods based on variational principles are one of the most efficient methods which provide high accuracy of results and relatively low labor-consuming and good algorithmization. That's why variational methods are widely used for now.

There are different classifications of numerical method and variety of methods itself.

We may categorize all the mathematical methods into three groups regardless the type of a mathematical model. These are *analytical*, *semi-analytical* and *discrete* groups of methods.

*Analytical* methods give precise solutions but for limited number of strain-stress states. These methods usually can be obtained for regular geometry, constraints and loads. *Discrete* methods can be divided into two common kinds: net methods and so called discrete volume methods. In net methods (collocation method, finite difference method (FDM), minimal squares method (MSM), etc.) we deal with certain body points and can know the function values only in those points. For discrete volume methods (Ritz method, Bubnov-Galerkin method, finite element method (FEM), boundary element method (BEM), finite volume method (FVM), etc.) we deal with discrete volumes, which together form the whole body. For every discrete volume the analytic solution of the required function is known. Usually net methods are simpler, less labor-consuming but less accurate in comparison with discrete volume methods.

Semi-analytical methods hold an intermediate position. There are a lot of semi-analytical methods so it’s hard to classify them. For example BEM mentioned above can be presented in this group because it demand some analytical solutions. These methods sometimes can be very effective, but sometimes it’s impossibly to apply them, because they require some analytic solutions of the given equations, which can be provided not always.

Numerical methods should be used in practice together with computer and appropriate software. Here below only several widely used methods are explained.

**Ritz Method.** Here is classic Ritz method is described. Ritz method is based on using of mathematical formulation of the *principle of virtual displacements* (or in some sources the *principle of virtual work*) which can be written

$$\delta Э = \delta П - \iiint\limits_V \left( X\delta u + Y\delta v + Z\delta w \right) dV -$$

$$- \iint\limits_{S_1} \left( p_{\nu x}\delta u + p_{\nu y}\delta v + p_{\nu z}\delta w \right) dS, \qquad (14.1)$$

where $\delta$ – variation; $\ddot{I}$ – is full potential energy of a body; $V$ – body volume; $S$ – body surface.

Also there are stated displacements on the part of a body $S_2$

$$u_2 = u_2(\bar{x}, \bar{y}, \bar{z}), \quad v_2 = v_2(\bar{x}, \bar{y}, \bar{z}), \quad w_2 = w_2(\bar{x}, \bar{y}, \bar{z}). \tag{14.2}$$

The solution of the problem trace to definition of displacements $u$, $v$, $w$ which provide (14.1) and (14.2). Let's present every displacement as a series

$$\left.\begin{aligned} u(x,y,z) &= u_0(x,y,z) + \sum_{k=1}^{N} a_k f_k(x,y,z), \\ v(x,y,z) &= v_0(x,y,z) + \sum_{k=1}^{N} b_k \varphi_k(x,y,z), \\ w(x,y,z) &= w_0(x,y,z) + \sum_{k=1}^{N} c_k \psi_k(x,y,z), \end{aligned}\right\} \tag{14.3}$$

where $u_0$, $v_0$, $w_0$ have to be chosen to satisfy the next relations on the part of surface $S_2$

$$u_0 = u_2, \quad v_0 = v_2, \quad w_0 = w_2 \text{ on } S_2, \tag{14.4}$$

where $f_k$, $\varphi_k$, $\psi_k$ ($k$ = 1, 2, … , $N$) – are so called coordinate functions satisfying to conditions of linear independency and completeness and also to the condition on $S_2$

$$f_k = 0, \quad \varphi_k = 0, \quad \psi_k = 0 \ \ (k = 1, 2, \ \dots, N) \text{ on } S_2, \tag{14.5}$$

where $a_k$, $b_k$, $c_k$ – are factors which have to be defined.

Expressions (14.4) and (14.5) means that displacements (14.3) have to provide kinematic boundary conditions on the part of the surface $S_2$.

On basis of expressions (14.3) displacement variations can be written

$$\delta u = \sum_k \delta a_k f_k(x, y, z),$$

$$\delta v = \sum_k \delta b_k \varphi_k(x, y, z), \quad \delta w = \sum_k \delta c_k \psi_k(x, y, z) . \quad (14.6)$$

To find constants $a_k$, $b_k$, $c_k$ it is necessary to have the expression for potential energy in terms of displacements. For isotropic body this expression can be presented

$$\left. \begin{aligned} &\Pi = G\iiint_V \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + \frac{\mu}{1-2\mu} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \right. \\ &\left. + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right] dxdydz. \right. \end{aligned} \right\} \quad (14.7)$$

If to substitute (14.3) into (14.1) considering (14.7) we shall have that *Ý* is the complex dependency of factors $a_k$, $b_k$, $c_k$.

Variation of the functional *Ý* is this

$$\delta Э = \sum_k \left( \frac{\partial Э}{\partial a_k} \delta a_k + \frac{\partial Э}{\partial b_k} \delta b_k + \frac{\partial Э}{\partial c_k} \delta c_k \right). \quad (14.8)$$

From (14.8) considering arbitrariness of $\delta a_k$, $\delta b_k$, $\delta c_k$ we have known system of Ritz equations to define unknown factors $a_k$, $b_k$, $c_k$

$$\frac{\partial Э}{\partial a_k} = 0, \quad \frac{\partial Э}{\partial b_k} = 0, \quad \frac{\partial Э}{\partial c_k} = 0, \quad k = 1, 2, \ \dots\ , N . \quad (14.9)$$

Inserting found values of $a_k$, $b_k$, $c_k$ into (14.3) we shall have expressions for displacements. Using known displacements it is possibly to find strains, stresses.

**Generalized Bubnov-Galerkin Method.** Expressions for generalized Bubnov-Galerkin method can be written

$$\left.\begin{aligned}
&\iiint_V \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X \right) f_k(x, y, z) dV - \\
&\quad - \iint_{S_1} (\sigma_x l + \tau_{yx} m + \tau_{zx} n - p_{\nu x}) f_k(x, y, z) dS = 0, \\
&\iiint_V \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y \right) \varphi_k(x, y, z) dV - \\
&\quad - \iint_{S_1} (\tau_{xy} l + \sigma_y m + \tau_{zy} n - p_{\nu y}) \varphi_k(x, y, z) dS = 0, \\
&\iiint_V \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z \right) \psi_k(x, y, z) dV - \\
&\quad - \iint_{S_1} (\tau_{xz} l + \tau_{yz} m + \sigma_z n - p_{\nu z}) \psi_k(x, y, z) dS = 0, \\
&k = 1, 2, \ \dots, N.
\end{aligned}\right\} \qquad (14.10)$$

On the basis of this method there is principle of virtual displacements, the same as for Ritz method. Expressions for displacements (14.3) have to satisfy to kinematic boundary conditions (14.4) and (14.5) and may not to satisfy to force boundary conditions on a part of a body.

That's why the generalized Bubnov-Galerkin method and (classic) Ritz methods are two different forms of the principle of virtual displacements. Using these two methods it is possibly to get the same system of resolution equations relatively factors $a_k$, $b_k$, $c_k$.

The procedure of using of Bubnov-Galerkin method is presented on the fig. 14.1.

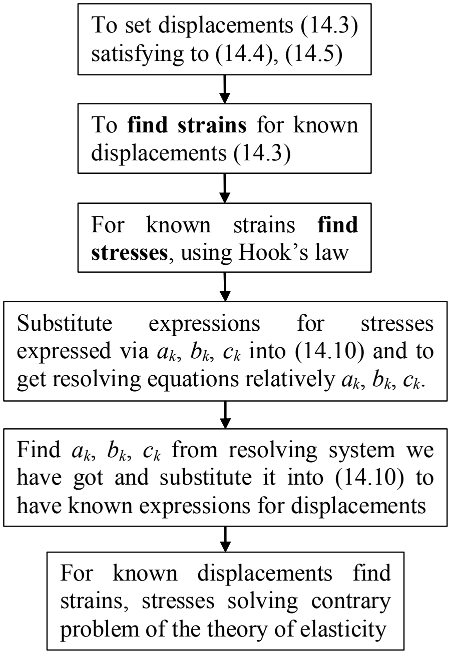



Fig. 14.1 – Algorithm of generalize Bubnov-Galerkin method

**The Finite-Difference Method.** Equations describing behavior of physical models may contain derivatives. There are different approximate formulas (so called-difference expressions) for derivatives, which define a derivative value of a given function and for a given order in some point via function values defined in neighboring points. Substituting these approximate formulas into describing equations we shall have so called a finite difference model or finite difference equations. Writing finite-difference equations for every body point we shall have the system of equations relatively function values in every point. The accuracy of FDM method depends much on number of body points and point's regularity and other factors.

We may obtain mentioned finite-difference expressions in a variety of ways. But expansion in a Taylor series is the mostly often used way. The central finite-difference approximation (FDA) is mostly used in practice, but other approximations (left FDA, right FDA) are also applied.

For example one of formulas for central FDA for the function depending on one variable can be written

$$\left.\begin{aligned}
f'(x_i) &= \frac{f(x_i+h)-f(x_i-h)}{2h};\\
f''(x_i) &= \frac{f(x_i+h)-2f(x_i)+f(x_i-h)}{h^2};\\
f'''(x_i) &= \frac{f(x_i+2h)-2f(x_i+h)+2f(x_i-h)-f(x_i-2h)}{2h^3};\\
f^{IV}(x_i) &= \frac{f(x_i+2h)-4f(x_i+h)+6f(x_i)-4f(x_i-h)+f(x_i-2h)}{h^4},
\end{aligned}\right\} \qquad (14.11)$$

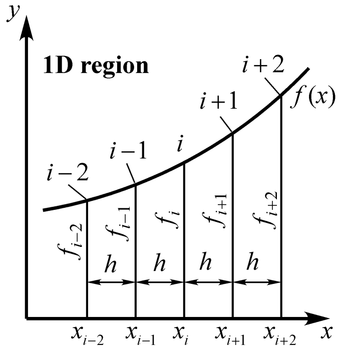
where $f$ – is a function depending on one variable; $x_i$ – is a fixed abscissa; $h$ – is a step.

The accuracy of expressions (14.11) is about $h^2$. For regular mesh expressions (14.11) can be written

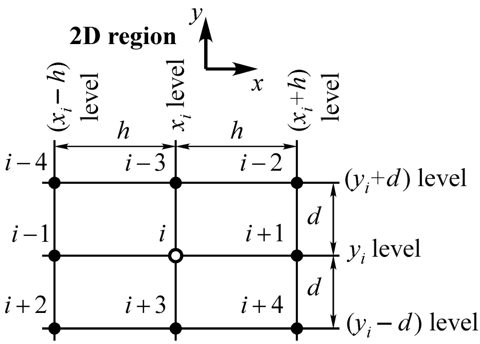
$$\left.\begin{aligned}
f_i' &= \frac{1}{2h}(f_{i+1}-f_{i-1});\\
f_i'' &= \frac{1}{h^2}(f_{i+1}-2f_i+f_{i-1});\\
f_i''' &= \frac{1}{2h^3}(f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2});\\
f_i^{IV} &= \frac{1}{h^4}(f_{i+2}-4f_{i+1}+6f_i-4f_{i-1}+f_{i-2}),
\end{aligned}\right\} \qquad (14.12)$$

where $f_i$ – is the function value in point $i$.

Explanations for expressions (14.11) and (14.12) are presented on the fig. 14.2,***a***.




***a***




***b***

Fig. 14.2 – For explanation of the finite-difference approximation

Finite-difference approximation also can be used for functions depending on several variables. Expressions (14.11) or/and (14.12) are also can be used to find derivative value in certain *i*-th point for function of several variables (see fig. 14.2,***b***).

Let's review on example how to find the mixed derivative $\partial^2 f/(\partial x \partial y)$ in *i*-th point (see fig. 14.2,***b***), where $f = f(x, y)$ is some function depending on two variables, but $x_i$ and $y_i$ are coordinates of $i$-th point. The given mixed derivative $\partial^2 f/(\partial x\, \partial y)$ can written

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right). \tag{14.13}$$

So first it's necessary to take derivative in $x$-direction $\partial f/\partial x$, and after it's necessary to take derivative in $y$-direction $\partial(\partial f/\partial x)/\partial y$ according to (14.13). To take derivative in $x$-direction means to take derivative for all fixed $y$-levels (see fig. 14.2,***b***). For some $y$-level $y$ coordinate in $f(x, y)$ is fixed, but $x$ coordinate is variable, so we deal with function depending only on one variable $x$, and 1-st equation of (14.11) or/and (14.12) can be applied for this function of one variable. To find derivative $\partial f/\partial x$ for $y_i$ level means to take derivative for $i$-th point. Analogously to take derivative $\partial f/\partial x$ for $(y_i - d)$ and for $(y_i + d)$ levels means to take this derivative for points $i+3$ and $i-3$ accordingly. On base of the 1-st expression of (14.11) and (14.12) derivative "*f*/"*x* for levels $(y_i - d)$, $y_i$, $(y_i + d)$ can be written

$$\left.\begin{aligned}
&\frac{\partial}{\partial x} f(x_i, y_i - d) = \frac{f(x_i + h, y_i - d) - f(x_i - h, y_i - d)}{2h} = \\
&\quad = \frac{1}{2h}(f_{i+4} - f_{i+2}) = Y_{j-1} \\
&\frac{\partial}{\partial x} f(x_i, y_i) = \frac{f(x_i + h, y_i) - f(x_i - h, y_i)}{2h} = \\
&\quad = \frac{1}{2h}(f_{i+1} - f_{i-1}) = Y_j \\
&\frac{\partial}{\partial x} f(x_i, y_i + d) = \frac{f(x_i + h, y_i + d) - f(x_i - h, y_i + d)}{2h} = \\
&\quad = \frac{1}{2h}(f_{i-2} - f_{i-4}) = Y_{j+1},
\end{aligned}\right\},\tag{14.14}$$

where $f_{i+4}$ – is the function value $f(x, y)$ in point $i + 4$, etc.; $Y_{j-1}$ – is derivative value in point $i + 3$ for $(y_i - d)$ level, etc.

To take derivative in $y$-direction we consider $y$-levels to be as "points". For every fixed $y$-level the derivative value $\partial f / \partial x$ is already known. These values are $Y_{j-1}$, $Y_j$, $Y_{j+1}$ for levels $(y_i - d)$, $y_i$, $(y_i + d)$ accordingly. Using derivative values for $y$-levels as values of the function of one variable $y$ on basis of 1-st expression of (14.11) or/and (14.12) we can write down the formula to find the demanded derivative $\partial(\partial f / \partial x) / \partial y$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{1}{2d}\left(Y_{j+1} - Y_{j-1}\right). \quad (14.15)$$

Substituting (14.14) into (14.15) we shall have the finite-difference approximation to find mixed derivative $\partial^2 f / (\partial x\ \partial y)$ in a point for the function $f = f(x, y)$, depending on two variables, which can be written

$$\left.\begin{aligned}
&\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \\
&= \frac{1}{4dh}[f(x_i + h, y_i + d) - f(x_i - h, y_i + d) - \\
&- f(x_i + h, y_i - d) + f(x_i - h, y_i - d)] = \\
&= \frac{1}{4dh}(f_{i-2} - f_{i-4} - f_{i+4} + f_{i+2}).
\end{aligned}\right\} \quad (14.16)$$

Analogously other FDA can be reached for other mixed derivatives.

**The Finite Element Method.** Finite element method (FEM) is based on the suppositional representation of a solid body (a continuum) as an assemblage of discrete finite elements (FE). These elements interact with each other in nodal points. Usually a body consists on one type of FEs. The FEs we approximate a body have stated numbers of degrees of freedom. For every degree of freedom there is the common nodal displacement $q_i$ and the common appropriate nodal reaction $R_i$ fit (see fig. 14.3).

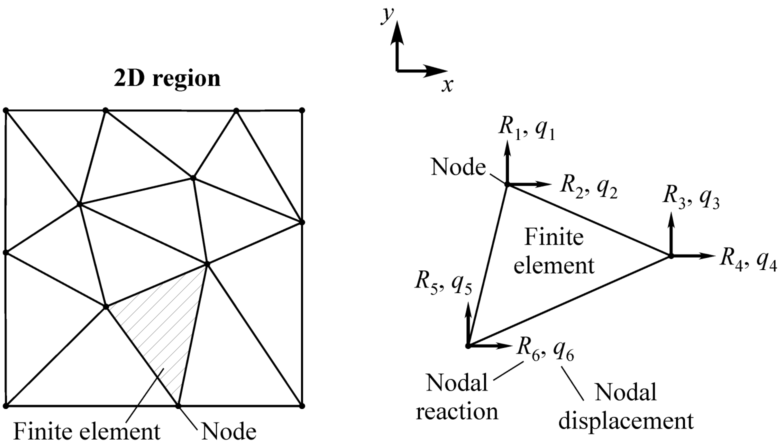



Fig. 14.3 – Approximation of the plate for the flat (2D) problem of the theory of elasticity by FE net

The required function within FE and other accompany functions can be defined via nodal displacements $\{q\}$ or nodal reactions $\{R\}$, where $\{q\} = \{q_1, q_2,\ldots, q_n\}$ – is a vector of nodal displacements for all $n$ number of nodes, but $\{R\} = \{R_1, R_2,\ldots, R_n\}$ – is a vector of nodal reactions. The required function usually is defined in local coordinate system connected with EF.

Between nodal displacements $\{q\}$ and nodal reactions $\{R\}$ there is a connection. In the most common occasion $\{q\}$ and $\{R\}$ are connected with some functions

$$\left.\begin{aligned} q_1 &= f_1(R_1, R_2, \dots, R_n); \\ q_2 &= f_2(R_1, R_2, \dots, R_n); \\ &\dots \\ q_n &= f_n(R_1, R_2, \dots, R_n); \end{aligned}\right\}; \qquad \left.\begin{aligned} R_1 &= \varphi_1(q_1, q_2, \dots, q_n); \\ R_2 &= \varphi_2(q_1, q_2, \dots, q_n); \\ &\dots \\ R_n &= \varphi_n(q_1, q_2, \dots, q_n). \end{aligned}\right\} \qquad (14.17)$$

where $n$ – is the number of degrees of freedom of the FE.

For physical models based on the classic linear theory of elasticity relations (14.17) can be much simplified and can be written

$$\{q\} = [A]\{R\}, \quad \{R\} = [K]\{q\}, \qquad (14.18)$$

where

$\{q\} = \{q_1, q_2, \dots, q_n\}$ – is a vector of nodal displacements within given FE;

$\{R\} = \{R_1, R_2, \dots, R_n\}$ – is a vector of nodal reactions within given FE;

$[A]$ – is a suppleness square matrix;

$[K]$ – is a stiffness square matrix.

Using 1-st equation in (14.18) we shall have FEM in so called "force method" performance, but using 2-nd equation in (14.18) we shall have FEM in "displacement method" performance. As practice show the "force method" for FEM realization is much less suitable for realization and algorithmization than "displacement method". That's why FEM in "displacement method" performance has always been using up till now.

The general steps for FEM in "displacement method" performance are presented below.

For the 1-st step it's necessary to develop the stiffness matrix $[K]$ for given type of FE using mainly variational and other (if possibly) principles. Factors of $[K]$ depend on geometrical sizes and physical (stiffness) properties of the given FE. If the stiffness matrix for the given FE is created in a FE local coordinate system it's necessary to develop additionally so called transform coordinate matrix $[T]$.

For the 2-nd step it's necessary to mesh the construction by FEs getting simultaneously information about node positions of FEs. Numbers/indexes of nodes and FEs have to be determined in global enumeration. Physical properties for every FE have to be defined and set.

For the 3-rd step it's necessary to compute stiffness matrix for every FE in global coordinate system, using expression

$$[K'] = [T]^T [K][T], \tag{14.19}$$

where $[T]$ – is a transform coordinate matrix; $[K]$ – is a stiffness matrix written in local FE coordinate system; $[K2\ ]$ – is a stiffness matrix written in global coordinate system of the whole construction; top index $T$ means transponition operation.

For the 4-th step it's necessary to compose equilibrium equations for every node. After all that we have to express nodal reactions $\{R\}$ via nodal displacements $\{q\}$ applying 2-nd equation of (14.18) regarding FE stiffness matrices we've got in 3-rd step. Within this step, using FE stiffness matrices of the 3-rd step, we form so called *total stiffness matrix* for the whole construction and total *vector of nodal loads* (the right part of the system) which don't consider kinematic boundary conditions. This step is voluminous and has several ways of realization. One of the most simple and clear methods to get total stiffness matrix and vector of nodal loads is based on so called *index matrix*.

The 5-th step consists on creation of so called *cutoff/trummed total stiffness matrix* for the whole construction taking into account kinematic boundary conditions. Within this procedure we eliminate some rows and columns in total stiffness matrix we've got in 4-th step. Indexes of eliminated rows and columns match the indexes of the known nodal displacements in global enumeration. Elimination concerns the right part too. As the result we have the resolution system in which only unknown displacements are taken into consideration.

For the 6-th step we solve the system of linear algebraic equations, we've got in 5-th step to get unknown displacements. Having all nodal displacements for every FE we can find the required function and other accompany functions within separate FE. FEM can be applied for all physic models (rods, plates, shells, massive bodies) in structural mechanics. Moreover FEM can be used almost in all problems of mathematical physics.

There are numerous software programs complexes which realize FEM for calculation of problems of solid mechanics, deformable solid mechanics, fluid mechanics, electric mechanics, magnetic and electro-magnetic mechanics. Some program complexes able to calculate only one problem but some of them are multi-physics. There are such widely known software complexes as ANSYS, ABAQUS, NASTRAN, SolidWorks Simulation, etc.

## LECTURE 15. EXPERIMENTAL METHODS IN THE THEORY OF ELASTICIY

**Basic Information about Experimental Methods of Explorations.** Practice and experiment are essential to all engineer sciences including structural mechanics and applied theory of elasticity. All the system of hypothesis and basic statements which serve as a base for the theory are obtained in a result of experience of practice investigations. Conclusions and recommendations of the theory are also checked by experiments. To improve and develop methods of strength science it is necessary to develop special methods of measuring and exploration of constructions and its elements in different conditions, i.e. it is necessary to develop experimental methods.

Here is incomplete list of problems which can be solved only by experimental methods:

- Definition of mechanic properties of materials in different conditions, influence of environment, temperature, kind of strain-stress state and other factors for strength and mechanical properties of construction materials.
- Checkout and searching of hypothesis which perform or able to perform basis of different theories and methods.
- Exploration of character and magnitudes of external forces acting into structures, its construction and elements.

• Search of the most rational constructive forms matching enough strength of structure minimizing its weight and cost.

• Investigation of influence of technology factors, deviations from correct geometrical forms, scattering of mechanical properties for strength and stability.

• Experimental definition of internal forces, strains and stresses instead of its calculation if calculative methods are not developed or too heavy.

On the way of solution of problems relatively structures or its parts and elements experimental methods can be divided into next general kinds: *natural* (full-size), *semi-natural*, *model* and *analogue* ones.

In *natural experiment* measuring and observations are realized on a nature (real) structure in usual or close to usual conditions of exploitation. Natural experiments have exceptional significance and importance, though its realization is often connected with great technical difficulties and material costs. In natural experiments they explore influence of all factors at its total action without exception. The only lack of this method is that it's not always possibly to define quite simply, from the total group of effects of different factors, the influence of some factor or group of factors. It's not always possibly get sufficient data for complete analysis and determination of true role of each factor in nature experiment.

*Semi-natural experiment* is realized using natural constructions or its large scale models on special test benches, which allow to recreate typical conditions for natural effects and actions of separate factors and groups of factors. In semi-natural experiment all the properties of object are saved completely or almost completely (using large scale models). External actions are reproduced with some degree of completeness and resemblance. The value of this kind of experiment is that it is possibly to explore different external factors sequentially and separately. In this case semi-natural experiment supplements possibilities of natural experiments.

In *model experiment* they use small scale models of explored constructions or its parts and elements made from the same or another material. Influence of a scale effect as a rule doesn't let to realize similarity of all properties of a model and nature construction. In this connection for one model it is possibly to investigate only one or several properties of a real construction. It is necessary to analyze carefully and eliminate factors and conditions, which may cause distortion of similarity of a model and a nature in a process of experiment. Model experiment in comparison with natural or semi-natural ones is more simple and inexpensive allowing multiply recurrence and any degree of particularization of explored effects. Model experiment is good for investigation of the problem of internal forces but eliminated for exploring of external forces and allowable stresses. Model experiments are widely used.

*Analogue experiment* is based on analogy of laws by which two or several physic phenomena are described. If one of analogue phenomena is reproduced simpler in an experiment, more obvious and availably for measuring it may be analogue for another phenomenon.

In summary on the fig. 15.1 there are kinds of experiment are presented.

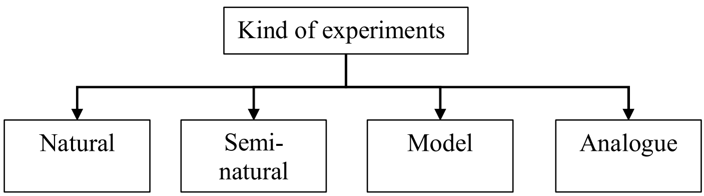



Fig. 15.1 – Kind of experiments

Analogies are established on basis of similarity or identity of mathematic equations describing physic phenomena. Since description of real phenomena by mathematic equations is as a rule approximate therefore all analogies are approximate. If magnitudes of some parameters exceed the bounds of allowable range the analogy may be fully or partially broken. This fact has always to be taken into account.

There are series of other analogies between problems of structural mechanics and the theory of elasticity and problems of electrician, physical optics, hydromechanics and other sciences. But these analogies are interest if usage of the analogy causes much simplification and extension of possibilities of experimental solution of the problems.

As example it is necessary to mention membrane analogy for pure torsion theory of rods. For definition of virtual masses when ship vibrating in water they use plane or volumetric electrostatic analogy etc. For exploration of strain-stress states a polarization-optical method is used.

The science of experimental methods include the theory of similarity and modeling, the theory of errors and reduction of observations, the theory of devices, apparatus and ways of measuring, special methods and organization of experiments according to problems and conditions of specific sciences.

**Polarization-Optical Method.** In the polarization-optical method they use the possibility of reproduce the strain-stress state in a model made of transparent, so called optically active material. This strain-stress state in a optical model is similar to strain-stress state of a natural body. In the process of loading the model made of an optical active material change its properties influencing for light which coming through it. This method is the most effective for plane strain-stress state. As it was shown above the solution of the two-dimensional problem of the theory of elasticity in stresses trace to integration of the system of three differential equations (11.23) considering boundary conditions (11.5). In the mentioned equations there are no elastic constants and body sizes. Hence in two geometrically similar ideally elastic bodies for identical intensity of external forces in similar points there are the same stresses independently from its elastic constants.

If for geometrically similar model and real body intensity of loads is not identical but similar that the dependency between external loads and stresses in model and in nature is next

$$k = \frac{P_m}{P_n} = \frac{\sigma_m}{\sigma_n}, \tag{15.1}$$

where $k$ – factor of similarity; $P_m$, $P_n$ – intensity of external loads in similar points of a model and a nature; $\sigma_m$, $\sigma_n$ – intensity of stresses in similar points of a model and a nature.

Thus for simulation of plain strain-stress state of a simply connected body it is necessary to keep geometrical similarity of a model and a nature and similarity of its external loads. For multiply connected body it is necessary to provide special conditions for displacements (4.21).

Let's review the optical part of the method. Models are made of optically active materials. After appearance of stress field in optically active materials it become optically anisotropic with orientation of principal optical axes along directions of principal stresses. If to cause plane strain-stress state in a plate and to shine out this plate by a beam of plane-polarized light perpendicularly to plate plane thus in every point of plane a light beam splits into two rays. Polarization planes of these two rays coincide with directions of principal stresses in a given point. At passage through depth of a plate both rays have different velocities. As experiments show difference of velocities of rays is proportional to algebraic difference of principal stresses. As a consequence on coming out of plate mentioned rays receive relative phase shift, which is proportional to the difference of principal stresses in a given point. Measuring phase difference, for example using interference of rays knowing the factor of proportionality it is possibly to define the difference of principal stresses. This is the principal basis of the method.

Let's review the simplest methodology on a basis of raying of the model by monochromatic light. For raying of the model it is possibly to use monochromatic light, i.e. light with fixed wave length and white light representing the totality of light rays with different wave lengths. Moreover the light may be plane-polarized or to have circular polarization.

There are different devices and apparatus of different complexity and cost which emit and receive light after its passage through a plate and which allow to define the relative phase shift for rays and as a consequence differences of principal stresses in a given point and other auxiliary operations.

Plane-polarized monochromatic light can be got using *polarizer*. Let we have vertical *polarizer* plane and as a consequence light beam is polarized in mentioned plane. Let's denote the ray coming out from polarizer as $I_0$. Hitting the plane of strained model plate the ray $I_0$ will be split up into two rays $I_1$, $I_2$ the polarization planes of which are mutually orthogonal and coincide with directions of principle stresses in a given point. If $\sigma_1$ ‘‘‘ $\sigma_2$ therefore at passing through plate with depth $h$, rays have different velocity and on coming out of the plate they obtain difference of phases. Further on movement in air velocities of both rays are equal, but obtained difference of phases and mutual orthogonality of planes of polarization is saved.

Path difference of rays can be measured using phenomena of interference. To do that it is necessary to coincide both rays $I_1$ and $I_2$ in one plane, passing them through second polarizer. The polarization plane of that second polarizer has to be perpendicular to polarization plane of the first polarizer. The second polarizer is called analyzer but the system of polarizer and analyzer with mutually perpendicular optical planes is called *polariscope*.

Path difference of rays $\delta$ is proportional to the difference of principle stresses and to a plate thickness $h$

$$\delta = hc(\sigma_1 - \sigma_2), \tag{15.2}$$

where $c$ – optical constant of a material.

As a result on a screen situated behind the second polarizer in frames of a lighted field of a model there are dark stripes appear, in all points of which one of principle stresses is situated in polarization plane of a light beam (in the plane of polarizer). These stripes are called *isoclinic lines*. Turning polariscope in the plane perpendicular to path rays saving mutual orthogonality of optical planes of polarizer and analyzer it is possibly to get isoclinic lines fitting for any angles of inclination of principal directions and according to it trajectories of principal stresses.

Thus the model field will be covered by dark and bright stripes with gradual transition from shade to light. Each stripe of identical illumination takes place in points with identical difference of principal stresses. Such stripes are called *isochromatic lines (isochromatics)*. The dark isochromatics line got for $\delta = 0$ ($\sigma_1 = \sigma_2 = 0$, or $\sigma_1 = \sigma_2$) is called *zero isochromatics*, for the value $\delta = \lambda$ – isochromatics of the 1-st order, for $\delta = \lambda$ – isochromatics of the 2-nd order etc. At rotation of polariscope isoclinic lines (dark lines) change its positions on a screen but the picture of isochromatics (bright lines) still to be unchangeable. This appearance can be used for getting of isochromatic lines in pure view not black-out by isoclinic lines what is necessary for measuring of the path difference of rays. If to rotate polariscope quite quickly we shall have isoclinic lines (dark lines) moving on a screen too quickly become unnoticeable.

The same effect can be got if to use light with circular polarization. A ray with circular polarization consists of two plane-polarized rays with mutual perpendicular planes of polarization, identical amplitudes and a path difference of rays in a quarter of a wave. Total vector of these two rays is uninterruptedly change its direction turning around axis of propagation of a ray, what is equivalent to revolving of a polariscope.

If the model is rayed not by monochromatic light but by the white one the picture of isoclinic lines in case of using of plane-polarized light still to be previous. Path differences of rays with different length of waves (for different colors) for the same value ($\sigma_1$–$\sigma_2$) are different. Therefore rays of different colors contained in white light go out not simultaneously and the picture of isochromatics will be colored. The color in every point will be defined by complementary colors regarding to rays died away fully or partially. That's why lines of equal values of ($\sigma_1 - \sigma_2$) are called isochromatics (isochromatic lines). Isochromatics take place sequentially on all colors of rainbow, but the most noticeable are red, yellow and green colors. It is important to notice that for zero isochromatic fringe on which there are isotropic points are situated ($\sigma_1 = \sigma_2 = 0$, or $\sigma_1 = \sigma_2$) all colors die away simultaneously and this isochromatic line as the separate isotropic points is always dark, that allows to distinguish it from isochromatic line of higher orders.

For different optical active material it is necessary to define *stripe scale interval*. Stripe scale interval has to be defined for the optical active material the model is going to be made. It is necessary to have simple model made of that material for which simple external loads are applied and there is analytical accurate solution for this case. For example it may be rectangular plate of constant thickness loaded by uniform tensile pressure σ ($\sigma_1 = \sigma$, $\sigma_2 = 0$, $\sigma_1 - \sigma_2 = \sigma$). If change of shadows (light intervals) took place *n* times at a change of stresses σ for a magnitude $\sigma_1$, but the plate thickness is *h*, the stripe scale interval is

$$\sigma_0 = \sigma_1 \frac{h}{n}. \tag{15.3}$$

Knowing the stripe scale interval for a given material the difference of principal stresses can be found using next formula

$$\sigma_1 - \sigma_2 = \frac{n\sigma_0}{h}. \tag{15.4}$$

The order of isochromatic line $n$ can be defined not only for integer but for fractional values of $n$ in semi-darken points. The less value of stripe scale interval $\sigma_0$ is, the more optical activity material has and the denser there are isochromatic lines situated for the same strain-stress state. It allows to find differences of principal stresses more precisely.

To define stripe scale interval there are often specimens in a view of a round plate are used compressed by diametric external forces. According to accurate solution of the theory of elasticity the stresses in the centre of this plate are

$$\sigma_1 = \frac{2P}{\pi hd}, \quad \sigma_2 = \frac{6P}{\pi hd}, \quad \sigma_1 - \sigma_2 = \frac{8P}{\pi hd}, \tag{15.5}$$

where $P$ – force value; $h$ – plate thickness; $d$ – plate diameter.

More precisely the path difference of rays can be measured using so called *compensator*. There are different constructions of compensators, but the most known of it are compensator of Cocker and Krasnov.

Thus polarized-optical method allow to define directly directions (isoclinic lines – dark lines) and algebraic differences (isochromatic lines – bright lines) in any point of a model.

But for the full solution of the plane problem it is necessary to find values of each principle stresses $\sigma_1$ and $\sigma_2$. There are two the simplest methods to do that:

1) In case of the plane stress-stain state a linear strain in the direction of thickness of a plate is defined by the well known formula

$$\varepsilon = -\frac{\mu}{E}(\sigma_1 + \sigma_2). \tag{15.6}$$

Thus changing the plate thickness $\Delta h$ in reviewing points and calculating $\varepsilon = \Delta h/h$ it is possibly to find the sum of principle stresses. Knowing difference and sum of principle stresses (having two equations) it is possibly to find principal stresses $\sigma_1$ and $\sigma_2$. To have satisfactory precision the value $\Delta h$ have to be measured using very sensitive tensometers what complicate experiment.

2) The second way is narrowed down to numerical integration of equilibrium equations using experimental data of optical method. Let's review some line $y = y_i$ and parallel to it lines $y_i + \Delta y$ and $y_i - \Delta y$ (see fig. 15.2). The position of comparison axes $xOy$ is arbitrary.

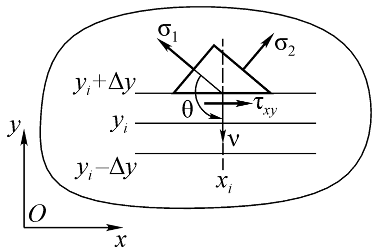



Fig. 15.2 – For explanation of analysis of observations

According to the first equation of (11.6) without consideration of volume and inertial forces there, for the line $y_i$ there is

$$\sigma_x = \sigma_x(0) + \int_0^x \frac{\partial \tau_{xy}}{\partial y} dy . \tag{15.7}$$

Derivative values $\partial \tau_{xy} / \partial y$ for points $x = x_i$ can be obtained using data of optical method. Indeed, according to (11.21) shear stresses on inclined plane relatively to principle planes are

$$\tau_{xy} = \frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta , \tag{15.8}$$

which allows to define shear stresses in any point and plane knowing difference of principle stresses. Calculating values of stresses $\tau_{xy}$ for points $(x_i, y_i + \Delta y)$ and $(x_i, y_i - \Delta y)$ it is possibly to find approximate value of derivative for the point $(x_i, y_i)$ using one of approximate formulae for example

$$\frac{\partial \tau_{xy}(x_i, y_i)}{\partial y} = \frac{\tau_{xy}(x_i, y_i + \Delta y) - \tau_{xy}(x_i, y_i - \Delta y)}{2\Delta y}. \quad (15.9)$$

The expression (15.9) is given as an example. Instead of (15.9) it is possibly to use more precise expressions for derivatives.

Knowing derivative values for sufficient number of equidistant points $x = x_i$ it is possibly using numerical integration for (15.7) to define stresses $\sigma_x$ for all the line $y = y_i$. The initial value $\sigma_x(0)$ can be found in the simplest way for a contour of a model using known values $\sigma_1$ and $\sigma_2$. Thus it is possibly to get values of stresses for any number of arbitrary sections of a model.

However without definition of principal stresses the picture of isochromatic lines gives a visual presentation about stress state of the model displaying stress concentration and understressed regions. Changing geometry of the model it is possibly to choose rational forms.

On the fig. 15.3 there is a round plate is shown made of an optical active material loaded by diametric loads and rayed in frames of polarized-optical method.

Polarization-optical method also is used for the three-dimensional strain-stress state, but for this case it becomes complicated and less precise.

Polarization-optical method is widespread. It is effective method of solution of static and dynamic problems of the theory of elasticity and structural mechanics.

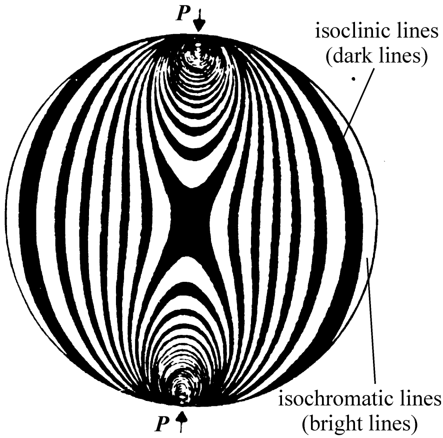



Fig. 15.3 – Round plate loaded by diametric loads

**Electrotensometry.** Electrotensometry is provided using *electrical tensometers*. Let's review physics fundamentals of it.

Electrical resistance of electric conductor having form of prismatic shaft can be defined using formula

$$R = \rho \frac{L}{F}, \tag{15.10}$$

where ρ – specific resistance of electric conductor; $L$ – conductor length; $F$ – area of the cross section of the conductor.

At tension of a conductor for magnitude $\Delta L$ the cross section area $F$ decreases simultaneously. Moreover, tests show in this case a specific resistance ρ increase. At compression of a conductor there are analogical changes take place but in opposite direction. Thus between relative change of electric resistance of a conductor and its relative longitudinal strain there is relation

$$\varepsilon = \frac{\Delta L}{L} = k \frac{\Delta R}{R}. \tag{15.11}$$

In consequence of inevitable change of ρ and $F$ at longitudinal deformation the factor $k$ increases and for different materials equals to 2.5–3. Dimensionless value $k$ is called *strain sensitivity factor* (*strain gauge factor*) which is defined in experimental way.

If electrical conductor is attached to any fiber of an explored body in the way for longitudinal strains to be passed to conductor completely, and then defining change of electrical resistance of a conductor it is possibly using expressions (15.11) to calculate linear strain of a fiber. This principle is the basis for measuring of strains and as a consequence of stresses applying electrical tensometers in the composition of which the described above conductor is used. Electrical resistance of tensometer is measured when there is transmission of electric current. In this connection tensometers have to be isolated from a body explored if the last is also electrical conductor.

Constructively tensometer (see fig. 15.4) consists of thin wire which is electrical conductor *1* which is stacked as bights as it is shown on the fig. 15.4, (but sometimes in another way) and glued on insulating foundation *2* such as polymeric or other one. For connection of tensometer into electrical measuring circuit there are conductors *3* are used. The length $L$ of a wire *1* is called the *grid length* (*active gage length*).

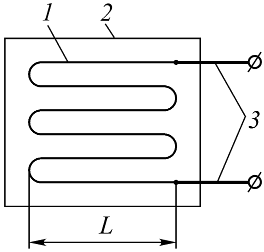



Fig. 15.4 – Electrical tensometer

Measurement of electrical resistance of tensometer is usually provided using circuits on basis of Wheatstone bridge. The principal model of the simplest set intended for this aim is shown of the fig. 15.5.

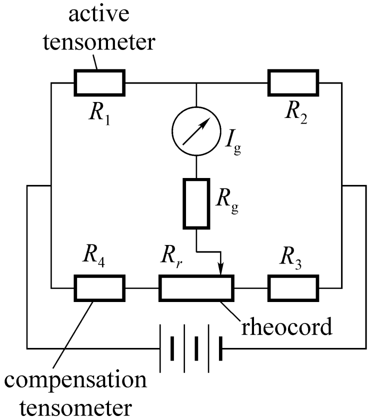



Fig. 15.5 – The principal model of reading on basis of Wheatstone bridge

Resistance $R_1$ match *active tensometer* which is attached to construction and by which we fix strains of a construction.

It is common knowledge that temperature changes cause deformation of a body. Temperature strains are fixed in the same way as strains caused by external forces. That's why providing measurements in conditions of changeable temperature it is necessary to exclude from gage reading temperature part of strains. It can be done using so called *compensation tensometer*, which have the same properties as an active tensometer, but have to be connected up as $R_4$. Compensation tensometer have to be glued on some specimen made of the same material an explored construction is made of and placed in a place where the temperature of an active tensometer and compensation tensometer is the same. In this case change of resistances of arms $R_1$ and $R_4$ caused by temperature strains is the same and no unbalance of bridge will take place.

There are two general methods of measuring of resistance of active tensometers: so called null-method (*balanced method*, *zero-method*) and *method of direct reading*. Each of these methods has its merits and demerits.

In industrial sets/apparatus for measuring of strains on basis of Wheatstone bridge instead of galvanometer there is a chain of logic circuits such as an amplifier, block of handling of reading data, computer output. All these circuits are controlled by control block. Rheocord is also managed automatically or present by itself semi-conductor circuit which is controlled by control block.

For a complex strain-stress state to define all components of plane strain-stress state in explored point they use so called *strain-gauge rosette*. Rosette present itself as a rule three-four tensometers glued on in one region (point) on three different directions. The most used there are orthogonal rosette (see fig. 15.6,***a***) and delta rosette (see fig. 15.6,***b***).

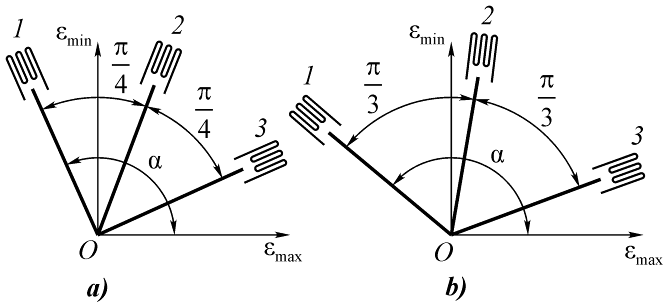



Fig. 15.6 – Strain-gauge rosettes

Orthogonal rosette is used when directions of principal strains are approximately known in a given point. In this case measurement of strains is more precise in comparison with delta rosette. Delta rosette is used for arbitrary orientation of rosette relatively principal strains.

For known strains read by tensometers *1*, *2*, *3* it is possibly to define principle strains $\varepsilon_{min}$, $\varepsilon_{max}$ and angle $\alpha$.

For orthogonal rosette $\varepsilon_{max}$, $\varepsilon_{min}$ and $\alpha$ can be found using expressions

$$\left.\begin{aligned} \varepsilon_{\substack{max\\min}} &= \frac{1}{2}(\varepsilon_1+\varepsilon_2) \pm \frac{\sqrt{2}}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_2-\varepsilon_3)^2}, \\ \operatorname{tg} 2\alpha &= \frac{2\varepsilon_2-\varepsilon_1-\varepsilon_3}{\varepsilon_1-\varepsilon_3}. \end{aligned}\right\} \quad (15.12)$$

For delta rosette $\varepsilon_{max}$, $\varepsilon_{min}$ and $\alpha$ can be found using expressions

$$\left.\begin{aligned} \varepsilon_{\substack{max\\min}} &= \frac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3) \pm \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_3-\varepsilon_1)^2}, \\ \operatorname{tg} 2\alpha &= \frac{\sqrt{3}(\varepsilon_2-\varepsilon_3)}{2\varepsilon_1-\varepsilon_2-\varepsilon_3}. \end{aligned}\right\} \quad (15.13)$$

Despite for a series of deficiencies electrical tensometers are the most universal and perfect means of experimental exploration of nature constructions and models.

There is variety of other experimental methods. Let’s review shortly some of it.

**String Tensometers.** The principle of action of string tensometers is based on change of natural frequency of free vibrations of tensed string under change of string tension. Measured strain of base is passed to string. Free vibrations of string are periodically perturbed by electromagnet impulse. These dumped oscillations induce the electric current of the same frequency in the electromagnet coil, that allow to measure frequency in any way. Usually the frequency of gauging string is compared with frequency of etalon string, the elongation strain of which can be changed for given quantities. For frequencies of gauging and etalon strings being identical a longitudinal strain of the first one equals to the known longitudinal strain of the second one. Base of string tensometers may be from 20 to 100 mm providing high accuracy of measuring of strains (1–2 %).

**Inductive Tensometers.** Tensometers of this type consists of two coils and keeper (core). On core moving the inductivity of one coil is increased but of the second one is decreased. Coils are connected with one point of measuring base but the core with the second point of measuring base. It allows gauging inductivity of coils to measure relative displacements of base points and thereby its strain. Change of inductivity of coils is measured with help of ac (alternative current) bridge. Base of inductive tensometers may be from 1–2 to 100 mm and more.

**Brittle Coating Method.** Using method of brittle coating there may be strains measured using special lacquer which is coated on an explored body with thin layer. After drying up of a lacquer there is a thin brittle layer appears. A lacquer coating has sensitivity $\varepsilon_0$ match relative linear strain when in a brittle coating there are cracks appear. Sensitivity $\varepsilon_0$ of applied brittle coating have to be less than expectable strains. Brittle coating method is applied mostly for qualitative investigations:

- for ascertainment of the most strained regions in a body where cracks appear first;
- for ascertainment of directions of principle stresses by crack directions;
- for ascertainment of character of deformation of body elements of construction.

In some points it is possibly to get quantitative results. So at ends of cracks linear strains of a body surface in direction perpendicular to crack direction equal to sensitivity $\varepsilon_0$ of coating. Brittle coating method is convenient to use with other methods. Qualitative analysis of a strain-stress state of explored object which can be simply provided by brittle coating method allows to create more correct program of quantitative investigations.

**Moire Fringe Method** is based on optical effect which takes place at relative shift of two systems dark and bright stripes.

Let for light surface of explored body there are parallel dense stripes are marked quite densely. The distance between those stripes and its thickness is equal. If the same system of stripes to mark on a transparent plate or a film and to match with stripes on a body surface then at deformation of a body matching of stripes will be broken. Depending on a degree of divergence of stripes of a body and a film in a given region this region may look more or less dark. Thus at deformation of a body surface there is instead of monotonous grey background on its field there appear dark and light stripes – moirés, the allocation and frequency of which depend on strain distribution on a body surface. For directions of moiré fringes (stripes) and distances between them it is possibly to define directions and magnitudes of principle strains.

There are numerous of other experimental methods: mechanical, optical, pneumatic, electrical, combined etc. meant for exploration of static and dynamic strain-stress states, vibration etc. There are usual gauges designed for constructions working in usual environmental conditions and specific gauges designed for specific constructions or special working conditions of constructions where usual devices can’t work or its work is inconvenient. Almost all modern gauges have computer outputs that allow to transfer read data from it into computer, providing quick handling of that data using appropriate software.

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## CONTENT

## ДЛЯ НОТАТОК

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Навчальне видання

**СОКОВ** Валерій Миколайович

**THEORY OF ELASTICITY**

Навчальний посібник

(*англійською мовою*)

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Комп'ютерне верстання *В. В. Москаленко*

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