МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ УКРАИНЫ
Национальный университет кораблестроения
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ПРАКТИКУМ ПО НЕОПРЕДЕЛЕННЫМ ИНТЕГРАЛАМ

Рекомендовано Методическим советом НУК
в качестве учебного пособия

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Практикум, содержащий более 550 неопределенных интегралов, наиболее часто встречающихся на практике, имеет целью максимально облегчить учебу, сделать ее побуждающей к самостоятельным исследованиям, что особенно важно при переходе на кредитно-модульную систему обучения.

Пособие будет полезно всем студентам и молодым преподавателям.

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ПРЕДИСЛОВИЕ

Для будущего инженера очень важно уметь применять определенные, криволинейные и многомерные интегралы для решения задач геометрии, физики, механики и техники.

В основе всех этих типов интегралов лежит неопределенный интеграл. От умения его отыскания лежит и решение поставленной задачи.

Данный практикум серии РЕШЕБНИК содержит нахождение более 550 интегралов. Все они взяты из известного "Сборника задач по курсу математического анализа" Г.Н. Бермана.

Каждому разделу предшествуют краткие теоретические положения. Это дает возможность студенту решать столько задач, сколько ему необходимо, чтобы приобрести устойчивые навыки по указанной тематике.

ТАБЛИЦА ИНТЕГРАЛОВ

1. $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$.

2. $\int \frac{dx}{x} = \ln|x| + C$.

3. $\int a^x dx = \frac{a^x}{\ln a} + C$.

4. $\int e^x dx = e^x + C$.

5. $\int \sin x\, dx = -\cos x + C$.

6. $\int \cos x\, dx = \sin x + C$.

7. $\int \operatorname{sh} x\, dx = \operatorname{ch} x + C$.

8. $\int \operatorname{ch} x\, dx = \operatorname{sh} x + C$.

9. $\int \operatorname{tg} x\, dx = -\ln|\cos x| + C$.

10. $\int \operatorname{ctg} x\, dx = \ln|\sin x| + C$.

11. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$.

12. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$.

13. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$.

14. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C$.

15. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$.

16. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$.

17. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$.

18. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$.

§ 1. ПРОСТЕЙШИЕ ПРИМЕРЫ ИНТЕГРИРОВАНИЯ

В задачах **1** (1676)–**27** (1702), воспользовавшись основной таблицей интегралов и простейшими правилами интегрирования, найти интегралы.

При решении этих задач будем также пользоваться теоремами:

1. Неопределенный интеграл от алгебраической суммы двух или нескольких функций равен сумме интегралов от этих функций.

2. Постоянный множитель можно выносить за знак интеграла.

1 (1676). $\int \sqrt{x}\,dx = \int x^{1/2}dx = \frac{x^{3/2}}{3/2} + C = \underline{\underline{\frac{2}{3}\sqrt{x^3} + C}}.$

2 (1677). $\int \sqrt[m]{x^n}\,dx = \int x^{n/m}dx = \frac{x^{(n/m)+1}}{(n/m)+1} + C = \underline{\underline{\frac{mx^{(n/m)+1}}{n+m} + C}}.$

3 (1678). $\int \frac{dx}{x^2} = \int x^{-2}dx = \frac{x^{-1}}{-1} + C = \underline{\underline{-\frac{1}{x} + C}}.$

4 (1679). $\int 10^x\,dx = \underline{\underline{\frac{10^x}{\ln 10} + C}}.$

5 (1680). $\int a^x e^x dx = \int (ae)^x\,dx = \frac{(ae)^x}{\ln ae} + C = \underline{\underline{\frac{a^x e^x}{1+\ln a} + C}}.$

6 (1681). $\int \frac{dx}{2\sqrt{x}} = \frac{1}{2}\int x^{-1/2}dx = \frac{1}{2}\cdot\frac{x^{1/2}}{\frac{1}{2}} + C = \underline{\underline{\sqrt{x} + C}}.$

7 (1682). $\int \frac{dh}{\sqrt{2gh}} = \frac{1}{\sqrt{2g}}\cdot\int\frac{dh}{\sqrt{h}} = \frac{1}{\sqrt{2g}}\cdot 2\sqrt{h} = \underline{\underline{\sqrt{\frac{2h}{g}} + C}}.$

8 (1683). $\int 3{,}4x^{-0{,}17}dx = 3{,}4\cdot\frac{x^{0{,}83}}{0{,}83} + C \approx \underline{\underline{4{,}1x^{0{,}83} + C}}.$

9 (1684). $\int (1-2u)\,du = \int du - 2\int u\,du = \underline{\underline{u - u^2 + C}}.$

10 (1685). $\int \left(\sqrt{x}+1\right)\left(x-\sqrt{x}+1\right)dx = \int \left(x\sqrt{x} + x - x - \sqrt{x} + \sqrt{x} + 1\right)dx =$

$$= \int (x^{3/2}+1)\;dx = \frac{x^{5/2}}{\frac{5}{2}} + x + C = \underline{\underline{\frac{2}{5}x^2\sqrt{x} + x + C}}.$$

11 (1686). $\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3} dx = \int \left(x^{-5/2} - e^x + x^{-1} \right) dx = \frac{x^{-3/2}}{-\frac{3}{2}} - e^x +$

$+ \ln|x| + C = \underline{\underline{- \frac{2}{3x\sqrt{x}} - e^{-x} + \ln|x| + C}}.$

12 (1687). $\int \left(2x^{-1,2} + 3x^{-0,8} - 5x^{0,38} \right) dx = 2\frac{x^{-0,2}}{-0,2} + 3\frac{x^{0,2}}{0,2} - 5\frac{x^{1,38}}{1,38} + C =$

$= \underline{\underline{-10x^{-0,2} + 15x^{0,2} - 3,62x^{1,38} + C}}.$

13 (1688). $\int \left(\frac{1-z}{z} \right)^2 dz = \int \frac{1 - 2z + z^2}{z^2} dz = \int \left(z^{-2} - \frac{2}{z} + 1 \right) dz =$

$= \underline{\underline{-\frac{1}{z} - 2\ln|z| + z + C}}.$

14 (1689). $\int \frac{(1-x)^2}{x\sqrt{x}} dx = \int \frac{1 - 2x + x^2}{x^{3/2}} dx = \int \left(x^{-3/2} - 2x^{-1/2} + x^{1/2} \right) dx =$

$\underline{\underline{= \frac{2x^2 - 12x - 6}{3\sqrt{x}} + C}}.$

15 (1690). $\int \frac{\left(1 + \sqrt{x}\right)^3}{\sqrt[3]{x}} dx = \int \frac{1 + 3\sqrt{x} + 3x + x^{3/2}}{\sqrt[3]{x}} dx = \int \left(x^{-1/3} + 3x^{1/6} + \right.$

$\left. + 3x^{2/3} + x^{7/6} \right) dx = \frac{x^{2/3}}{\frac{2}{3}} + 3\frac{x^{7/6}}{\frac{7}{6}} + 3\frac{x^{5/3}}{\frac{5}{3}} + \frac{x^{13/6}}{\frac{13}{6}} + C =$

$= \underline{\underline{\frac{3}{2}\sqrt[3]{x^2} + \frac{18}{7}x\sqrt[6]{x} + \frac{9}{5}x\sqrt[3]{x^2} + \frac{6}{13}x^2\sqrt[6]{x} + C}}.$

16 (1691). $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}dx = \int \left(x^{1/6}-x^{-1/4}\right)dx = \frac{x^{7/6}}{\frac{7}{6}} - \frac{x^{3/4}}{\frac{3}{4}} + C =$

$= \underline{\underline{\frac{6}{7}\sqrt[6]{x^7} - \frac{4}{3}\sqrt[4]{x^3} + C.}}$

17 (1692). $\int \frac{dx}{\sqrt{3-3x^2}} = \underline{\underline{\frac{1}{\sqrt{3}}\arcsin x + C.}}$

18 (1693). $\int \frac{3\cdot 2^x - 2\cdot 3^x}{2^x}dx = \int \left(3-2\left(\frac{3}{2}\right)^x\right)dx = 3x - 2\frac{\left(\frac{3}{2}\right)^x}{\ln\frac{3}{2}} + C =$

$= \underline{\underline{3x - \frac{2\cdot 1{,}5^x}{\ln 1{,}5} + C.}}$

19 (1694). $\int \frac{1+\cos^2 x}{1+\cos 2x}dx = \int \frac{1+\cos^2 x}{\cos^2 x + \sin^2 x + \cos^2 x - \sin^2 x}dx =$

$= \frac{1}{2}\int\left(\frac{1}{\cos^2 x}+1\right)dx = \underline{\underline{\frac{1}{2}(\operatorname{tg} x + x) + C.}}$

20 (1695). $\int \frac{\cos 2x}{\cos^2 x\cdot\sin^2 x}dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x\cdot \sin^2 x}dx = \int\left(\frac{1}{\sin^2 x} -\right.$

$\left. - \frac{1}{\cos^2 x}\right)dx = \underline{\underline{C - \operatorname{ctg} x - \operatorname{tg} x.}}$

21 (1696). $\int \operatorname{tg}^2 x\, dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \underline{\underline{\operatorname{tg} x - x + C.}}$

22 (1697). $\int \operatorname{ctg}^2 x\, dx = \int \frac{1-\sin^2 x}{\sin^2 x}dx = \underline{\underline{C - \operatorname{ctg} x - x.}}$

23 (1698). $\int 2\sin^2\frac{x}{2}\,dx=\int(1-\cos x)\,dx=\underline{\underline{x-\sin x+C}}.$

24 (1699). $\int\frac{(1+2x^2)}{x^2(1+x^2)}\,dx=\int\frac{(1+x^2)+x^2}{x^2(1+x^2)}\,dx=\int\left(\frac{1}{x^2}+\frac{1}{1+x^2}\right)dx=$

$=\underline{\underline{\frac{-1}{x}+\operatorname{arctg}x+C}}.$

25 (1700). $\int\frac{(1+x)^2}{x(1+x^2)}\,dx=\int\frac{(1+x^2)+2x}{x(1+x^2)}\,dx=\int\left(\frac{1}{x}+\frac{2}{1+x^2}\right)dx=$

$=\underline{\underline{\ln|x|+2\operatorname{arctg}x+C}}.$

26 (1701). $\int\frac{dx}{\cos 2x+\sin^2 x}=\int\frac{dx}{\cos^2 x-\sin^2 x+\sin^2 x}=\underline{\underline{\operatorname{tg}x+C}}.$

27 (1702). $\int\underbrace{(\arcsin x+\arccos x)}_{\pi/2}\,dx=\underline{\underline{\frac{\pi}{2}x+C}}.$

В задачах **28** (1703)–**105** (1780) найти интегралы, воспользовавшись теоремой об инвариантности формул интегрирования: если $\int f(x)\,dx=F(x)+C$ и $u=u(x)$ – произвольная функция, имеющая производную, то $\int f(u)\,du=F(u)+C$.

28 (1703). $\int\sin x\,d(\sin x)=\underline{\underline{\frac{\sin^2 x}{2}+C}}.$

29 (1704). $\int\operatorname{tg}^3x\,d\,(\operatorname{tg}x)=\underline{\underline{\frac{\operatorname{tg}^4x}{4}+C}}.$

30 (1705). $\int\frac{d(1+x^2)}{\sqrt{1+x^2}}=\underline{\underline{2\sqrt{1+x^2}+C}}.$

31 (1706). $\int (x+1)^{15}\,dx = \int (x+1)^{15}\,d(x+1) = \underline{\underline{\dfrac{(x+1)^{16}}{16} + C}}.$

32 (1707). $\int \dfrac{dx}{(2x-3)^5} = \dfrac{1}{2}\int \dfrac{d(2x-3)}{(2x-3)^5} = \underline{\underline{-\dfrac{1}{8(2x-3)^4} + C}}.$

33 (1708). $\int \dfrac{dx}{(ax+b)^c} = \dfrac{1}{b}\int \dfrac{d(a+bx)}{(a+bx)^c} = \underline{\underline{\dfrac{(a+bx)^{1-c}}{b(1-c)} + C}} \quad (c \neq 1).$

34 (1709). $\int \sqrt[5]{(8-3x)^6}\,dx = -\dfrac{1}{3}\int (8-3x)^{6/5}\,d(8-3x) = -\dfrac{(8-3x)^{11/5}}{3\cdot\dfrac{11}{5}} +$

$+C = \underline{\underline{C - \dfrac{5}{33}(8-3x)^{11/5}}}.$

35 (1710). $\int \sqrt{8-2x}\,dx = -\dfrac{1}{2}\int (8-2x)^{1/2}\,d(8-2x) = -\dfrac{1}{2}\cdot\dfrac{(8-2x)^{3/2}}{\dfrac{3}{2}} + C =$

$= \underline{\underline{C - \dfrac{\sqrt{(8-2x)^3}}{3}}}.$

36 (1711). $\int \dfrac{m\,dx}{\sqrt[3]{(a+bx)^2}} = \dfrac{m}{b}\int (a+bx)^{-2/3}\,d(a+bx) = \dfrac{m}{b}\cdot\dfrac{(a+bx)^{1/3}}{\dfrac{1}{3}} + C =$

$= \underline{\underline{\dfrac{3m}{b}\sqrt[3]{a+bx} + C}}.$

37 (1712). $\int 2x\sqrt{x^2+1}\,dx = \int (x^2+1)^{1/2}\,d(x^2+1) = \underline{\underline{\dfrac{2}{3}\sqrt{(x^2+1)^3} + C}}.$

38 (1713). $\int x\sqrt{1-x^2}\,dx = -\dfrac{1}{2}\int (1-x^2)^{1/2}\,d(1-x^2) = \underline{\underline{C - \dfrac{1}{3}\sqrt{(1-x^2)^3}}}.$

39 (1714). $\displaystyle\int x^2\sqrt[5]{x^3+2}\,dx=\frac{1}{3}\int (x^3+2)^{1/5}\,d(x^3+2)=\underline{\underline{\frac{5}{18}\sqrt[5]{(x^2+2)^6}+C}}.$

40 (1715). $\displaystyle\int\frac{x\,dx}{\sqrt{x^2+1}}=\frac{1}{2}\int\frac{d(x^2+1)}{\sqrt{x^2+1}}=\underline{\underline{\sqrt{x^2+1}+C}}.$

41 (1716). $\displaystyle\int\frac{x^4dx}{\sqrt{4+x^5}}=\frac{1}{5}\int\frac{d(4+x^5)}{\sqrt{4+x^5}}=\underline{\underline{\frac{2}{5}\sqrt{4+x^5}+C}}.$

42 (1717). $\displaystyle\int\frac{x^3dx}{\sqrt[3]{x^4+1}}=\frac{1}{4}\int (x^4+1)^{-1/3}\,d(x^4+1)=\underline{\underline{\frac{3}{8}\sqrt[3]{(x^4+1)^2}+C}}.$

43 (1718). $\displaystyle\int\frac{(6x-5)\,dx}{2\sqrt{3x^2-5x+6}}=\frac{1}{2}\int\frac{d(3x^2-5x+6)}{\sqrt{3x^2-5x+6}}=\underline{\underline{\sqrt{3x^2-6x+6}+C}}.$

44 (1719). $\displaystyle\int\sin^3 x\cdot\cos x\,dx=\int\sin^3 x\,d(\sin x)=\underline{\underline{\frac{\sin^4 x}{4}+C}}.$

45 (1720). $\displaystyle\int\frac{\sin x\,dx}{\cos^2 x}=-\int\frac{d(\cos x)}{\cos^2 x}=\frac{1}{\cos x}+C=\underline{\underline{\sec x+C}}.$

46 (1721). $\displaystyle\int\frac{\cos x\,dx}{\sqrt[3]{\sin^2 x}}=\int (\sin x)^{-2/3}\,d(\sin x)=\underline{\underline{3\sqrt[3]{\sin x}+C}}.$

47 (1722). $\displaystyle\int\cos^3 x\cdot\sin 2x\,dx=2\int\cos^3 x\cdot\sin x\cdot\cos x\,dx=$

$$=-2\int\cos^4 x\,d(\cos x)=\underline{\underline{C-\frac{2}{5}\cos^5 x}}.$$

48 (1723). $\displaystyle\int\frac{\sqrt{\ln x}\,dx}{x}=\int(\ln x)^{1/2}\,d(\ln x)=\underline{\underline{\frac{2}{3}\sqrt{(\ln x)^3}+C}}.$

49 (1724). $\displaystyle\int\frac{(\operatorname{arctg} x)^2\,dx}{1+x^2}=\int(\operatorname{arctg} x)^2\,d(\operatorname{arctg} x)=\underline{\underline{\frac{(\operatorname{arctg} x)^3}{3}+C}}.$

50 (1725). $\displaystyle\int \frac{dx}{(\arcsin x)^3\sqrt{1-x^2}} = \int \frac{d(\arcsin x)}{(\arcsin x)^3} = \underline{\underline{-\frac{1}{2(\arcsin x)^2}+C}}.$

51 (1726). $\displaystyle\int \frac{dx}{\cos^2 x\sqrt{1+\operatorname{tg} x}} = \int \frac{d(1+\operatorname{tg} x)}{\sqrt{1+\operatorname{tg} x}} = \underline{\underline{2\sqrt{1+\operatorname{tg} x}+C}}.$

52 (1727). $\displaystyle\int \cos 3x\, d(3x) = \underline{\underline{\sin 3x + C}}.$

53 (1728). $\displaystyle\int \frac{d(1+\ln x)}{\cos^2(1+\ln x)} = \underline{\underline{\operatorname{tg}(1+\ln x)+C}}.$

54 (1729). $\displaystyle\int \cos 3x\, dx = \frac{1}{3}\int \cos 3x\, d(3x) = \underline{\underline{\frac{1}{3}\sin 3x + C}}.$

55 (1730). $\displaystyle\int (\cos\alpha - \cos 2x)\, dx = \underline{\underline{x\cos\alpha - \frac{\sin 2x}{2} + C}}.$

56 (1731). $\displaystyle\int \sin(2x-3)\, dx = \frac{1}{2}\int \sin(2x-3)\; d(2x-3) =$

$\displaystyle= \underline{\underline{-\frac{1}{2}\cos(2x-3)+C}}.$

57 (1732). $\displaystyle\int \cos(1-2x)\, dx = -\frac{1}{2}\int \cos(1-2x)\; d(1-2x) =$

$\displaystyle= \underline{\underline{-\frac{1}{2}\sin(1-2x)+C}}.$

58 (1733). $\displaystyle\int \left(\cos\left(2x-\frac{\pi}{4}\right)\right)^2 dx = \frac{1}{2}\int \left(\cos\left(2x-\frac{\pi}{4}\right)\right)^2 d\left(2x-\frac{\pi}{4}\right) =$

$\displaystyle= \underline{\underline{\frac{1}{2}\operatorname{tg}\left(2x-\frac{\pi}{4}\right)+C}}.$

59 (1734). $\displaystyle\int e^x \sin(e^x)\, dx = \int \sin(e^x)\, d(e^x) = \underline{\underline{-\cos(e^x)+C}}.$

60 (1735). $\int \frac{d(1+x^2)}{1+x^2} = \underline{\underline{\ln(1+x^2)+C}}.$

61 (1736). $\int \frac{d(\arcsin x)}{\arcsin x} = \underline{\underline{\ln|\arcsin x| + C}}.$

62 (1737). $\int \frac{(2x-3)\,dx}{x^2-3x+8} = \int \frac{d(x^2-3x+8)}{x^2-3x+8} = \underline{\underline{\ln\left|x^2-3x+8\right| + C}}.$

63 (1738). $\int \frac{dx}{2x-1} = \frac{1}{2}\int \frac{d\,(2x-1)}{2x-1} = \underline{\underline{\frac{1}{2}\ln|2x-1| + C}}.$

64 (1739). $\int \frac{dx}{cx+m} = \frac{1}{c}\int \frac{d\,(cx+m)}{cx+m} = \underline{\underline{\frac{1}{c}\ln|\,cx+m| + C}}.$

65 (1740). $\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{d\,(x^2+1)}{x^2+1} = \underline{\underline{\frac{1}{2}\ln(x^2+1) + C}}.$

66 (1741). $\int \frac{x^2\,dx}{x^3+1} = \frac{1}{3}\int \frac{d\,(x^3+1)}{x^3+1} = \underline{\underline{\frac{1}{3}\ln\left|x^3+1\right| + C}}.$

67 (1742). $\int \frac{e^x\,dx}{e^x+1} = \int \frac{d\,(e^x+1)}{e^x+1} = \underline{\underline{\ln(e^x+1) + C}}.$

68 (1743). $\int \frac{e^{2x}\,dx}{e^{2x}+a^2} = \frac{1}{2}\int \frac{d\,(e^{2x}+a^2)}{e^{2x}+a^2} = \underline{\underline{\frac{1}{2}\ln(e^{2x}+a^2) + C}}.$

69 (1744). $\int \operatorname{tg} x\,dx = -\int \frac{d\,(\cos x)}{\cos x} = \underline{\underline{-\ln|\cos x| + C}}.$

70 (1745). $\int \operatorname{ctg} x\,dx = \int \frac{d\,(\sin x)}{\sin x} = \underline{\underline{\ln|\sin x| + C}}.$

71 (1746). $\int \operatorname{tg} 3x\,dx = \frac{1}{3}\int \operatorname{tg} 3x\,d\,(3x) = \underline{\underline{-\frac{1}{3}\ln|\cos 3x\,| + C}}.$

72 (1747). $\int \operatorname{ctg}(2x+1)\,dx = \frac{1}{2}\int \operatorname{ctg}(2x+1)\,d(2x+1) =$

$= \underline{\underline{\frac{1}{2}\ln\left|\sin(2x+1)\right| + C}}.$

73 (1748). $\int \frac{\sin 2x}{1+\cos^2 x}\,dx = -\int \frac{d(1+\cos^2 x)}{1+\cos^2 x} = \underline{\underline{-\ln(1+\cos^2 x) + C}}.$

74 (1749). $\int \frac{dx}{x\ln x} = \int \frac{d(\ln x)}{\ln x} = \underline{\underline{\ln\left|\ln x\right| + C}}.$

75 (1750). $\int \frac{(\ln x)^m\,dx}{x} = \int (\ln x)^m\,d(\ln x) = \underline{\underline{\frac{(\ln x)^{m+1}}{m+1} + C}}.$

76 (1751). $\int e^{\sin x}\,d(\sin x) = \underline{\underline{e^{\sin x} + C}}.$

77 (1752). $\int e^{\sin x}\cos x\,dx = \int e^{\sin x}\,d(\sin x) = \underline{\underline{e^{\sin x} + C}}.$

78 (1753). $\int e^{3x}dx = \frac{1}{3}\int e^{3x}\,d(3x) = \underline{\underline{\frac{1}{3}e^{3x} + C}}.$

79 (1754). $\int a^{-x}\,dx = -\int a^{-x}\,d(-x) = \underline{\underline{-\frac{a^{-x}}{\ln a} + C}}.$

80 (1755). $\int e^{-3x+1}\,dx = -\frac{1}{3}\int e^{-3x+1}\,d(-3x+1) = \underline{\underline{-\frac{1}{3}e^{-3x+1} + C}}.$

81 (1756). $\int e^{x^2}x\,dx = \frac{1}{2}\int e^{x^2}\,d(x^2) = \underline{\underline{\frac{1}{2}e^{x^2} + C}}.$

82 (1757). $\int e^{-x^3}x^2\,dx = -\frac{1}{3}\int e^{-x^3}\,d(-x^3) = \underline{\underline{-\frac{1}{3}e^{-x^3} + C}}.$

83 (1758). $\int \frac{d\left(\frac{x}{3}\right)}{\sqrt{1-\left(\frac{x}{3}\right)^2}} = \underline{\underline{\arcsin\frac{x}{3} + C}}.$

84 (1759). $\displaystyle\int \frac{dx}{\sqrt{1-25x^2}} = \frac{1}{5}\int \frac{d(5x)}{\sqrt{1-(5x)^2}} = \underline{\underline{\frac{1}{5}\arcsin 5x + C}}.$

85 (1760). $\displaystyle\int \frac{dx}{1+9x^2} = \frac{1}{3}\int \frac{d(3x)}{1+(3x)^2} = \underline{\underline{\frac{1}{3}\arcsin 3x + C}}.$

86 (1761). $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \underline{\underline{\arcsin \frac{x}{2} + C}}.$

87 (1762). $\displaystyle\int \frac{dx}{2x^2+9} = \frac{1}{\sqrt{2}}\int \frac{d\left(\sqrt{2}x\right)}{\left(\sqrt{2}x\right)^2+3^2} = \underline{\underline{\frac{1}{3\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2}x}{3} + C}}.$

88 (1763). $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{d(3x)}{\sqrt{2^2-(3x)^2}} = \underline{\underline{\frac{1}{3}\arcsin\frac{3x}{2} + C}}.$

89 (1764). $\displaystyle\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{d(x^2)}{(x^2)^2+1} = \underline{\underline{\frac{1}{2}\operatorname{arctg} x^2 + C}}.$

90 (1765). $\displaystyle\int \frac{x\,dx}{\sqrt{a^2-x^4}} = \frac{1}{2}\int \frac{d(x^2)}{\sqrt{a^2-(x^2)^2}} = \underline{\underline{\frac{1}{2}\arcsin\frac{x^2}{a} + C}}.$

91 (1766). $\displaystyle\int \frac{x^2\,dx}{x^6+4} = \frac{1}{3}\int \frac{d(x^3)}{(x^3)^2+4} = \underline{\underline{\frac{1}{6}\operatorname{arctg} x^3 + C}}.$

92 (1767). $\displaystyle\int \frac{x^3\,dx}{\sqrt{1-x^8}} = \frac{1}{4}\int \frac{d(x^4)}{\sqrt{1-(x^4)^2}} = \underline{\underline{\frac{1}{4}\arcsin x^4 + C}}.$

93 (1768). $\displaystyle\int \frac{e^x\,dx}{e^{2x}+4} = \int \frac{de^x}{(e^x)^2+2^2} = \underline{\underline{\frac{1}{2}\operatorname{arctg}\frac{e^x}{2} + C}}.$

94 (1769). $\displaystyle\int \frac{2^x\,dx}{\sqrt{1-(2^x)^2}} = \frac{1}{\ln 2}\int \frac{d(2^x)}{\sqrt{1-(2^x)^2}} = \underline{\underline{\frac{\arcsin 2^x}{\ln 2} + C}}.$

95 (1770). $\displaystyle\int \frac{\cos\alpha\, d\alpha}{a^2+\sin^2\alpha} = \int \frac{d\,(\sin\alpha)}{a^2+\sin^2\alpha} = \underline{\underline{\frac{1}{a}\operatorname{arctg}\frac{\sin\alpha}{a}+C}}.$

96 (1771). $\displaystyle\int \frac{e^{2x}-1}{e^x}\,dx = \int e^x\,dx - \int e^{-x}\,dx = \underline{\underline{e^x+e^{-x}+C}}.$

97 (1772). $\displaystyle\int (e^x+1)^3\,dx = \int (e^{3x}+3e^{2x}+3e^x+1)\,dx =$

$$= \underline{\underline{\frac{1}{3}e^{3x}+\frac{3}{2}e^{2x}+3e^x+x+C}}.$$

98 (1773). $\displaystyle\int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} =$

$$= \arcsin x - \frac{1}{2}\cdot 2\sqrt{1-x^2}+C = \underline{\underline{\arcsin x - \sqrt{1-x^2}+C}}.$$

99 (1774). $\displaystyle\int \frac{3x-1}{x^2+9}\,dx = \frac{3}{2}\int \frac{d\,(x^2+9)}{x^2+9} - \int \frac{dx}{x^2+3^2} =$

$$= \underline{\underline{\frac{3}{2}\ln(x^2+9) - \frac{1}{3}\arcsin\frac{x}{3}+C}}.$$

100 (1775). $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{\left(\sqrt{1-x}\right)^2}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int \frac{d\,(1-x^2)}{\sqrt{1-x^2}} =$

$$= \underline{\underline{\arcsin x + \sqrt{1-x^2}+C}}.$$

101 (1776). $\displaystyle\int \frac{x(1-x^2)}{1+x^4}\,dx = \frac{1}{2}\int \frac{d(x^2)}{1+(x^2)^2} - \frac{1}{4}\int \frac{d(1+x^4)}{1+x^4} =$

$$= \underline{\underline{\frac{1}{2}\operatorname{arctg} x^2 - \frac{1}{4}\ln(1+x^4)+C}}.$$

102 (1777). $$\int \frac{1+x-x^2}{\sqrt{(1-x^2)^3}}\,dx=\int \frac{(1-x^2)\,dx}{\sqrt{(1-x^2)^3}}+\int \frac{x\,dx}{\sqrt{(1-x^2)^3}}=$$

$$=\int \frac{dx}{\sqrt{1-x^2}}-\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{(1-x^2)^3}}=\arcsin x-\frac{1}{2}\frac{(1-x^2)^{-1/2}}{-\frac{1}{2}}+C=$$

$$=\underline{\underline{\arcsin x+\frac{1}{\sqrt{1-x^2}}+C}}.$$

103 (1778). $$\int \frac{dx}{\left(x+\sqrt{x^2-1}\right)^2}=\int \frac{\left(x+\sqrt{x^2-1}\right)\left(x-\sqrt{x^2-1}\right)}{\left(x+\sqrt{x^2-1}\right)^2}dx=$$

$$=\int \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\,dx=\int \frac{\left(x-\sqrt{x^2-1}\right)^2}{\left(x+\sqrt{x^2-1}\right)\left(x-\sqrt{x^2-1}\right)}\,dx=$$

$$=\int\left(x^2-2x\sqrt{x^2-1}+x^2-1\right)dx=\frac{2}{3}x^3-\frac{2}{3}\sqrt{(x^2-1)^3}-$$

$$-x+C=\underline{\underline{\frac{2}{3}\left(x^3-\sqrt{\left(x^2-1\right)^3}\right)-x+C}}.$$

104 (1779). $$\int \frac{2x-\sqrt{\arcsin x}}{\sqrt{1-x^2}}\,dx=-\int \frac{d\,(1-x^2)}{\sqrt{1-x^2}}-$$

$$-\int(\arcsin x)^{1/2}\,d\,(\arcsin x)=\underline{\underline{C-2\sqrt{1-x^2}-\frac{2}{3}\sqrt{(\arcsin x)^3}}}.$$

105 (1780). $$\int \frac{x+(\arccos 3x)^2}{\sqrt{1-9x^2}}\,dx=-\frac{1}{18}\int \frac{d\,(1-9x^2)}{\sqrt{1-9x^2}}-$$

$$-\frac{1}{3}\int(\arccos 3x)^2\,d\,(\arccos 3x)=\underline{\underline{C-\frac{1}{9}\left(\sqrt{1-9x^2}+(\arccos 3x)^3\right)}}.$$

В задачах **106** (1781)–**115** (1790) найти интегралы, выделив целую часть подынтегральной функции.

106 (1781). $\int \frac{x}{x+4}\,dx = \int \frac{(x+4)-4}{x+4}\,dx = \int dx - 4\int \frac{dx}{x+4} =$

$= \underline{\underline{x - 4\ln|x+4| + C}}.$

107 (1782). $\int \frac{x\,dx}{2x+1} = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{2x+1}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \frac{dx}{2x+1} = \frac{1}{2}x -$

$- \frac{1}{4}\ln|2x+1| + C = \underline{\underline{\frac{1}{2}\left(1 - \frac{1}{2}\ln|2x+1|\right) + C}}.$

108 (1783). $\int \frac{Ax\,dx}{a+bx} = A\int \frac{(a+bx)\frac{1}{b} - \frac{a}{b}}{a+bx}\,dx = \frac{A}{b}\int dx - \frac{Aa}{b}\int \frac{dx}{a+bx} =$

$= \frac{A}{b}x - \frac{Aa}{b^2}\ln|a+bx| + C = \underline{\underline{\frac{A}{b}\left(x - \frac{a}{b}\ln|a+bx|\right) + C}}.$

109 (1784). $\int \frac{3+x}{3-x}\,dx = \int \frac{-(3-x)+6}{3-x}\,dx = -\int dx + 6\int \frac{dx}{3-x} =$

$= \underline{\underline{-x - 6\ln|3-x| + C}}.$

110 (1785). $\int \frac{(2x-1)\,dx}{x-2} = \int \frac{2(x-2)+3}{x-2}\,dx = 2\int dx + 3\int \frac{dx}{x-2} =$

$= \underline{\underline{2x + 3\ln|x-2| + C}}.$

111 (1786). $\int \frac{x+2}{2x-1}\,dx = \int \frac{\frac{1}{2}(2x-1) + \frac{5}{2}}{2x-1}\,dx = \frac{1}{2}\int dx + \frac{5}{2}\int \frac{dx}{2x-1} =$

$= \underline{\underline{\frac{1}{2}x + \frac{5}{4}\ln|2x-1| + C}}.$

112 (1787). $\int \frac{(1+x)^2}{x^2+1}\,dx = \int \frac{(1+x^2)+2x}{x^2+1}\,dx = \int dx + \int \frac{2x\,dx}{x^2+1} =$

$= \underline{\underline{x + \ln(x^2+1) + C}}.$

113 (1788). $\int \frac{x^2-1}{x^2+1}\,dx = \int \frac{(x^2+1)-2}{x^2+1}\,dx = \int dx - 2\int \frac{dx}{x^2+1} =$

$= \underline{\underline{x - 2\,\mathrm{arctg}\ x + C}}.$

114 (1789). $\int \frac{x^4}{1-x}\,dx = \int \frac{(x^4-1)+1}{1-x}\,dx = \int \frac{(x-1)(x+1)(x^2+1)}{1-x}\,dx +$

$+ \int \frac{dx}{1-x} = -\int (x^3+x^2+x+1)\,dx - \ln|1-x| + C =$

$= \underline{\underline{C - \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1-x|}}.$

115 (1790). $\int \frac{x^4\,dx}{x^2+1} = \int \frac{(x^4-1)+1}{x^2+1}\,dx = \int (x^2-1)\,dx + \int \frac{dx}{x^2+1} =$

$= \underline{\underline{\frac{x^3}{3} - x + \mathrm{arctg}\,x + C}}.$

В задачах **116** (1791)–**132** (1807) найти интегралы, использовав прием разложения подынтегрального выражения и прием выделения полного квадрата.

116 (1791). $\int \frac{dx}{x(x-1)} = \int\left(\frac{1}{x-1} - \frac{1}{x}\right)dx = \underline{\underline{\ln\left|\frac{x-1}{x}\right| + C}}.$

117 (1792). $\int \frac{dx}{x(x+1)} = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \underline{\underline{\ln\left|\frac{x}{x+1}\right| + C}}.$

118 (1793). $\int \frac{dx}{(x+1)\,(2x-3)} = I$. Разложим подынтегральное выражение:

$$\frac{1}{(x+1)(2x-3)} = \frac{A}{x+1} + \frac{B}{2x-3}, \quad 1 = A(2x-3) + B(x+1);$$

$$x = -1: \quad 1 = -5A, \quad A = -\frac{1}{5};$$

$$x = \frac{3}{2}: \quad 1 = B \cdot \frac{5}{2}, \quad B = \frac{2}{5}.$$

Тогда $I = -\frac{1}{5}\int \frac{dx}{x+1} + \frac{2}{5}\int \frac{dx}{2x-3} = -\frac{1}{5}\ln|x+1| + \frac{2}{10}\ln|2x-3| + C =$

$$= \frac{1}{5}\ln\left|\frac{2x-3}{x+1}\right| + C.$$

119 (1794). $\int \frac{dx}{(a-x)(b-x)} = \frac{1}{a-b}\int\left(\frac{1}{b-x} - \frac{1}{a-x}\right)dx =$

$$= \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| + C.$$

120 (1795). $\int \frac{x^2+1}{x^2-1}\,dx = \int \frac{(x^2-1)+2}{x^2-1}\,dx = \int dx + 2\int \frac{dx}{x^2-1} = x +$

$$+ \int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = x + \ln\left|\frac{x-1}{x+1}\right| + C.$$

121 (1796). $\int \frac{dx}{x^2-7x+10} = \left[\begin{array}{l} x^2 - 7x + 10 = 0, \\ x_{1,2} = \dfrac{7 \pm \sqrt{49-40}}{2}, \\ x_1 = \dfrac{7+3}{2} = 5, \quad x_2 = \dfrac{7-3}{2} = 2 \end{array}\right] =$

$$= \int \frac{dx}{(x-5)(x-2)} = \frac{1}{3}\int\left(\frac{1}{x-5} - \frac{1}{x-2}\right)dx = \frac{1}{3}\ln\left|\frac{x-5}{x-2}\right| + C.$$

122 (1797). $$\int \frac{dx}{x^2+3x-10} = \left[\begin{matrix} x^2+3x-10=0, \\ x_1=-5, \quad x_2=2 \end{matrix}\right] = \int \frac{dx}{(x+5)(x-2)} =$$

$$= \frac{1}{7}\int\left(\frac{1}{x-2}-\frac{1}{x+5}\right)dx = \underline{\underline{\frac{1}{7}\ln\left|\frac{x-2}{x+5}\right|+C}}.$$

123 (1798). $$\int \frac{dx}{4x^2-9} = \frac{1}{2}\int \frac{d(2x)}{(2x)^2-3^2} = \underline{\underline{\frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right|+C}}.$$

124 (1799). $$\int \frac{dx}{2-3x^2} = \frac{1}{\sqrt{3}}\int \frac{d\left(\sqrt{3}x\right)}{\left(\sqrt{2}\right)^2-\left(\sqrt{3}x\right)^2} = \underline{\underline{\frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}+\sqrt{3}x}{\sqrt{2}-\sqrt{3}x}\right|+C}}.$$

125 (1800). $$\int \frac{dx}{(x-1)^2+4} = \int \frac{d(x-1)}{(x-1)^2+2^2} = \underline{\underline{\frac{1}{2}\operatorname{arctg}\frac{x-1}{2}+C}}.$$

126 (1801). $$\int \frac{dx}{x^2+2x+3} = \int \frac{d(x+1)}{(x+1)^2+\left(\sqrt{2}\right)^2} = \underline{\underline{\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+1}{\sqrt{2}}+C}}.$$

127 (1802). $$\int \frac{dx}{x-x^2-2{,}5} = -\int \frac{dx}{x^2-x+\frac{5}{2}} = -\int \frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2} =$$

$$= \underline{\underline{\frac{2}{3}\operatorname{arctg}\frac{1-2x}{3}+C}}.$$

128 (1803). $$\int \frac{dx}{4x^2+4x+5} = \frac{1}{2}\int \frac{d(2x+1)}{(2x+1)^2+4} = \underline{\underline{\frac{1}{4}\operatorname{arctg}\frac{(2x+1)}{2}+C}}.$$

129 (1804). $$\int \frac{dx}{\sqrt{1-(2x+3)^2}} = \frac{1}{2}\int \frac{d(2x+3)}{\sqrt{1-(2x+3)^2}} = \underline{\underline{\frac{1}{2}\arcsin(2x+3)+C}}.$$

130 (1805). $\int \frac{dx}{\sqrt{4x-3-x^2}} = \int \frac{dx}{\sqrt{-(x^2-4x+4)+1}} = \int \frac{d(x-2)}{\sqrt{1-(x-2)^2}} =$

$= \underline{\underline{\arcsin(x-2)+C}}.$

131 (1806). $\int \frac{dx}{\sqrt{8+6x-9x^2}} = \int \frac{dx}{\sqrt{-(3x-1)^2+9}} = \frac{1}{3}\int \frac{d(3x-1)}{\sqrt{3^2-(3x-1)^2}} =$

$= \underline{\underline{\frac{1}{3}\arcsin\frac{3x-1}{3}+C}}.$

132 (1807). $\int \frac{dx}{\sqrt{2-6x-9x^2}} = \int \frac{dx}{\sqrt{-(3x-1)^2+3}} = \frac{1}{3}\int \frac{d(3x-1)}{\sqrt{\left(\sqrt{3}\right)^2-(3x-1)^2}} =$

$= \underline{\underline{\frac{1}{3}\arcsin\frac{3x-1}{\sqrt{3}}+C}}.$

В задачах **133** (1808)–**156** (1831) найти интегралы, используя формулы тригонометрии для преобразования подынтегральных выражений:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x),$$

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),$$

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)),$$

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)).$$

133 (1808). $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \underline{\underline{\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C}}.$

134 (1809). $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \underline{\underline{\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+C}}.$

135 (1810). $$\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\sin^2 \frac{x}{2}} = \int \frac{d\left(\frac{x}{2}\right)}{\sin^2 \frac{x}{2}} = \underline{\underline{-\operatorname{ctg}\frac{x}{2} + C}}.$$

136 (1811). $$\int \frac{dx}{1+\sin x} = \int \frac{dx}{1+\cos\left(x - \frac{\pi}{2}\right)} = \int \frac{dx}{2\cos^2\left(\frac{x}{2} - \frac{\pi}{4}\right)} =$$

$$= \int \frac{d\left(\frac{x}{2} - \frac{\pi}{4}\right)}{\cos^2\left(\frac{x}{2} - \frac{\pi}{4}\right)} = \underline{\underline{\operatorname{tg}\left(\frac{x}{2} - \frac{\pi}{4}\right) + C}}.$$

137 (1812). $$\int \frac{1-\cos x}{1+\cos x}\,dx = \int \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}\,dx = \int \frac{1-\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}\,dx = 2\int \frac{d\left(\frac{x}{2}\right)}{\cos^2 \frac{x}{2}} -$$

$$- \int dx = \underline{\underline{2\operatorname{tg}\frac{x}{2} - x + C}}.$$

138 (1813). $$\int \frac{1+\sin x}{1-\sin x}\,dx = \int \frac{1-\cos\left(x + \frac{\pi}{2}\right)}{1+\cos\left(x + \frac{\pi}{2}\right)}\,dx = \int \frac{\sin^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}{\cos^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}\,dx =$$

$$= \int \frac{1-\cos^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}{\cos^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}\,dx = \int \frac{dx}{\cos^2\left(\frac{x}{2} + \frac{\pi}{4}\right)} - \int dx =$$

$$= \underline{\underline{\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right) - x + C}}.$$

139 (1814). $$\int(\mathrm{tg}^2 x+\mathrm{tg}^4 x)\,dx=\int \mathrm{tg}^2 x(1+\mathrm{tg}^2 x)\,dx=\int \mathrm{tg}^2 x\cdot\frac{dx}{\cos^2 x}=$$

$$=\int \mathrm{tg}^2 x\,d(\mathrm{tg}\,x)=\underline{\underline{\frac{\mathrm{tg}^3 x}{3}}}+C.$$

140 (1815). $$\int\frac{\cos 2x\,dx}{1+\sin x\cdot\cos x}=2\int\frac{\cos 2x\,dx}{2+\sin 2x}=\int\frac{d(2+\sin 2x)}{2+\sin 2x}=$$

$$=\underline{\underline{\ln(2+\sin 2x)+C}}.$$

141 (1816). $$\int\cos x\cdot\sin 3x\,dx=\frac{1}{2}\int(\sin 4x+\sin 2x)\,dx=\frac{1}{2}\left(-\frac{1}{4}\cos 4x-\right.$$

$$\left.-\frac{1}{2}\cos 2x\right)+C=\underline{\underline{C-\frac{1}{4}\left(\frac{\cos 4x}{2}+\cos 2x\right)}}.$$

142 (1817). $$\int\cos 2x\cdot\cos 3x\,dx=\frac{1}{2}\int(\cos 5x+\cos x)\,dx=$$

$$=\underline{\underline{\frac{1}{2}\left(\frac{\sin 5x}{5}+\sin x\right)+C}}.$$

143 (1818). $$\int\sin 2x\cdot\sin 5x\,dx=\frac{1}{2}\int(\cos 3x-\cos 7x)\,dx=$$

$$=\underline{\underline{\frac{1}{2}\left(\frac{\sin 3x}{3}-\frac{\sin 7x}{7}\right)+C}}.$$

144 (1819). $$\int\cos x\cdot\cos 2x\cdot\cos 3x\,dx=\int(\cos x\cdot\cos 3x)\cos 2x\,dx=$$

$$=\frac{1}{2}\int(\cos 4x+\cos 2x)\cos 2x\,dx=\frac{1}{2}\int\cos 4x\cdot\cos 2x\,dx+$$

$$+\frac{1}{2}\int\cos^2 2x\,dx=\frac{1}{4}\int(\cos 6x+\cos 2x)\,dx+$$

$$+\frac{1}{4}\int(1+\cos 4x)\,dx=\frac{1}{4}\left(\frac{\sin 6x}{6}+\frac{\sin 2x}{2}\right)+$$

$$+\frac{1}{4}\left(x+\frac{\sin 4x}{4}\right)+C=\underline{\underline{\frac{1}{8}\left(\frac{\sin 6x}{6}+\sin 2x+2x+\frac{\sin 4x}{2}\right)+C}}.$$

145 (1820). $\displaystyle\int\frac{dx}{\cos x}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{\sin\left(x+\frac{\pi}{2}\right)}=\underline{\underline{\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+C}}.$

146 (1821). $\displaystyle\int\frac{(1-\sin x)\,dx}{\cos x}=\left[\frac{1-\sin x}{\cos x}=\frac{\cos x}{1+\sin x}\right]=\int\frac{\cos x\,dx}{1+\sin x}=$

$$=\underline{\underline{\ln|1+\sin x|+C}}.$$

147 (1822). $\displaystyle\int\frac{\sin^3 x}{\cos x}dx=\int\frac{\sin x(1-\cos^2 x)}{\cos x}\,dx=-\int\frac{d(\cos x)}{\cos x}+$

$$+\int\cos x\,d(\cos x)=\underline{\underline{-\ln|\cos x|+\frac{\cos^2 x}{2}+C}}.$$

148 (1823). $\displaystyle\int\frac{\cos^3 x\,dx}{\sin^4 x}=\int\frac{\cos x(1-\sin^2 x)}{\sin^4 x}\,dx=\int\frac{d(\sin x)}{\sin^4 x}-\int\frac{d(\sin x)}{\sin^2 x}=$

$$=\underline{\underline{-\frac{1}{3\sin^3 x}+\frac{1}{\sin x}}}+C.$$

149 (1824). $\displaystyle\int\frac{\sin^3\alpha}{\sqrt{\cos\alpha}}\,d\alpha=\int\frac{\sin\alpha(1-\cos^2\alpha)}{\sqrt{\cos\alpha}}\,d\alpha=-\int\frac{d(\cos\alpha)}{\sqrt{\cos\alpha}}+$

$$+\int\cos^{3/2}\alpha\,d(\cos\alpha)=-2\sqrt{\cos\alpha}+\frac{2}{5}\cos^{5/2}\alpha+C=$$

$$=\underline{\underline{2\sqrt{\cos\alpha}\left(\frac{\cos^2\alpha}{5}-1\right)+C}}.$$

150 (1825). $\displaystyle\int \frac{dx}{\cos^4 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^4 x}\, dx = \int \operatorname{tg}^2 x\, d(\operatorname{tg} x) + \int \frac{dx}{\cos^2 x} =$

$$= \underline{\underline{\frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x + C}}.$$

151 (1826). $\displaystyle\int \cos^3 x\, dx = \int \cos x (1 - \sin^2 x)\, dx = \int \cos x\, dx -$

$$- \int \sin^2 x\, d(\sin x) = \underline{\underline{\sin x - \frac{\sin^3 x}{3} + C}}.$$

152 (1827). $\displaystyle\int \operatorname{tg}^4 x\, dx = \int \operatorname{tg}^2 x \cdot \operatorname{tg}^2 x\, dx = \int \operatorname{tg}^2 x \cdot \frac{1 - \cos^2 x}{\cos^2 x}\, dx =$

$$= \int \operatorname{tg}^2 x\, d(\operatorname{tg} x) - \int \operatorname{tg}^2 x\, dx = \frac{\operatorname{tg}^3 x}{3} - \int \frac{1 - \cos^2 x}{\cos^2 x}\, dx =$$

$$= \underline{\underline{\frac{\operatorname{tg}^3 x}{3} - \operatorname{tg} x + x + C}}.$$

153 (1828). $\displaystyle\int \sin^5 d\, x = \int \sin x (1 - \cos^2 x)^2\, dx =$

$$= -\int (1 - 2\cos^2 x + \cos^4 x)\, d(\cos x) =$$

$$= \underline{\underline{- \cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C}}.$$

154 (1829). $\displaystyle\int \sin^4 x\, dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx = \frac{1}{4} \int (1 - 2\cos 2x +$

$$+ \cos^2 2x)\, dx = \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx =$$

$$= \underline{\underline{\frac{1}{4} \left(\frac{3}{2} x - \sin 2x + \frac{1}{8} \sin 4x \right) + C}}.$$

155 (1830). $\int \mathrm{tg}^3 x\,dx = \int \mathrm{tg}\,x \frac{\sin^2 x}{\cos^2 x}\,dx = \int \mathrm{tg}\,x\left(\frac{1}{\cos^2 x} - 1\right)dx =$

$$= \int \mathrm{tg}\,x\,d\,(\mathrm{tg}\,x) + \int \frac{d\,(\cos x)}{\cos x} = \underline{\underline{\frac{\mathrm{tg}^2 x}{2} + \ln|\cos x| + C}}.$$

156 (1831). $\int \frac{dx}{\sin^6 x} = \int \frac{(\sin^2 x + \cos^2 x)^2}{\sin^6 x}\,dx = \int \frac{\sin^4 x}{\sin^6 x}\,dx +$

$$+ 2\int \mathrm{ctg}^2 x \cdot \frac{dx}{\sin^2 x} + \int \mathrm{ctg}^4 x \cdot \frac{dx}{\sin^2 x} = -\int d\,(\mathrm{ctg}\,x) -$$

$$- 2\int \mathrm{ctg}^2 x\,d\,(\mathrm{ctg}\,x) - \int \mathrm{ctg}^4 x\,d\,(\mathrm{ctg}\,x) =$$

$$= \underline{\underline{- \mathrm{ctg}\,x - \frac{2}{3}\,\mathrm{ctg}^3 x - \frac{1}{5}\mathrm{ctg}^5 x + C}}.$$

§ 2. ОСНОВНЫЕ МЕТОДЫ ИНТЕГРИРОВАНИЯ

2.1. Интегрирование по частям

Интегрирование по частям осуществляется согласно формуле

$$\int u\,dv = uv - \int v\,du,$$

где $u = u(x)$ и $v = v(x)$ – две дифференцируемые функции от x.

Этим методом решить задачи **157** (1832)–**193** (1868).

157 (1832). $\int x \cdot \sin 2x\,dx = \left[\begin{matrix} u = x, & du = dx, \\ \sin 2x\,dx = dv, & v = -\frac{1}{2}\cos 2x \end{matrix}\right] = -\frac{x}{2}\cos 2x +$

$$+ \frac{1}{2}\int \cos 2x\,dx = \underline{\underline{-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C}}.$$

158 (1833). $\int x\cos x\,dx = \left[\begin{matrix} u = x, & du = dx, \\ \cos x\,dx = dv, & v = \sin x \end{matrix}\right] = x\sin x - \int \sin x\,dx =$

$$= \underline{\underline{x\sin x + \cos x + C}}.$$

159 (1834). $\int xe^{-x}\,dx = \begin{bmatrix} u = x, & du = dx, \\ e^{-x}\,dx = dv, & v = -e^{-x} \end{bmatrix} = -xe^{-x} + \int e^{-x}\,dx =$

$= -xe^{-x} - e^{-x} + C = \underline{\underline{C - e^{-x}(x+1)}}.$

160 (1835). $\int x \cdot 3^x\,dx = \begin{bmatrix} u = x, & du = dx, \\ 3^x\,dx = dv, & v = \dfrac{3^x}{\ln 3} \end{bmatrix} = \dfrac{x \cdot 3^x}{\ln 3} - \dfrac{1}{\ln 3}\int 3^x\,dx =$

$= \underline{\underline{\dfrac{3^x}{\ln^2 3}(x \cdot \ln 3 - 1) + C}}.$

161 (1836). $\int x^n \ln x\,dx = \begin{bmatrix} u = \ln x, & du = \dfrac{dx}{x}, \\ x^n\,dx = dv, & v = \dfrac{x^{n+1}}{n+1} \end{bmatrix} = \dfrac{x^{n+1}\ln x}{n+1} -$

$- \dfrac{1}{n+1}\int x^n\,dx = \underline{\underline{\dfrac{x^{n+1}}{n+1}\left(\ln x - \dfrac{1}{n+1}\right) + C}}.$

162 (1837). $\int x \cdot \operatorname{arctg} x\,dx = \begin{bmatrix} u = \operatorname{arctg} x, & du = \dfrac{dx}{1+x^2}, \\ x\,dx = dv, & v = \dfrac{x^2}{2} \end{bmatrix} = \dfrac{x^2}{2}\operatorname{arctg} x -$

$- \dfrac{1}{2}\int \dfrac{x^2\,dx}{1+x^2} = \dfrac{x^2}{2}\operatorname{arctg} x - \dfrac{1}{2}\int \dfrac{(x^2+1)-1}{1+x^2}\,dx = \dfrac{x^2}{2}\operatorname{arctg} x -$

$- \dfrac{1}{2}x + \dfrac{1}{2}\operatorname{arctg} x + C = \underline{\underline{\dfrac{x^2+1}{2}\operatorname{arctg} x - \dfrac{x}{2} + C}}.$

163 (1838). $\int \arccos x\,dx = \left[\begin{array}{ll} u = \arccos x, & du = -\dfrac{1}{\sqrt{1-x^2}}, \\ dv = dx, & v = x \end{array}\right] = x\arccos x +$

$+\int \dfrac{x\,dx}{\sqrt{1-x^2}} = x\arccos x - \dfrac{1}{2}\int \dfrac{d(1-x^2)}{\sqrt{1-x^2}} = \underline{\underline{x\arccos x - \sqrt{1-x^2} + C}}.$

164 (1839). $\int \operatorname{arctg}\sqrt{x}\,dx = \left[\begin{array}{ll} u = \operatorname{arctg}\sqrt{x}, & du = \dfrac{dx}{2\sqrt{x}\,(1+x)}, \\ dv = dx, & v = x \end{array}\right] =$

$= x\operatorname{arctg}\sqrt{x} - \dfrac{1}{2}\int \dfrac{\sqrt{x}\,dx}{1+x} = x\operatorname{arctg}\sqrt{x} - \dfrac{1}{2}\int \dfrac{x\,dx}{\sqrt{x}\,(1+x)} = x\operatorname{arctg}\sqrt{x} -$

$-\dfrac{1}{2}\int \dfrac{(x+1)-1}{\sqrt{x}\,(1+x)}\,dx = x\operatorname{arctg}\sqrt{x} - \dfrac{1}{2}\int \dfrac{dx}{\sqrt{x}} + \int \dfrac{d\left(\sqrt{x}\right)}{1+\left(\sqrt{x}\right)^2} =$

$= \underline{\underline{x\operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x} + C}}.$

165 (1840). $\int \dfrac{\arcsin x}{\sqrt{x+1}}\,dx = \left[\begin{array}{ll} u = \arcsin x, & du = \dfrac{dx}{\sqrt{1-x^2}}, \\ dv = \dfrac{dx}{\sqrt{1+x}}, & v = 2\sqrt{1+x} \end{array}\right] =$

$= 2\sqrt{1+x}\arcsin x - 2\int \dfrac{\sqrt{1+x}}{\sqrt{1-x^2}}\,dx = 2\sqrt{1+x}\cdot\arcsin x + 2\int \dfrac{d(1-x)}{\sqrt{1-x}} =$

$= \underline{\underline{2\sqrt{1+x}\arcsin x + 4\sqrt{1-x} + C}}.$

166 (1841). $\int x\operatorname{tg}^2 x\,dx = \left[\begin{array}{l} u = x, \quad du = dx, \quad \operatorname{tg}^2 x\,dx = dv, \\ v = \int \operatorname{tg}^2 x\,dx = \int \dfrac{1-\cos^2 x}{\cos^2 x}\,dx = \operatorname{tg} x - x \end{array}\right] =$

$= x\operatorname{tg} x - x^2 - \int (\operatorname{tg} x - x)\,dx = \underline{\underline{x\operatorname{tg} x - \dfrac{x^2}{2} - \ln|\cos x| + C}}.$

167 (1842). $\int x\cos^2 x\,dx =$

$$=\left[\begin{array}{l} u=x,\quad du=dx,\quad \cos^2 x\,dx=dv,\\ v=\int\cos^2 x\,dx=\frac{1}{2}\int(1+\cos 2x)\,dx=\frac{1}{2}x+\frac{1}{4}\sin 2x\end{array}\right]=\frac{1}{2}x^2+$$

$$+\frac{1}{4}x\sin 2x-\int\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)dx=\underline{\underline{\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C}}.$$

168 (1843). $\int\frac{\lg x}{x^3}\,dx=\left[\begin{array}{ll}\lg x=u, & du=\dfrac{dx}{x\ln 10},\\ \dfrac{dx}{x^3}=dv, & v=-\dfrac{1}{2x^2}\end{array}\right]=-\frac{\lg x}{2x^2}+\frac{1}{2}\int\frac{dx}{x^3\ln 10}=$

$$=-\frac{\lg x}{2x^2}-\frac{1}{4x^2\ln 10}=-\frac{1}{2x^2\ln 10}\left(\lg x+\lg\sqrt{e}\right)=\underline{\underline{-\frac{\lg x\sqrt{e}}{2x^2}+C}}.$$

169 (1844). $\int\frac{x\operatorname{arctg} x}{\sqrt{1+x^2}}\,dx=\left[\begin{array}{ll}u=\operatorname{arctg} x, & du=\dfrac{dx}{1+x^2},\\ \dfrac{x\,dx}{\sqrt{1+x^2}}=dv, & v=\sqrt{1+x^2}\end{array}\right]=$

$$=\sqrt{1+x^2}\operatorname{arctg} x-\int\frac{dx}{\sqrt{1+x^2}}=\underline{\underline{\sqrt{1+x^2}\operatorname{arctg} x-\ln\left|x+\sqrt{1+x^2}\right|+C}}.$$

170 (1845). $\int\frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx=\left[\begin{array}{ll}u=\arcsin\sqrt{x}, & du=\dfrac{dx}{2\sqrt{x}\sqrt{1-x}},\\ \dfrac{dx}{\sqrt{1-x}}=dv, & v=\int\dfrac{dx}{\sqrt{1-x}}=-2\sqrt{1-x}\end{array}\right]=$

$$=-2\sqrt{1-x}\arcsin\sqrt{x}+\int\frac{dx}{\sqrt{x}}=\underline{\underline{2\left(\sqrt{x}-\sqrt{1-x}\arcsin\sqrt{x}\right)+C}}.$$

171 (1846). $\int \ln(x^2+1)\,dx = \left[\begin{array}{c} u=\ln(x^2+1), \quad du=\dfrac{2x\,dx}{x^2+1}, \\ dv=dx, \quad v=x \end{array}\right] = x\ln(x^2+1) -$

$-2\int \dfrac{x^2dx}{x^2+1} = \underline{\underline{x\ln(x^2+1)-2x+2\,\mathrm{arctg}\,x+C}}.$

172 (1847). $\int \dfrac{x^2dx}{(1+x^2)^2} = \left[\begin{array}{c} x=u, \quad du=dx, \quad \dfrac{x\,dx}{(1+x^2)^2}=dv, \\ v=\dfrac{1}{2}\int \dfrac{d(1+x^2)}{(1+x^2)^2} = -\dfrac{1}{2(1+x^2)} \end{array}\right] =$

$= -\dfrac{x}{2(1+x^2)} + \dfrac{1}{2}\int \dfrac{dx}{1+x^2} = \underline{\underline{C - \dfrac{x}{2(1+x^2)} + \dfrac{1}{2}\,\mathrm{arctg}\,x}}.$

173 (1848). $\int \dfrac{x^3dx}{\sqrt{1+x^2}} = \left[\begin{array}{c} u=x^2, \quad du=2x\,dx, \quad dv=\dfrac{x\,dx}{\sqrt{1+x^2}}, \\ v=\dfrac{1}{2}\int \dfrac{d(1+x^2)}{\sqrt{1+x^2}} = \sqrt{1+x^2} \end{array}\right] =$

$= x^2\sqrt{1+x^2} - 2\int x\sqrt{1+x^2}\,dx = x^2\sqrt{1+x^2} - \int (1+x^2)^{1/2}\,d(1+x^2) =$

$= \underline{\underline{x^2\sqrt{1+x^2} - \dfrac{2}{3}\sqrt{(1+x^2)^3} + C}}.$

174 (1849). $\int x^2\ln(1+x)\,dx = \left[\begin{array}{c} u=\ln(1+x), \quad du=\dfrac{dx}{1+x}, \\ x^2dx=dv, \quad v=\dfrac{x^3}{3} \end{array}\right] = \dfrac{x^3}{3}\ln(1+x) -$

$-\dfrac{1}{3}\int \dfrac{x^3dx}{1+x} = \dfrac{x^3}{3}\ln(1+x) - \dfrac{1}{3}\int \dfrac{(x^3+1)-1}{1+x}\,dx = \dfrac{x^3}{3}\ln(1+x) -$

$-\dfrac{1}{3}\int (x^2-x+1)\,dx + \dfrac{1}{3}\int \dfrac{dx}{1+x} = \underline{\underline{\dfrac{x^3+1}{3}\ln(1+x) - \dfrac{1}{9}x^3 + \dfrac{1}{6}x^2 - \dfrac{1}{3}x + C}}.$

175 (1850). $\int x^2 e^{-x}dx = \begin{bmatrix} u = x^2, & du = 2x\,dx, \\ e^{-x}dx = dv, & v = -e^{-x} \end{bmatrix} = -x^2e^{-x} +$

$+2\int xe^{-x}dx = \begin{bmatrix} u = x, & du = dx, \\ dv = e^{-x}dx, & v = -e^{-x} \end{bmatrix} = -x^2e^{-x} + \left(-xe^{-x} + \int e^{-x}dx \right) =$

$= \underline{\underline{C - e^{-x}(x^2 + 2x + 2)}}.$

176 (1851). $\int x^3 e^x dx = \begin{bmatrix} u = x^3, & du = 3x^2dx, \\ e^x dx = dv, & v = e^x \end{bmatrix} = x^3e^x - 3\int x^2e^x dx =$

$= \begin{bmatrix} x^2 = u, & du = 2x\,dx, \\ e^x dx = dv, & v = e^x \end{bmatrix} = x^3e^x - 3\left(x^2e^x - 2\int xe^x dx \right) = x^3e^x -$

$-3x^2e^x + 6\int xe^x dx = \begin{bmatrix} x = u, & dx = du, \\ e^x dx = dv, & v = e^x \end{bmatrix} = x^3e^x - 3x^2e^x + 6xe^x -$

$= 6e^x + C = \underline{\underline{e^x(x^3 - 3x^2 + 6x - 6) + C}}.$

177 (1852). $\int x^2 a^x dx = \begin{bmatrix} x^2 = u, & du = 2x\,dx, \\ a^x dx = dv, & v = \dfrac{a^x}{\ln a} \end{bmatrix} = \dfrac{x^2a^x}{\ln a} - \dfrac{2}{\ln a}\int xa^x dx =$

$= \begin{bmatrix} u = x, & du = dx, \\ dv = a^x dx, & v = \dfrac{a^x}{\ln a} \end{bmatrix} = \dfrac{x^2a^x}{\ln a} - \dfrac{2}{\ln a}\left(\dfrac{xa^x}{\ln a} - \dfrac{1}{\ln a}\int a^x dx \right) =$

$= \underline{\underline{a^x\left(\dfrac{x^2}{\ln a} - \dfrac{2x}{\ln^2 a} + \dfrac{2}{\ln^3 a} \right) + C}}.$

178 (1853). $\int x^3 \sin x\,dx = \begin{bmatrix} u = x^3, & du = 3x^2dx, \\ dv = \sin x\,dx, & v = -\cos x \end{bmatrix} = -x^3\cos x +$

$+3\int x^2\cos x\,dx = \begin{bmatrix} u = x^2, & du = 2x\,dx, \\ dv = \cos x\,dx, & v = \sin x \end{bmatrix} = -x^3\cos x + 3\Big(x^2\sin x -$

$$-2\int x\sin x\,dx\Bigg)=\begin{bmatrix} u=x, & du=dx, \\ dv=\sin x\,dx, & v=-\cos x \end{bmatrix}=-x^3\cos x+3x^2\sin x-$$

$$-6\left(-x\cos x+\int\cos x\,dx\right)=\underline{\underline{-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+C}}.$$

179 (1854). $\displaystyle\int x^2\cos^2 x\,dx=\frac{1}{2}\int x^2dx+\frac{1}{2}\int x^2\cos 2x\,dx=$

$$=\begin{bmatrix} u=x^2, & du=2x\,dx, \\ dv=\cos 2x\,dx, & v=\dfrac{1}{2}\sin 2x \end{bmatrix}=\frac{1}{6}x^3+\frac{1}{2}\left(\frac{x^2}{2}\sin 2x-\int x\sin 2x\,dx\right)=$$

$$=\begin{bmatrix} u=x, & du=dx, \\ dv=\sin 2x\,dx, & v=-\dfrac{1}{2}\cos 2x \end{bmatrix}=\frac{1}{6}x^3+\frac{1}{2}\left(\frac{x^2}{2}\sin 2x-\left(-\frac{x}{2}\cos 2x+\right.\right.$$

$$\left.\left.+\frac{1}{2}\int\cos 2x\,dx\right)\right)=\underline{\underline{\frac{1}{6}x^3+\frac{x^2}{4}\sin 2x+\frac{x}{4}\cos 2x-\frac{1}{8}\sin 2x+C}}.$$

180 (1855). $\displaystyle\int\ln^2 x\,dx=\begin{bmatrix} u=\ln x, & du=\dfrac{dx}{x}, \\ dv=\ln x dx, & v=\displaystyle\int\ln x\,dx \end{bmatrix}=$

$$=\begin{bmatrix} u_1=\ln x, & du_1=\dfrac{dx}{x}, \\ dv_1=dx, & v_1=x \end{bmatrix}=\begin{bmatrix} u=\ln x, \quad du=\dfrac{dx}{x}, \quad dv=\ln x\,dx, \\ v=x\ln x-\displaystyle\int dx=x\ln x-x \end{bmatrix}=$$

$$=x(\ln x-1)\cdot\ln x-\int(\ln x-1)\,dx=x(\ln x-1)\cdot\ln x-\int\ln x\,dx+x=$$

$$=x(\ln x-1)\cdot\ln x-x(\ln x-1)+x+C=\underline{\underline{x(\ln^2 x-2\ln x+2)+C}}.$$

181 (1856). $$\int \frac{\ln^3 x\,dx}{x^2} = \left[\begin{matrix} u = \ln^3 x, & du = \dfrac{3\ln^2 x}{x}\,dx, \\ dv = \dfrac{dx}{x^2}, & v = -\dfrac{1}{x} \end{matrix}\right] = -\frac{\ln^3 x}{x} +$$

$$+3\int \frac{\ln^2 x}{x^2}\,dx = \left[\begin{matrix} u = \ln^2 x, & du = \dfrac{2\ln x}{x}\,dx, \\ dv = \dfrac{dx}{x^2}, & v = -\dfrac{1}{x} \end{matrix}\right] = -\frac{\ln^3 x}{x} + 3\left(-\frac{\ln^2 x}{x} +\right.$$

$$\left. +2\int \frac{\ln x}{x^2}\,dx\right) = \left[\begin{matrix} u = \ln x, & du = \dfrac{dx}{x}, \\ dv = \dfrac{dx}{x^2}, & v = -\dfrac{1}{x} \end{matrix}\right] = -\frac{\ln^3 x}{x} - 3\,\frac{\ln^2 x}{x} +$$

$$+6\left(-\frac{\ln x}{x} + \int \frac{dx}{x^2}\right) = \underline{\underline{-\frac{1}{x}(\ln^3 x + 3\ln^2 x + 6\ln x + 6) + C}}.$$

182 (1857). $$\int \frac{\ln^2 x}{\sqrt{x^5}}\,dx = \left[\begin{matrix} u = \ln^2 x, & du = \dfrac{2\ln x}{x}\,dx, \\ dv = \dfrac{dx}{x^{5/2}}, & v = \displaystyle\int x^{-5/2}\,dx = -\dfrac{2}{3}\,x^{-2/3} \end{matrix}\right] =$$

$$= -\frac{2}{3\sqrt{x^3}}\ln^2 x + \frac{4}{3}\int \frac{\ln x\,dx}{\sqrt{x^5}} = \left[\begin{matrix} u = \ln x, & du = \dfrac{dx}{x}, \\ dv = \dfrac{dx}{x^{5/2}}, & v = -\dfrac{2}{3}x^{-2/3} \end{matrix}\right] =$$

$$= -\frac{2}{3\sqrt{x^3}}\ln^2 x - \frac{8\ln x}{9\sqrt{x^3}} - \frac{4}{9\sqrt{x^3}} + C = \underline{\underline{C - \frac{8}{27\sqrt{x^3}}\left(\frac{9}{4}\ln^2 x + 3\ln x + 2\right)}}.$$

183 (1858). $\int (\arcsin x)^2\,dx = \left[\begin{array}{ll} u=(\arcsin x)^2, & du=\dfrac{2\arcsin x}{\sqrt{1-x^2}}\,dx, \\ dv=dx, & v=x \end{array}\right] =$

$$= x(\arcsin x)^2 - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx =$$

$$= \left[\begin{array}{ll} u=\arcsin x, & du=\dfrac{dx}{\sqrt{1-x^2}}, \\ dv=\dfrac{x\,dx}{\sqrt{1-x^2}}, & v=-\dfrac{1}{2}\displaystyle\int\dfrac{d(1-x^2)}{\sqrt{1-x^2}}=-\sqrt{1-x^2} \end{array}\right] = x(\arcsin x)^2 -$$

$$-2\left(-\sqrt{1-x^2}\arcsin x + \int dx\right) = \underline{\underline{x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C}}.$$

184 (1859). $\int (\operatorname{arctg} x)^2 x\,dx = \left[\begin{array}{ll} u=(\operatorname{arctg} x)^2, & du=\dfrac{2\operatorname{arctg} x}{1+x^2}\,dx, \\ dv=x\,dx, & v=\dfrac{x^2}{2} \end{array}\right] =$

$$= \frac{x^2}{2}(\operatorname{arctg} x)^2 - \int \frac{(x^2+1)-1}{1+x^2}\operatorname{arctg} x\,dx = \frac{x^2}{2}(\operatorname{arctg} x)^2 - \int \operatorname{arctg} x\,dx +$$

$$+\int \operatorname{arctg} x\ d(\operatorname{arctg} x) = \left[\begin{array}{ll} u=\operatorname{arctg} x, & du=\dfrac{dx}{1+x^2}, \\ dv=dx, & v=x \end{array}\right] = \frac{x^2}{2}(\operatorname{arctg} x)^2 -$$

$$-\left(x\operatorname{arctg} x - \int \frac{x\,dx}{1+x^2}\right) + \frac{(\operatorname{arctg} x)^2}{2} =$$

$$= \underline{\underline{\frac{x^2+1}{2}(\operatorname{arctg} x)^2 - x\operatorname{arctg} x + \frac{1}{2}\ln(1+x^2) + C}}.$$

185 (1860). $\int e^x \sin x\, dx = \begin{bmatrix} u = e^x, & du = e^x dx, \\ dv = \sin x\, dx, & v = -\cos x \end{bmatrix} = -e^x \cos x +$

$+\int e^x \cos x\, dx = \begin{bmatrix} u = e^x, & du = e^x dx, \\ dv = \cos x\, dx, & v = \sin x \end{bmatrix} = -e^x \cos x + e^x \sin x -$

$-\int e^x \sin x\, dx \quad \Rightarrow \quad \int e^x \sin x\, dx = \underline{\underline{\dfrac{e^x(\sin x - \cos x)}{2} + C}}.$

186 (1861). $\int e^{3x}(\sin 2x - \cos 2x)\, dx = \int e^{3x} \sin 2x\, dx - \int e^{3x} \cos 2x\, dx =$

$= I_1 - I_2;$

$I_1 = \int e^{3x} \sin 2x\, dx = \begin{bmatrix} u = e^{3x}, & du = 3e^{3x} dx, \\ dv = \sin 2x\, dx, & v = -\dfrac{1}{2}\cos 2x \end{bmatrix} = -\dfrac{1}{2} e^{3x} \cos 2x +$

$+\dfrac{3}{2}\int e^{3x} \cos 2x\, dx = \begin{bmatrix} u = e^{3x}, & du = 3e^{3x} dx, \\ dv = \cos 2x\, dx, & v = \dfrac{1}{2}\sin 2x \end{bmatrix} = -\dfrac{1}{2} e^{3x} \cos 2x +$

$+\dfrac{3}{2}\left(\dfrac{1}{2} e^{3x} \sin 2x - \dfrac{3}{2}\int e^{3x} \sin 2x dx\right) \quad \Rightarrow \quad \dfrac{13}{4}\int e^{3x} \sin 2x\, dx =$

$= -\dfrac{1}{2} e^{3x} \cos 2x + \dfrac{3}{4} e^{3x} \sin 2x \quad \Rightarrow \quad \int e^{3x} \sin 2x\, dx = -\dfrac{2}{13} e^{3x} \cos 2x +$

$+\dfrac{3}{13} e^{3x} \sin 2x;$

$I_2 = \int e^{3x} \cos 2x\, dx = \begin{bmatrix} u = e^{3x}, & du = 3e^{3x} dx, \\ dv = \cos 2x\, dx, & v = \dfrac{1}{2}\sin 2x \end{bmatrix} = \dfrac{1}{2} e^{3x} \sin 2x -$

$$-\frac{3}{2}\int e^{3x}\sin 2x\,dx=\begin{bmatrix} u=e^{3x}, & du=3e^{3x}dx, \\ dv=\sin 2x\,dx, & v=-\frac{1}{2}\cos 2x \end{bmatrix}=\frac{1}{2}e^{3x}\sin 2x-$$

$$-\frac{3}{2}\left(-\frac{1}{2}e^{3x}\cos 2x+\frac{3}{2}\int e^{3x}\cos 2x\,dx\right)\ \Rightarrow\ \frac{13}{4}\int e^{3x}\cos 2x\,dx=$$

$$=\frac{1}{2}e^{3x}\sin 2x+\frac{3}{4}e^{3x}\cos 2x\ \Rightarrow\ \int e^{3x}\sin 2x\,dx=\frac{2}{13}e^{3x}\sin 2x+$$

$$+\frac{3}{13}e^{3x}\cos 2x;$$

$$\int e^{3x}(\sin 2x-\cos 2x)\,dx=I_1-I_2=-\frac{2}{13}e^{3x}\cos 2x+\frac{3}{13}e^{3x}\sin 2x-$$

$$-\frac{2}{13}e^{3x}\sin 2x-\frac{3}{13}e^{3x}\cos 2x+C=\underline{\underline{\frac{e^{3x}}{13}(\sin 2x-5\cos 2x)+C}}.$$

187 (1862). $\displaystyle\int e^{ax}\cos nx\,dx=\begin{bmatrix} u=e^{ax}, & du=ae^{ax}dx, \\ dv=\cos nx\,dx, & v=\frac{1}{n}\sin nx \end{bmatrix}=$

$$=\frac{e^{ax}}{n}\sin nx-\frac{a}{n}\int e^{ax}\sin nx\,dx=\begin{bmatrix} u=e^{ax}, & du=ae^{ax}dx, \\ dv=\sin nx\,dx, & v=-\frac{1}{n}\cos nx \end{bmatrix}=$$

$$=\frac{e^{ax}}{n}\sin nx-\frac{a}{n}\left(-\frac{e^{ax}}{n}\cos nx+\frac{a}{n}\int e^{ax}\cos nx\,dx\right)\ \Rightarrow$$

$$\left(1+\frac{a^2}{n^2}\right)\int e^{ax}\cos nx\,dx=\frac{e^{ax}}{n}\left(\sin nx+\frac{a}{n}\cos nx\right)\ \Rightarrow$$

$$\int e^{ax}\cos nx\,dx=\underline{\underline{\frac{e^{ax}}{a^2+n^2}(n\sin nx+a\cos nx)+C}}.$$

188 (1863). $\int \sin\ln x\,dx = \left[\begin{array}{ll} u=\sin\ln x, & du=\cos\ln x\cdot\dfrac{dx}{x}, \\ dv=dx, & v=x \end{array}\right] = x\sin\ln x -$

$$-\int x\frac{\cos\ln x}{x}\,dx = x\sin\ln x - \int \cos\ln x\,dx =$$

$$= \left[\begin{array}{ll} u=\cos\ln x, & du=-\sin\ln x\cdot\dfrac{dx}{x}, \\ dv=dx, & v=x \end{array}\right] = x\sin\ln x - x\cos\ln x -$$

$$-\int \sin\ln x\,dx \quad\Rightarrow\quad \int \sin\ln x\,dx = \underline{\underline{\frac{1}{2}x\,(\sin\ln x-\cos\ln x)+C}}.$$

189 (1864). $\int \cos\ln x\,dx = \left[\begin{array}{ll} u=\cos\ln x, & du=-\sin\ln x\cdot\dfrac{dx}{x}, \\ dv=dx, & v=x \end{array}\right] =$

$$= x\cos\ln x + \int \sin\ln x\,dx = \left[\begin{array}{ll} u=\sin\ln x, & du=\cos\ln x\cdot\dfrac{dx}{x}, \\ dv=dx, & v=x \end{array}\right] =$$

$$= x\cos\ln x + x\sin\ln x - \int \cos\ln x\,dx \quad\Rightarrow\quad \int \cos\ln x\,dx =$$

$$= \underline{\underline{\frac{1}{2}x\,(\cos\ln x+\sin\ln x)+C}}.$$

190 (1865). $\int \dfrac{x^2dx}{\sqrt{1-x^2}} = \left[\begin{array}{ll} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{\sqrt{1-x^2}}, & v=-\sqrt{1-x^2} \end{array}\right] = -x\sqrt{1-x^2}\,+$

$$+\int\sqrt{1-x^2}\,dx = -x\sqrt{1-x^2}+\int\frac{1-x^2}{\sqrt{1-x^2}}\,dx = -x\sqrt{1-x^2}+\arcsin x -$$

$$-\int\frac{x^2dx}{\sqrt{1-x^2}} \quad\Rightarrow\quad \int\frac{x^2dx}{\sqrt{1-x^2}} = \underline{\underline{-\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x + C}}.$$

191 (1866). $\int \sqrt{a^2+x^2}\,dx = \int \frac{a^2+x^2}{\sqrt{a^2+x^2}}\,dx = a^2 \int \frac{dx}{\sqrt{a^2+x^2}} +$

$$+\int \frac{x^2dx}{\sqrt{a^2+x^2}} = a^2 \ln\left|x+\sqrt{a^2+x^2}\right| + \int \frac{x^2dx}{\sqrt{a^2+x^2}} =$$

$$=\begin{bmatrix} u = x, & du = dx, \\ dv = \dfrac{x\,dx}{\sqrt{a^2+x^2}}, & v = \sqrt{a^2+x^2} \end{bmatrix} = a^2 \ln\left|x+\sqrt{a^2+x^2}\right| + x\sqrt{a^2+x^2} -$$

$$-\int \sqrt{a^2+x^2}\,dx \Rightarrow \int \sqrt{a^2+x^2}\,dx =$$

$$=\underline{\underline{\frac{a^2}{2} \ln\left|x+\sqrt{a^2+x^2}\right| + \frac{x}{2}\sqrt{a^2+x^2} + C}}.$$

192 (1867). $\int \frac{x^2e^xdx}{(x+2)^2} = I.$

$$\frac{x^2e^x}{(x+2)^2} = \frac{((x^2-4)+4)\,e^x}{(x+2)^2} = \frac{x-2}{x+2}e^x + \frac{4e^x}{(x+2)^2} = \frac{(x+2)-4}{x+2}e^x +$$

$$+\frac{4e^x}{(x+2)^2} = e^x - \frac{4e^x}{x+2} + \frac{4e^x}{(x+2)^2};$$

$$I = \int e^xdx - 4\int \frac{e^xdx}{x+2} + 4\int \frac{e^xdx}{(x+2)^2} = \begin{bmatrix} u = \dfrac{1}{x+2}, & du = -\dfrac{dx}{(x+2)^2}, \\ dv = e^xdx, & v = e^x \end{bmatrix} = e^x -$$

$$-4\left(\frac{e^x}{x+2} + \int \frac{e^xdx}{(x+2)^2}\right) + 4\int \frac{e^xdx}{(x+2)^2} = e^x - \frac{4e^x}{x+2} + C =$$

$$=\underline{\underline{\frac{x-2}{x+2}e^x + C}}.$$

193 (1868). $\int x^2e^x \sin x\,dx =$

$$=\left[\begin{matrix} u=x^2, & du=2x\,dx, \\ dv=e^x\sin x\,dx, & v=\int e^x\sin x\,dx=\dfrac{e^x(\sin x-\cos x)}{2}\ \text{(см. задачу 185 (1860))}\end{matrix}\right]=$$

$$=\frac{1}{2}e^x\cdot x^2(\sin x-\cos x)-\int xe^x\sin x\,dx+\int xe^x\cos x\,dx=$$

$$=\left[\begin{matrix} u=x, & du=dx, \\ dv=e^x\sin x\,dx, & v=\dfrac{e^x}{2}(\sin x-\cos x)\end{matrix}\right]=$$

$$=\left[\begin{matrix} u=x, & du=dx, \\ dv=e^x\cos x\,dx, & v=\dfrac{e^x}{2}(\sin x+\cos x)\ \text{(см. задачу 187 (1862)}\end{matrix}\right]=$$

$$=\frac{x^2}{2}e^x(\sin x-\cos x)-\frac{x}{2}e^x(\sin x-\cos x)+\frac{1}{2}\int e^x\sin x\,dx-$$

$$-\frac{1}{2}\int e^x\cos x\,dx+\frac{x}{2}e^x(\sin x+\cos x)-\frac{1}{2}\int e^x\sin x\,dx-\frac{1}{2}\int e^x\cos x\,dx=$$

$$=\frac{x^2}{2}e^x(\sin x-\cos x)-\frac{x}{2}e^x(\sin x-\cos x)-\frac{1}{2}e^x(\sin x+\cos x)+$$

$$+\frac{x}{2}e^x(\sin x+\cos x)=\underline{\underline{\frac{1}{2}e^x((x^2-1)\sin x-(x-1)^2\cos x).}}$$

2.2. Замена переменной

Если в неопределенном интеграле $\int f(x)\,dx$ первообразная для $f(x)$ существует, но непосредственно найти ее мы не можем, то сделаем замену переменной, положив

$$x=\varphi(t),$$

где $\varphi(t)$ – непрерывная функция с непрерывной производной, имеющая обратную функцию; тогда

$$\int f(x)\,dx=\int f(\varphi(t))\,\varphi'(t)\,dt.$$

На практике часто бывает целесообразнее подобрать замену переменной в виде не $x = \varphi(t)$, а $t = \psi(x)$.

194 (1869). $$\int \frac{dx}{1+\sqrt{x+1}} = \begin{bmatrix} \sqrt{x+1} = t,\ x+1 = t^2, \\ dx = 2t\,dt \end{bmatrix} = 2\int \frac{t\,dt}{1+t} =$$

$$= 2\int \frac{(t+1)-1}{1+t}\,dt = 2\left(\int dt - \int \frac{dt}{1+t}\right) = 2\,(t - \ln|1+t|) + C =$$

$$= 2\left(\sqrt{x+1} - \ln\left|1+\sqrt{x+1}\right| + C\right).$$

195 (1870). $$\int \frac{x^3 dx}{\sqrt{x-1}} = \begin{bmatrix} \sqrt{x-1} = t,\ x-1 = t^2, \\ dx = 2t\,dt,\ x^3 = t^6 + 3t^4 + 3t^2 + 1 \end{bmatrix} =$$

$$= 2\int \frac{t^6 + 3t^4 + 3t^2 + 1}{t}\,t\,dt = 2\left(\frac{1}{7}t^7 + \frac{3}{5}t^5 + t^3 + t\right) + C = \frac{2}{35}\,t\,(5t^6 + 21t^4 +$$

$$+ 35t^2 + 35) + C = \frac{2}{35}\sqrt{x-1}\,(5x^3 - 15x^2 + 15x - 15 + 21x^2 - 42x + 21 +$$

$$+ 35x - 35 + 35) + C = \frac{2}{35}\sqrt{x-1}\,(5x^3 + 6x^2 + 8x + 16) + C.$$

196 (1871). $$\int \frac{4x+3}{(x-2)^3}\,dx = \begin{bmatrix} x-2 = t, \\ dx = dt \end{bmatrix} = \int \frac{4t+11}{t^3}\,dt = 4\int \frac{dt}{t^2} + 11\int \frac{dt}{t^3} =$$

$$= -\frac{4}{t} - \frac{11}{2t^2} + C = -\frac{4}{x-2} - \frac{11}{2\,(x-2)^2} + C.$$

197 (1872). $$\int \frac{dx}{x\sqrt{x+1}} = \begin{bmatrix} \sqrt{x+1} = t,\ x+1 = t^2, \\ dx = 2t\,dt \end{bmatrix} = 2\int \frac{t\,dt}{(t^2-1)\,t} =$$

$$= \ln\left|\frac{t-1}{t+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

198 (1873). $\displaystyle\int \frac{x+1}{x\sqrt{x-2}}\,dx = \left[\begin{array}{c}\sqrt{x-2}=t,\ x-2=t^2,\\ dx=2t\,dt\end{array}\right] = 2\int \frac{t^2+3}{(t^2+2)\,t}\,t\,dt =$

$$= 2\int dt + 2\int \frac{dt}{t^2+2} = 2\left(t + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}}\right) + C =$$

$$= \underline{\underline{2\left(\sqrt{x-2} + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{x-2}}{\sqrt{2}}\right) + C}}.$$

199 (1874). $\displaystyle\int \frac{dx}{1+\sqrt{x}} = \left[\begin{array}{c}\sqrt{x}=t,\ x=t^2,\\ dx=2t\,dt\end{array}\right] = 2\int \frac{t\,dt}{1+t} = 2\left(\int dt - \int \frac{dt}{1+t}\right) =$

$$= 2(t - \ln|1+t|) + C = \underline{\underline{2\left(\sqrt{x} - \ln\left|1+\sqrt{x}\right|\right) + C}}.$$

200 (1875). $\displaystyle\int \frac{\sqrt{x}\,dx}{x(x+1)} = \left[\begin{array}{c}\sqrt{x}=t,\ x=t^2,\\ dx=2t\,dt\end{array}\right] = 2\int \frac{t^2dt}{t^2(1+t^2)} = 2\operatorname{arctg} t + C =$

$$= \underline{\underline{2\operatorname{arctg}\sqrt{x} + C}}.$$

201 (1876). $\displaystyle\int \frac{\sqrt{x}\,dx}{x+1} = \left[\begin{array}{c}\sqrt{x}=t,\ x=t^2,\\ dx=2t\,dt\end{array}\right] = 2\int \frac{t^2dt}{1+t^2} = 2\left(\int dt - \int \frac{dt}{1+t^2}\right) =$

$$= 2(t - \operatorname{arctg} t) + C = \underline{\underline{2\left(\sqrt{x} - \operatorname{arctg}\sqrt{x}\right) + C}}.$$

202 (1877). $\displaystyle\int \frac{dx}{1+\sqrt[3]{x+1}} = \left[\begin{array}{c}\sqrt[3]{x+1}=t,\ x+1=t^3,\\ dx=3t^2dt\end{array}\right] = 3\int \frac{t^2dt}{1+t} =$

$$= 3\int \frac{(t^2-1)+1}{t+1}\,dt = 3\left(\int (t-1)\,dt + \int \frac{dt}{t+1}\right) = 3\left(\frac{(t-1)^2}{2} + \ln|t+1|\right) + C =$$

$$= \underline{\underline{3\left(\frac{\left(\sqrt[3]{x+1}-1\right)^2}{2} + \ln\left|\sqrt[3]{x+1}+1\right|\right) + C}}.$$

203 (1878). $\displaystyle\int\frac{dx}{\sqrt{ax+b}+m}=\left[\begin{array}{c}\sqrt{ax+b}=t,\ ax+b=t^2,\\ dx=\dfrac{2t\,dt}{a}\end{array}\right]=\frac{2}{a}\int\frac{t\,dt}{t+m}=$

$$=\frac{2}{a}\int\frac{(t+m)-m}{t+m}\,dt=\frac{2}{a}\left(\int dt-m\int\frac{dt}{t+m}\right)=\frac{2}{a}(t-m\ln|t+m|)+C=$$

$$=\underline{\underline{\frac{2}{a}\left(\sqrt{ax+b}-m\ln\left|\sqrt{ax+b}+m\right|\right)+C}}.$$

204 (1879). $\displaystyle\int\frac{\sqrt{x}\,dx}{\sqrt{x}-\sqrt[3]{x}}=\left[\begin{array}{c}x=t^6,\\ dx=6t^5dt\end{array}\right]=\int\frac{6t^8dt}{t^3-t^2}=6\int\frac{t^6dt}{t-1}=$

$$=6\int\frac{t^6-1}{t-1}\,dt+6\int\frac{dt}{t-1}=6\int\frac{(t^3-1)\,(t^3+1)}{t-1}\,dt+6\ln|t-1|+C=$$

$$=6\int(t^2+t+1)\,(t^3+1)\,dt+6\ln|t-1|+C=6\left(\frac{t^6}{6}+\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}+\frac{t^2}{2}+t\right)+$$

$$+6\ln|t-1|+C=\underline{\underline{x+\frac{6}{5}\sqrt[6]{x^5}+\frac{3}{2}\sqrt[3]{x^2}+2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C}}.$$

205 (1880). $\displaystyle\int\frac{dx}{\sqrt[3]{x}\left(\sqrt[3]{x}-1\right)}=\left[\begin{array}{c}x=t^3,\\ dx=3t^2dt\end{array}\right]=\int\frac{3t^2dt}{t(t-1)}=3\int\frac{(t-1)+1}{t-1}\,dt=$

$$=3\int dt+3\int\frac{dt}{t-1}=3\,(t+\ln|t-1|)+C=\underline{\underline{3\left(\sqrt[3]{x}+\ln\left|\sqrt[3]{x}-1\right|\right)+C}}.$$

206 (1881). $\displaystyle\int\frac{dx}{\sqrt{x}+\sqrt[4]{x}}=\left[\begin{array}{c}x=t^4,\\ dx=4t^3dt\end{array}\right]=4\int\frac{t^3dt}{t^2+t}=4\int\frac{(t^2-1)}{t+1}\,dt=$

$$=4\int(t-1)\,dt+4\int\frac{dt}{t+1}=4\left(\frac{t^2}{2}-t+\ln|t+1|\right)=\underline{\underline{2\sqrt{x}-4\sqrt[4]{x}+\ln\left|\sqrt[4]{x}+1\right|+C}}.$$

207 (1882). $\displaystyle\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}\,dx=\left[\begin{array}{c} x=t^{12}, \\ dx=12t^{11}dt \end{array}\right]=\int \frac{12t^{17}dt}{t^8-t^3}=12\int \frac{t^{14}}{t^5-1}\,dt=I.$

$$\begin{array}{r|l} t^{14} & t^5-1 \\ \cline{2-2} -\;\underline{t^{14}-t^9} & t^9+t^4 \\ t^9 & \\ -\;\underline{t^9-t^4} & \\ t^4 & \end{array}$$

$$I=12\int (t^9+t^4)\,dt+12\int \frac{t^4dt}{t^5-1}=12\left(\frac{t^{10}}{10}+\frac{t^5}{5}\right)+\frac{12}{5}\ln\left|t^5-1\right|+C=$$

$$=\underline{\underline{\frac{6}{5}\left(\sqrt[6]{x^5}+2\sqrt[12]{x^5}+2\ln\left|\sqrt[12]{x^5}-1\right|+C\right)}}.$$

208 (1883). $\displaystyle\int \frac{e^{2x}dx}{\sqrt[4]{e^x+1}}=\left[\begin{array}{c} e^x+1=t^4,\ e^{2x}=(t^4-1)^2, \\ e^xdx=4t^3dt,\ dx=\dfrac{4t^3dt}{t^4-1} \end{array}\right]=4\int \frac{(t^4-1)t^3dt}{t}=$

$$=4\int (t^6-t^2)\,dt=4\left(\frac{t^7}{7}-\frac{t^3}{3}\right)+C=4\left(\frac{\sqrt[4]{(e^x+1)^7}}{7}-\frac{\sqrt[4]{(e^x+1)^3}}{3}\right)+C=$$

$$=\underline{\underline{\frac{4}{21}\sqrt[4]{(e^x+1)^3}\,(3e^x-4)+C}}.$$

209 (1884). $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}=\left[\begin{array}{c} 1+e^x=t^2,\ e^xdx=2t\,dt, \\ dx=\dfrac{2t\,dt}{t^2-1} \end{array}\right]=\int \frac{2t\,dt}{t(t^2-1)}=\ln\left|\frac{t-1}{t+1}\right|+$

$$+C=\underline{\underline{\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|+C}}.$$

210 (1885). $$\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx = \left[\begin{array}{l} 1+\ln x = t^2, \\ \dfrac{dx}{x} = 2t\,dt \end{array}\right] = 2\int \frac{t^2 dt}{t^2-1} = 2\int \frac{(t^2-1)+1}{t^2-1}\,dt =$$

$$= 2t + \ln\left|\frac{t-1}{t+1}\right| + C = 2\sqrt{1+\ln x} + \ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right| + C = 2\sqrt{1+\ln x} +$$

$$+\ln\left|\frac{\left(\sqrt{1+\ln x}-1\right)^2}{\left(\sqrt{1+\ln x}+1\right)\left(\sqrt{1+\ln x}-1\right)}\right| + C =$$

$$= \underline{\underline{2\sqrt{1+\ln x} - \ln|\ln x| + 2\ln\left|\sqrt{1+\ln x}-1\right| + C}}.$$

211 (1886). $$\int \sqrt{1+\cos^2 x}\,\sin 2x\cdot\cos 2x\,dx =$$

$$= \left[\begin{array}{l} 1+\cos^2 x = t^2,\ -2\cos x\cdot\sin x\,dx = 2t\,dt, \\ \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 2t^2 - 3 \end{array}\right] = -\int 2t^2(2t^2-3)\,dt =$$

$$= -\frac{4}{5}t^5 + 2t^3 + C = 2t^3(1-0{,}4t^2) + C = 2\sqrt{(1+\cos^2 x)^3}\,(0{,}6 - 0{,}4\cos^2 x) +$$

$$+ C = \underline{\underline{0{,}4\sqrt{(1+\cos^2 x)^3}\,(3-2\cos^2 x) + C}}.$$

212 (1887). $$\int \frac{\ln \operatorname{tg} x}{\sin x\cdot\cos x}\,dx = \left[\begin{array}{c} \ln\operatorname{tg} x = t, \\ dt = \dfrac{dx}{\operatorname{tg} x\cdot\cos^2 x} = \dfrac{dx}{\sin x\cdot\cos x} \end{array}\right] =$$

$$= \int t\,dt = \frac{t^2}{2} + C = \underline{\underline{\frac{(\ln\operatorname{tg} x)^2}{2} + C}}.$$

213 (1888). $$\int \frac{x^5 dx}{\sqrt{a^3-x^3}} = \int \frac{x^2\cdot x^3 dx}{\sqrt{a^3-x^3}} = \left[\begin{array}{c} a^3 - x^3 = t^2,\ x^3 = a^3 - t^2, \\ 3x^2 dx = -2t\,dt \end{array}\right] =$$

$$=-\frac{2}{3}\int\frac{t(a^3-t^2)}{t}\,dt=-\frac{2}{3}\left(a^3t-\frac{t^3}{3}\right)+C=-\frac{2}{3}\left(a^3\sqrt{a^3-x^3}-\right.$$

$$\left.-\frac{1}{3}\sqrt{(a^3-x^3)^3}\right)+C=\underline{\underline{C-\frac{2}{9}\sqrt{a^3-x^3}\,(2a^3+x^3)}}.$$

214 (1889). $\displaystyle\int\frac{x^5dx}{(x^2-4)^2}=I.$

$$\begin{array}{r|l} -\,x^5 & x^4-8x^2+16 \\ \cline{2-2} \underline{x^5-8x^3+16x} & x \\ 8x^3-16x & \end{array}$$

$$I=\int x\,dx+8\int\frac{x(x^2-2)}{(x^2-4)^2}\,dx=\frac{x^2}{2}+8\int\frac{x((x^2-4)+2)}{(x^2-4)^2}\,dx=\frac{x^2}{2}+$$

$$+8\int\frac{x\,dx}{x^2-4}+16\int\frac{x\,dx}{(x^2-4)^2}=\frac{x^2}{2}+4\int\frac{d\,(x^2-4)}{x^2-4}+8\int\frac{d\,(x^2-4)}{(x^2-4)^2}=$$

$$=\underline{\underline{\frac{x^2}{2}+4\ln\left|x^2-4\right|-\frac{8}{x^2-4}+C}}.$$

215 (1890). $\displaystyle\int\frac{dx}{x^2\sqrt{x^2+a^2}}=I.$

Способ 1. $x=\dfrac{a}{z},\ dx=-\dfrac{a}{z^2}dz.$

$$I=-\int\frac{az^3\,dz}{z^2a^3\sqrt{1+z^2}}=-\int\frac{z\,dz}{a^2\sqrt{1+z^2}}=-\frac{1}{2}\int\frac{d\,(1+z^2)}{a^2\sqrt{1+z^2}}=$$

$$=-\frac{\sqrt{1+z^2}}{a^2}+C=\underline{\underline{-\frac{\sqrt{x^2+a^2}}{a^2x}+C}}.$$

Способ 2. $x = a\,\mathrm{tg}\,z,\ dx = \dfrac{a\,dz}{\cos^2 z}.$

$$I = \int \frac{a\cos^2 z \cdot \cos z}{\cos^2 z \cdot a^3 \sin^2 z}\,dz = \frac{1}{a^2}\int \frac{d\,(\sin z)}{\sin^2 z} = -\frac{1}{a^2 \sin z} + C =$$

$$= -\frac{1}{a^2} \cdot \frac{\sqrt{1+\mathrm{tg}^2 z}}{\mathrm{tg}\,z} + C = -\frac{1}{a^2} \cdot \frac{\sqrt{1+\dfrac{x^2}{a^2}}}{\dfrac{x}{a}} + C = \underline{\underline{-\frac{\sqrt{x^2+a^2}}{a^2 x} + C}}.$$

Способ 3. $x = a\,\mathrm{sh}\,z,\ dx = a\,\mathrm{ch}\,z\,dz,\ \sqrt{x^2+a^2} = a\sqrt{1+\mathrm{sh}^2 z} = a\,\mathrm{ch}\,z.$

$$I = \int \frac{a\,\mathrm{ch}\,z\,dz}{a^2\,\mathrm{sh}^2 z \cdot a\,\mathrm{ch}\,z} = -\frac{1}{a^2}\mathrm{cth}\,z + C = -\frac{1}{a^2} \cdot \frac{\mathrm{ch}\,z}{\mathrm{sh}\,z} + C =$$

$$= -\frac{1}{a^2} \cdot \frac{\sqrt{1+\mathrm{sh}^2 z}}{\mathrm{sh}\,z} = \underline{\underline{-\frac{\sqrt{x^2+a^2}}{a^2 x}}} + C.$$

216 (1891). $\displaystyle\int \frac{x^2 dx}{\sqrt{a^2-x^2}} = \left[\begin{matrix} x = a\sin z,\ dx = a\cos z\,dz, \\ \sqrt{a^2-x^2} = a\cos z \end{matrix}\right] = a^2 \int \sin^2 z\,dz =$

$$= \frac{a^2}{2}\int (1-\cos 2z)\,dz = \frac{a^2}{2}\left(z - \frac{1}{2}\sin 2z \right) + C =$$

$$= \frac{a^2}{2}\left(z - \sin z\sqrt{1-\sin^2 z} \right) + C = \underline{\underline{\frac{a^2}{2}\left(\arcsin\frac{x}{a} - \frac{x}{a^2}\sqrt{a^2-x^2} \right) + C}}.$$

217 (1892). $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \left[\begin{matrix} x = \dfrac{a}{z},\ dx = -\dfrac{a}{z^2}\,dz, \\ \sqrt{x^2-a^2} = \sqrt{\dfrac{a^2}{z^2} - a^2} = \dfrac{a}{z}\sqrt{1-z^2} \end{matrix}\right] =$

$$= -\int \frac{a\,dz}{z^2 \cdot \frac{a}{z} \cdot \frac{a}{z}\sqrt{1-z^2}} = -\frac{1}{a}\int \frac{dz}{\sqrt{1-z^2}} = -\frac{1}{a}\arcsin z + C =$$

$$= \underline{\underline{-\frac{1}{a}\arcsin\frac{a}{x} + C}}.$$

218 (1893). $\int \frac{\sqrt{1+x^2}}{x^4}\,dx = \left[x = \frac{1}{z},\ z = \frac{1}{x},\ dx = -\frac{dz}{z^2}\right] = -\int \frac{z^4\sqrt{1+z^2}}{z^3}\,dz =$

$$= -\int z\sqrt{1+z^2}\,dz = -\frac{1}{2}\int\sqrt{1+z^2}\,d(1+z^2) = -\frac{1}{3}\sqrt{(1+z^2)^3} + C =$$

$$= \underline{\underline{C - \frac{\sqrt{(1+x^2)^3}}{3x^3}}}.$$

Примечание. Интеграл можно взять подстановкой $x = \operatorname{tg} z$.

219 (1894). $\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \left[\begin{matrix} x = \sin z, \\ dx = \cos z\,dz \end{matrix}\right] = \int \frac{\cos^2 z\,dz}{\sin^2 z} = \int \frac{dz}{\sin^2 z} - \int dz =$

$$= -\operatorname{ctg} z - z + C = -\frac{\cos(\arcsin x)}{\sin(\arcsin x)} - \arcsin x + C =$$

$$= C - \frac{\sqrt{1-\sin^2(\arcsin x)}}{\sin(\arcsin x)} - \arcsin x = \underline{\underline{C - \frac{\sqrt{1-x^2}}{x} - \arcsin x}}.$$

220 (1895). $\int \frac{dx}{\sqrt{(a^2+x^2)^3}} = \left[\begin{matrix} x = a\operatorname{tg} z, \\ dx = \frac{a\,dz}{\cos^2 z} \end{matrix}\right] = \int \frac{a\,dz}{\cos^2 z \cdot \frac{a^3}{\cos^3 z}} =$

$$=\frac{1}{a^2}\int\cos z\,dz=\frac{1}{a^2}\sin z+C=\frac{1}{a^2}\frac{\operatorname{tg} z}{\sqrt{1+\operatorname{tg}^2 z}}+C=$$

$$=\frac{1}{a^2}\cdot\frac{x}{a\sqrt{1+\dfrac{x^2}{a^2}}}=\underline{\underline{\frac{x}{a^2\sqrt{a^2+x^2}}}}+C.$$

Примечание. Интеграл можно взять подстановкой $x=\dfrac{1}{z}$.

221 (1896). $\displaystyle\int\frac{\sqrt{(9-x^2)^3}}{x^6}\,dx=\left[\begin{array}{c} x=\dfrac{3}{z},\ z=\dfrac{3}{x},\ dx=-\dfrac{3dz}{z^2}, \\ \sqrt{(9-x^2)^3}=\sqrt{\left(9-\dfrac{9}{z^2}\right)^3}=\dfrac{27}{z^3}\sqrt{z^2-1} \end{array}\right]=$

$$=-\int\frac{3\cdot 27\cdot\sqrt{(z^2-1)^3}\cdot z^6\,dz}{z^2\cdot 27^2\cdot z^3}=-\frac{1}{9\cdot 2}\int\sqrt{(z^2-1)^3}\,d\,(z^2-1)=$$

$$=-\frac{2}{5\cdot 9\cdot 2}(z^2-1)^{5/2}+C=C-\frac{\sqrt{(z^2-1)^5}}{45}=\underline{\underline{C-\frac{\sqrt{(9-x^2)^5}}{45x^5}}}.$$

222 (1897). $\displaystyle\int\frac{dx}{x^2\sqrt{x^2-9}}=\left[\begin{array}{c} x=\dfrac{3}{z},\ dx=-\dfrac{3}{z^2}\,dz, \\ \sqrt{x^2-9}=\dfrac{3\sqrt{1-z^2}}{z} \end{array}\right]=-\int\frac{3\,dz}{\dfrac{9}{z^2}\cdot\dfrac{3\sqrt{1-z^2}}{z}\cdot z^2}=$

$$=-\frac{1}{9}\int\frac{z\,dz}{\sqrt{1-z^2}}=\frac{1}{18}\int\frac{d\,(1-z^2)}{\sqrt{1-z^2}}=\frac{1}{9}\sqrt{1-z^2}+C=\underline{\underline{\frac{\sqrt{x^2-9}}{9x}}}+C.$$

223 (1898). $\displaystyle\int\frac{dx}{x\sqrt{1+x^2}}=\left[x=\frac{1}{z},\ dx=-\frac{dz}{z^2}\right]=-\int\frac{dz}{z^2\cdot\dfrac{1}{z}\cdot\dfrac{\sqrt{z^2+1}}{z}}=$

$$=-\int\frac{dz}{\sqrt{z^2+1}}=-\ln\left|z+\sqrt{z^2+1}\right|+C=$$

$$=C-\ln\left|\frac{1}{x}+\frac{\sqrt{1+x^2}}{x}\right|=\ln\frac{|x|}{1+\sqrt{1+x^2}}+C.$$

224 (1899). $\displaystyle\int\frac{dx}{\sqrt{(x^2-a^2)^3}}=\left[\begin{array}{c} x=\dfrac{a}{\cos z},\ dx=\dfrac{a\sin z}{\cos^2 z}\,dz, \\ \sqrt{(x^2-a^2)^3}=\dfrac{a^3\sin^3 z}{\cos^3 z},\ z=\arccos\dfrac{a}{x}\end{array}\right]=$

$$=\int\frac{a\sin z\cdot\cos^3 z\,dz}{\cos^2 z\cdot a^3\sin^3 z}=\frac{1}{a^2}\int\frac{d\,(\sin z)}{\sin^2 z}=-\frac{1}{a^2\sin z}+C=-\frac{1}{a^2\sin z}+C=$$

$$=C-\frac{1}{a^2\sqrt{1-\cos^2 z}}=C-\frac{1}{a^2\sqrt{1-\dfrac{a^2}{x^2}}}=C-\frac{x}{a^2\sqrt{x^2-a^2}}.$$

225 (1900). $\displaystyle\int x^2\sqrt{4-x^2}\,dx=[x=2\sin z,\ dx=2\cos z\,dz]=$

$$=\int 16\sin^2 z\cdot\cos^2 z\,dz=4\int\sin^2 2z\,dz=2\int(1-\cos 4z)\,dz=$$

$$=2z-\frac{1}{2}\sin 4z+C=2z-\sin 2z\cdot\cos 2z+C=2z-2\sin z\cdot\cos z(\cos^2 z-$$

$$-\sin^2 z)+C=2z-2\sin z\sqrt{1-\sin^2 z}\,(1-2\sin^2 z)+C=2\arcsin\frac{x}{2}-$$

$$-x\sqrt{1-\frac{x^2}{4}}\left(1-\frac{x^2}{x}\right)+C=2\arcsin\frac{x}{2}-\frac{x}{4}\sqrt{4-x^2}\,(2-x^2)+C=$$

$$=2\arcsin\frac{x}{2}+\frac{x}{4}\sqrt{4-x^2}\,(x^2-2)+C.$$

226 (1901). $\displaystyle\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}} = \left[x = \frac{1}{2}\operatorname{tg} t,\ dx = \frac{dt}{2\cos^2 t} \right] =$

$$= 2\int \frac{\cos t\,dt}{(\operatorname{tg}^2 t + 16)\cos^2 t} = 2\int \frac{\cos t\;dt}{\sin^2 t + 16\cos^2 t} = \frac{2}{\sqrt{15}}\int \frac{d\left(\sqrt{15}\sin t\right)}{16 - 15\sin^2 t} =$$

$$= \frac{1}{4\sqrt{15}}\ln\left|\frac{4+\sqrt{15}\sin t}{4-\sqrt{15}\sin t}\right| + C = \frac{1}{4\sqrt{15}}\ln\left|\frac{4+\sqrt{15}\,\dfrac{\operatorname{tg} t}{\sqrt{1+\operatorname{tg}^2 t}}}{4-\sqrt{15}\,\dfrac{\operatorname{tg} t}{\sqrt{1+\operatorname{tg}^2 t}}}\right| + C =$$

$$= \underline{\underline{\frac{1}{4\sqrt{15}}\ln\left|\frac{2\sqrt{1+4x^2}+\sqrt{15}x}{2\sqrt{1+4x^2}-\sqrt{15}x}\right| + C}}.$$

227 (1902). $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,\frac{dx}{x^2} = \left[x = \frac{1}{z},\ dx = -\frac{dz}{z^2} \right] = -\int \sqrt{\frac{1-z}{1+z}}\,dz =$

$$= \begin{bmatrix} z = \cos 2t,\ dz = -2\sin 2t\,dt, \\ t = \dfrac{1}{2}\arccos z \end{bmatrix} = 2\int \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}\,\sin 2t\,dt =$$

$$= 4\int \frac{\sin t}{\cos t}\sin t\cdot\cos t\,dt = 2\int (1-\cos 2t)\,dt = 2\left(t - \frac{1}{2}\sin 2t \right) + C =$$

$$= \arccos t - \sin(\arccos z) + C = \arccos\frac{1}{x} - \sqrt{1-\frac{1}{x^2}} + C =$$

$$= \underline{\underline{\arccos\frac{1}{x} - \frac{\sqrt{x^2-1}}{x} + C}}.$$

228 (1903). $\displaystyle\int \frac{dx}{\sqrt{x-x^2}} = \begin{bmatrix} x = \sin^2 z,\ z = \arcsin\sqrt{x}, \\ dx = 2\sin z\cdot\cos z\,dz \end{bmatrix} = 2\int \frac{\sin z\cdot\cos z\,dz}{\sin z\cdot\cos z} =$

$$= 2z + C = \underline{\underline{2\arcsin\sqrt{x} + C}}.$$

229 (1904). $\displaystyle\int \frac{(x+1)\,dx}{x(1+xe^x)} = \left[\begin{array}{c} xe^x = z, \\ dz = e^x(1+x)\,dx \end{array}\right] = \int \frac{e^x(x+1)\,dx}{xe^x(1+xe^x)} =$

$$= \int \frac{dz}{z(1+z)} = \int \frac{dz}{z} - \int \frac{dz}{1+z} = \ln\left|\frac{z}{1+z}\right| + C = \underline{\underline{\ln\left|\frac{xe^x}{1+xe^x}\right| + C}}.$$

В задачах **230** (1905)–**234** (1909) найти интегралы, применив сначала замену переменной, а потом интегрирование по частям.

230 (1905). $\displaystyle\int e^{\sqrt{x}}dx = \left[\begin{array}{c} \sqrt{x} = z,\ x = z^2, \\ dx = 2z\,dz \end{array}\right] = 2\int ze^z dz =$

$$= \left[\begin{array}{ll} u = z, & du = dz, \\ dv = e^z dz, & v = e^z \end{array}\right] = 2\left(ze^z - \int e^z dz\right) = 2(z-1)e^z + C =$$

$$= \underline{\underline{2\left(\sqrt{x}-1\right)e^{\sqrt{x}} + C}}.$$

231 (1906). $\displaystyle\int \sin\sqrt[3]{x}\,dx = \left[\begin{array}{c} \sqrt[3]{x} = z,\ x = z^3, \\ dx = 3z^2dz \end{array}\right] = 3\int z^2\sin z\,dz =$

$$= \left[\begin{array}{ll} u = z^2, & du = 2z\,dz, \\ dv = \sin z\,dz, & v = -\cos z \end{array}\right] = 3\left(-z^2\cos z + 2\int z\cos z\,dz\right) =$$

$$= \left[\begin{array}{ll} u = z, & du = dz, \\ dv = \cos z\,dz, & v = \sin z \end{array}\right] = -3z^2\cos z + 6\left(z\sin z - \int \sin z\,dz\right) =$$

$$= -3z^2\cos z + 6z\sin z + 6\cos z + C = -3\sqrt[3]{x^2}\cos\sqrt[3]{x} + 6\sqrt[3]{x}\sin\sqrt[3]{x} +$$

$$+ 6\cos\sqrt[3]{x} + C = \underline{\underline{3\left(\left(2-\sqrt[3]{x^2}\right)\cos\sqrt[3]{x} + 2\sqrt[3]{x}\sin\sqrt[3]{x}\right) + C}}.$$

232 (1907). $\displaystyle\int \frac{\arcsin x}{\sqrt{(1-x^2)^3}}\,dx =$

$$=\left[\begin{array}{ll} u=\arcsin x, & du=\dfrac{dx}{\sqrt{1-x^2}}, \\ dv=\dfrac{dx}{\sqrt{(1-x^2)^3}}, & v=\displaystyle\int\frac{dx}{\sqrt{(1-x^2)^3}}=\frac{x}{\sqrt{1-x^2}} \end{array}\right]=$$

$$=\frac{x}{\sqrt{1-x^2}}\arcsin x-\int\frac{x\,dx}{1-x^2}=\underline{\underline{\frac{x}{\sqrt{1-x^2}}\arcsin x+\frac{1}{2}\ln\left|1-x^2\right|+C}}.$$

233 (1908). $\displaystyle\int\frac{x^2\operatorname{arctg}x}{1+x^2}\,dx=\left[\operatorname{arctg}x=z,\ x=\operatorname{tg}z,\ \frac{dx}{1+x^2}=dz\right]=$

$$=\int z\operatorname{tg}^2 z\,dz=\left[\begin{array}{ll} u=z, & du=dz, \\ dv=\operatorname{tg}^2 z\,dz, & v=\displaystyle\int\frac{1-\cos^2 z}{\cos^2 z}\,dz=\operatorname{tg}z-z \end{array}\right]=$$

$$=z(\operatorname{tg}z-z)-\int(\operatorname{tg}z-z)\,dz=z\operatorname{tg}z-z^2+\ln\left|\cos z\right|+\frac{z^2}{2}+C=$$

$$=z\operatorname{tg}z-\frac{z^2}{2}+\ln\left|\cos z\right|+C=z\operatorname{tg}z-\frac{z^2}{2}+\ln\frac{1}{\sqrt{1+\operatorname{tg}^2 z}}+C=$$

$$=\underline{\underline{x\operatorname{arctg}x-\frac{1}{2}(\operatorname{arctg}x)^2-\frac{1}{2}\ln(1+x^2)+C}}.$$

234 (1909). $\displaystyle\int\frac{\operatorname{arctg}x}{x^2(1+x^2)}\,dx=$

$$=\left[\begin{array}{ll} u=\operatorname{arctg}x, & du=\dfrac{dx}{1+x^2}, \\ dv=\dfrac{dx}{x^2(1+x^2)}, & v=\displaystyle\int\frac{dx}{x^2}-\int\frac{dx}{1+x^2}=-\frac{1}{x}-\operatorname{arctg}x \end{array}\right]=$$

$$=-\frac{\operatorname{arctg}x}{x}-(\operatorname{arctg}x)^2+\int\frac{x\,dx}{x^2(1+x^2)}+\int(\operatorname{arctg}x)\,d(\operatorname{arctg}x)=$$

$$= -\frac{\operatorname{arctg} x}{x} - (\operatorname{arctg} x)^2 + \int \frac{x\,dx}{x^2} - \int \frac{x\,dx}{1+x^2} + \frac{(\operatorname{arctg} x)^2}{2} + C =$$

$$= -\frac{\operatorname{arctg} x}{x} - \frac{(\operatorname{arctg} x)^2}{2} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C =$$

$$= \underline{\underline{\ln\frac{|x|}{1+x^2} - \frac{\operatorname{arctg} x}{x} - \frac{(\operatorname{arctg} x)^2}{x} + C}}.$$

2.3. Разные задачи

235 (1910). $\int (x+1)\sqrt{x^2+2x}\,dx = \int (x+1)\sqrt{(x+1)^2-1}\,dx = \begin{bmatrix} x+1=z, \\ dx=dz \end{bmatrix} =$

$$= \int z\sqrt{z^2-1}\,dz = \frac{1}{2}\int \sqrt{z^2-1}\,d(z^2-1) = \frac{1}{3}\sqrt{(z^2-1)^3} +$$

$$+ C = \underline{\underline{\frac{1}{3}\sqrt{(x^2+2x)^3} + C}}.$$

236 (1911). $\int (1+e^{3x})^2 e^{3x} dx = \begin{bmatrix} 1+e^{3x}=z, \\ 3e^{3x}dx=dz \end{bmatrix} = \frac{1}{3}\int z^2 dz = \frac{1}{3}\cdot\frac{z^3}{3} + C =$

$$= \underline{\underline{\frac{1}{9}(1+e^{3x})^3 + C}}.$$

237 (1912). $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \left[e^{\sqrt{x}} = z,\ \frac{1}{2\sqrt{x}}e^{\sqrt{x}}dx = dz\right] = \int 2\,dz = 2z + C =$

$$= \underline{\underline{2e^{\sqrt{x}} + C}}.$$

238 (1913). $\int \frac{\sin x}{e^{\cos x}}\,dx = \begin{bmatrix} \cos x = z, \\ -\sin x\,dx = dz \end{bmatrix} = -\int \frac{dz}{e^z} = \int e^{-z} d(-z) =$

$$= e^{-z} + C = \underline{\underline{e^{-\cos x} + C}}.$$

239 (1914). $\int \sqrt{1-e^x}\, e^x dx = \left[\begin{matrix} 1-e^x = z, \\ -e^x dx = dz \end{matrix}\right] = -\int z^{1/2} dz = -\frac{2}{3}\sqrt{z^3} + C =$

$= \underline{\underline{-\frac{2}{3}\sqrt{(1-e^x)^3} + C}}.$

240 (1915). $\int x \cos x^2 \, dx = \frac{1}{2}\int \cos x^2 \, d(x^2) = \underline{\underline{\frac{1}{2}\sin x^2 + C}}.$

241 (1916). $\int (2-3x^{4/3})^{1/5} x^{1/3} dx = \left[\begin{matrix} 2-3x^{4/3} = z, \\ -4x^{1/3} dx = dz \end{matrix}\right] = -\frac{1}{4}\int z^{1/5} dz =$

$= C - \frac{5}{24} z^{6/5} = \underline{\underline{C - \frac{5}{24}\cdot(2-3x^{4/3})^{6/5}}}.$

242 (1917). $\int \frac{2x^5 - 3x^2}{1+3x^3 - x^6}\, dx = -\frac{1}{3}\int \frac{d(1+3x^3-x^6)}{1+3x^3-x^6} = \underline{\underline{C - \frac{1}{3}\ln\left|1+3x^3-x^6\right|}}.$

243 (1918). $\int \frac{\sqrt{x}\, dx}{1+x^{3/2}} = \frac{2}{3}\int \frac{d(1+x^{3/2})}{1+x^{3/2}} = \underline{\underline{\frac{2}{3}\ln\left|1+x^{3/2}\right| + C}}.$

244 (1919). $\int \frac{dx}{e^x(3+e^{-x})} = -\int \frac{d(3+e^{-x})}{3+e^{-x}} = \underline{\underline{C - \ln(3+e^{-x})}}.$

245 (1920). $\int \frac{dx}{e^x\sqrt{1-e^{-2x}}} = \left[\begin{matrix} e^{-x} = z, \\ -e^{-x} dx = dz \end{matrix}\right] = -\int \frac{dz}{\sqrt{1-z^2}} = -\arcsin z + C =$

$= \underline{\underline{C - \arcsin e^{-x}}}.$

246 (1921). $\int \frac{2x+3}{\sqrt{1+x^2}}\, dx = \int \frac{d(1+x^2)}{\sqrt{1+x^2}} + 3\int \frac{dx}{\sqrt{1+x^2}} =$

$= \underline{\underline{2\sqrt{1+x^2} + 3\ln\left|x+\sqrt{1+x^2}\right| + C}}.$

247 (1922). $\int \frac{2x-1}{\sqrt{9x^2-4}}\,dx = \frac{1}{9}\int \frac{d(9x^2-4)}{\sqrt{9x^2-4}} - \frac{1}{3}\int \frac{d(3x)}{\sqrt{9x^2-4}} =$

$$= \underline{\underline{\frac{1}{9}\left(2\sqrt{9x^2-4} - 3\ln\left|3x+\sqrt{9x^2-4}\right|\right) + C}}.$$

248 (1923). $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \left[\sqrt{x}=z,\ \frac{dx}{2\sqrt{x}} = dz\right] = 2\int \cos z\,dz =$

$$= 2\sin z + C = \underline{\underline{2\sin\sqrt{x} + C}}.$$

249 (1924). $\int \frac{dx}{x\sqrt{3-\ln^2 x}} = \left[\ln x = z,\ \frac{dx}{x} = dz\right] = \int \frac{dz}{\sqrt{3-z^2}} = \arcsin\frac{z}{\sqrt{3}} +$

$$+ C = \underline{\underline{\arcsin\frac{\ln x}{\sqrt{3}} + C}}.$$

250 (1925). $\int \frac{\ln x\,dx}{x(1-\ln^2 x)} = \left[\ln x = z,\ \frac{dx}{x} = dz\right] = \int \frac{z\,dz}{1-z^2} =$

$$= -\frac{1}{2}\int \frac{d(1-z^2)}{1-z^2} = C - \frac{1}{2}\ln\left|1-z^2\right| = \underline{\underline{C - \frac{1}{2}\ln\left|1-\ln^2 x\right|}}.$$

251 (1926). $\int \frac{x^2-x+1}{\sqrt{(x^2+1)^3}}\,dx = \int \frac{x^2+1}{\sqrt{(x^2+1)^3}}\,dx - \int \frac{x\,dx}{\sqrt{(x^2+1)^3}} =$

$$= \int \frac{dx}{\sqrt{x^2+1}} - \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{(x^2+1)^3}} = \underline{\underline{\ln\left|x+\sqrt{x^2+1}\right| + \frac{1}{\sqrt{x^2+1}} + C}}.$$

252 (1927). $\int \frac{(\operatorname{arctg} x)^n}{1+x^2}\,dx = \left[\operatorname{arctg} x = z,\ \frac{dx}{1+x^2} = dz\right] = \int z^n dz =$

$$= \frac{z^{n+1}}{n+1} + C = \underline{\underline{\frac{(\operatorname{arctg} x)^{n+1}}{n+1} + C}}.$$

253 (1928). $\displaystyle\int \frac{d\varphi}{\sin^2\varphi\cdot\cos^2\varphi} = 4\int\frac{d\varphi}{\sin^2 2\varphi} = 2\int\frac{d(2\varphi)}{\sin^2 2\varphi} = \underline{\underline{-2\operatorname{ctg}2\varphi + C}}.$

254 (1929). $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx = \int\frac{\cos^2 x - \sin^2 x}{\cos^2 x}\,dx = \int\frac{2\cos^2 x - 1}{\cos^2 x}\,dx =$

$\displaystyle = 2\int\frac{dx}{\cos^2 x}\cdot\cos^2 x - \int\frac{dx}{\cos^2 x} = \underline{\underline{2x - \operatorname{tg}x + C}}.$

255 (1930). $\displaystyle\int \frac{\sin^4 x}{\cos^6 x}\,dx = \int\frac{\sin^4 x}{\cos^4 x}\,d(\operatorname{tg}x) = \underline{\underline{\frac{\operatorname{tg}^5 x}{5} + C}}.$

256 (1931). $\displaystyle\int\sqrt{\operatorname{tg}^3 x}\,\sec^4 x\,dx = \int\sqrt{\operatorname{tg}^3 x}\,\frac{\cos^2 x + \sin^2 x}{\cos^4 x}\,dx =$

$\displaystyle = \int\sqrt{\operatorname{tg}^3 x}\,d(\operatorname{tg}x) + \int\sqrt{\operatorname{tg}^7 x}\,d(\operatorname{tg}x) = \frac{2}{5}\operatorname{tg}^{5/2}x + \frac{2}{9}\operatorname{tg}^{9/2}x + C =$

$\displaystyle \underline{\underline{= \frac{2}{45}\sqrt{\operatorname{tg}^5 x}\,(5\operatorname{tg}^2 x + 9) + C}}.$

257 (1932). $\displaystyle\int(1 - \operatorname{tg}3x)^2\,dx = \int\frac{(\cos 3x - \sin 3x)^2}{\cos^2 3x}\,dx = \int\frac{dx}{\cos^2 3x} +$

$\displaystyle + \frac{2}{3}\int\frac{d(\cos 3x)}{\cos 3x} = \underline{\underline{\frac{1}{3}(\operatorname{tg}3x + \ln\cos^2 3x) + C}}.$

258 (1933). $\displaystyle\int\frac{x^3dx}{x+1} = \int\frac{(x^3+1)-1}{x+1}\,dx = \int(x^2 - x + 1)\,dx - \int\frac{dx}{x+1} =$

$\displaystyle \underline{\underline{= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C}}.$

259 (1934). $\displaystyle\int\frac{x\,dx}{(x-1)^3} = \int\frac{(x-1)+1}{(x-1)^3}\,dx = \int\frac{d(x-1)}{(x-1)^2} + \int\frac{d(x-1)}{(x-1)^3} =$

$\displaystyle = -\frac{1}{x-1} - \frac{1}{2(x-1)^2} + C = \underline{\underline{C - \frac{1}{x-1} - \frac{1}{2(x-1)^2}}}.$

260 (1935). $\int \frac{x\,dx}{\sqrt{2+4x}} = \int \frac{\frac{1}{4}(2+4x)-\frac{1}{2}}{\sqrt{2+4x}}\,dx = \frac{1}{4^2}\int \sqrt{2+4x}\,d(2+4x) -$

$- \frac{1}{2\cdot 4}\int \frac{d(2+4x)}{\sqrt{2+4x}} = \frac{1}{24}\sqrt{(2+4x)^3} - \frac{1}{4}\sqrt{2+4x} + C =$

$= \underline{\underline{\frac{\sqrt{2+4x}\,(x-1)}{6} + C}}.$

261 (1936). $\int \frac{x\,dx}{\sqrt{1+2x}} = \int \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{\sqrt{1+2x}}\,dx = \frac{1}{4}\int \sqrt{1+2x}\,d(1+2x) -$

$- \frac{1}{4}\int \frac{d(1+2x)}{\sqrt{1+2x}} = \frac{1}{4}\left(\frac{2}{3}\sqrt{(1+2x)^3} - 2\sqrt{1+2x}\right) + C =$

$= \frac{1}{2}\sqrt{1+2x}\left(\frac{1}{3} + \frac{2}{3}x - 1\right) = \frac{1}{3}\sqrt{1+2x}\,(x-1) + C =$

$= \underline{\underline{x\sqrt{1+2x} - \frac{1}{3}\sqrt{(1+2x)^3} + C}}.$

262 (1937). $\int x\sqrt{a+x}\,dx = \int (a+x-a)\sqrt{a+x}\,dx = \int (a+x)^{3/2}\,dx -$

$- a\int (a+x)^{1/23}\,dx = \frac{2}{5}\sqrt{(a+x)^5} - \frac{2}{3}a\sqrt{(a+x)^3} + C =$

$= \underline{\underline{\frac{2}{15}\sqrt{(a+x)^3}\,(3x-2a) + C}}.$

263 (1938). $\int \left(\sqrt{\sin x} + \cos x\right)^2 dx = \int \sin x\,dx + 2\int \sqrt{\sin x}\,d(\sin x) +$

$+ \int \cos^2 x\,dx = \underline{\underline{-\cos x + \frac{4}{3}\sqrt{\sin^3 x} + \frac{1}{2}x + \frac{1}{4}\sin 2x + C}}.$

264 (1939). $\int a^{mx}b^{nx}dx=\int (a^m b^n)^x dx=\dfrac{a^{mx}b^{nx}}{\ln a^m b^n}+C=\underline{\underline{\dfrac{a^{mx}b^{nx}}{m\ln a+n\ln b}}}+C.$

265 (1940). $\int \dfrac{dx}{\sqrt{5-2x+x^2}}=\int \dfrac{d(x-1)}{\sqrt{4+(x-1)^2}}=$

$=\underline{\underline{\ln\left|x-1+\sqrt{5-2x+x^2}\right|+C}}=\underline{\underline{-\ln\left|1-x+\sqrt{5-2x+x^2}\right|+C}}.$

266 (1941). $\int \dfrac{dx}{\sqrt{9x^2-6x+2}}=\int \dfrac{dx}{\sqrt{(3x-1)^2+1}}=\dfrac{1}{3}\int \dfrac{d(3x-1)}{\sqrt{(3x-1)^2+1}}=$

$=\underline{\underline{\dfrac{1}{3}\ln\left|3x-1+\sqrt{9x^2-6x+2}\right|+C.}}$

267 (1942). $\int \dfrac{dx}{\sqrt{12x-9x^2-2}}=\int \dfrac{dx}{\sqrt{2-(3x-2)^2}}=\dfrac{1}{3}\int \dfrac{d(3x-2)}{\sqrt{2-(3x-2)^2}}=$

$=\underline{\underline{\dfrac{1}{3}\arcsin\dfrac{3x-2}{\sqrt{2}}+C.}}$

268 (1943). $\int \dfrac{(8x-11)\,dx}{\sqrt{5+2x-x^2}}=\int \dfrac{-4(-2x+2)-3}{\sqrt{5+2x-x^2}}\,dx=-4\int \dfrac{d(5+2x-x^2)}{\sqrt{5+2x-x^2}}-$

$-3\int \dfrac{d(5+2x-x^2)}{\sqrt{6-(x-1)^2}}=\underline{\underline{-8\sqrt{5+2x-x^2}-3\arcsin\dfrac{x-1}{\sqrt{6}}+C.}}$

269 (1944). $\int \dfrac{(x+2)\,dx}{x^2+2x+2}=\int \dfrac{\frac{1}{2}(2x+2)+1}{x^2+2x+2}\,dx=\dfrac{1}{2}\int \dfrac{d(x^2+2x+2)}{x^2+2x+2}+$

$+\int \dfrac{d(x+1)}{(x+1)^2+1^2}=\underline{\underline{\dfrac{1}{2}\ln\left|x^2+2x+2\right|+\operatorname{arctg}(x+1)+C.}}$

270 (1945). $\displaystyle\int \frac{(x-3)\,dx}{\sqrt{3-2x-x^2}} = \int \frac{-\frac{1}{2}(-2x-2)-4}{\sqrt{3-2x-x^2}}\,dx =$

$$= -\frac{1}{2}\int \frac{d\,(3-2x-x^2)}{\sqrt{3-2x-x^2}} - 4\int \frac{dx}{\sqrt{4-(x+1)^2}} =$$

$$= \underline{\underline{-\sqrt{3-2x-x^2} - 4\arcsin\frac{x+1}{2} + C}}.$$

271 (1946). $\displaystyle\int \frac{(3x-1)\,dx}{4x^2-4x+17} = \int \frac{\frac{3}{8}(8x-4)+\left(\frac{3}{2}-1\right)}{4x^2-4x+17}\,dx =$

$$= \frac{3}{8}\int \frac{d\,(4x^2-4x+17)}{4x^2-4x+17} + \frac{1}{4}\int \frac{d\,(2x-1)}{(2x-1)^2+4^2} = \frac{3}{8}\ln\left|4x^2-4x+17\right| +$$

$$+ \frac{1}{16}\operatorname{arctg}\frac{2x-1}{4} + C = \underline{\underline{\frac{3}{8}\left(\ln\left|4x^2-4x+17\right| + \frac{1}{6}\operatorname{arctg}\frac{2x-1}{4}\right) + C}}.$$

272 (1947). $\displaystyle\int \frac{(3x-1)\,dx}{\sqrt{x^2+2x+2}} = \int \frac{\frac{3}{2}(2x+2)+(-3-1)}{\sqrt{x^2+2x+2}}\,dx =$

$$= \frac{3}{2}\int \frac{d(x^2+2x+2)}{\sqrt{x^2+2x+2}} + 2\cdot(-2)\int \frac{d\,(x+1)}{\sqrt{(x+1)^2+1}} =$$

$$= \underline{\underline{3\sqrt{x^2+2x+2} - 4\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C}}.$$

273 (1948). $\displaystyle\int \frac{(x-2)\,dx}{x^2-7x+12} = \int \frac{\frac{1}{2}(2x-7)+\left(\frac{7}{2}-2\right)}{x^2-7x+12}\,dx =$

$$=\frac{1}{2}\int\frac{d\,(x^2-7x+12)}{x^2-7x+12}+\frac{3}{2}\int\frac{dx}{\left(x-\frac{7}{2}\right)^2-\frac{1}{4}}=\frac{1}{2}\ln\left|x^2-7x+12\right|+$$

$$+\frac{3}{2}\cdot\frac{1}{2\cdot\frac{1}{2}}\ln\frac{x-\frac{7}{2}-\frac{1}{2}}{x-\frac{7}{2}+\frac{1}{2}}=\frac{1}{2}\ln\left|x^2-7x+12\right|+\frac{3}{2}\ln\frac{x-4}{x-3}+C=$$

$$=\ln\frac{\sqrt{(x-3)(x-4)}\ (x-4)\sqrt{x-4}}{|x-3|\sqrt{x-3}}+C=\underline{\underline{\ln\frac{(x-4)^2}{|x-3|}}}+C.$$

274 (1949). $$\int\frac{2x+5}{\sqrt{9x^2+6x+2}}\,dx=\int\frac{\frac{1}{9}(18x+6)+\left(5-\frac{2}{3}\right)}{\sqrt{9x^2+6x+2}}\,dx=$$

$$=\frac{1}{9}\int\frac{d\,(9x^2+6x+2)}{\sqrt{9x^2+6x+2}}+\frac{13}{3\cdot 3}\int\frac{d(3x+1)}{\sqrt{(3x+1)^2+1}}=$$

$$\underline{\underline{=\frac{2}{9}\sqrt{9x^2+6x+2}+\frac{13}{9}\ln\left|3x+1+\sqrt{9x^2+6x+2}\right|+C.}}$$

275 (1950). $$\int\frac{3-4x}{2x^2-3x+1}\,dx=-\int\frac{(4x-3)\,dx}{2x^2-3x+1}=\underline{\underline{C-\ln\left|2x^2-3x+1\right|}}.$$

276 (1951). $$\int\frac{(4-3x)\,dx}{5x^2+6x+18}=\int\frac{-\frac{3}{10}(10x+6)+\left(\frac{9}{5}+4\right)}{5x^2+6x+18}\,dx=$$

$$=-\frac{3}{10}\int\frac{d\,(5x^2+6x+18)}{5x^2+6x+18}+29\int\frac{dx}{(5x+3)^2+81}=$$

$$\underline{\underline{=-\frac{3}{10}\ln\left|5x^2+6x+18\right|+\frac{29}{45}\operatorname{arctg}\frac{5x+3}{9}+C.}}$$

277 (1952). $\displaystyle\int \frac{(2-5x)\,dx}{\sqrt{4x^2+9x+1}} = \int \frac{-\frac{5}{8}(8x+9)+\left(\frac{45}{8}+2\right)}{\sqrt{4x^2+9x+1}}\,dx =$

$$= -\frac{5}{8}\cdot 2\sqrt{4x^2+9x+1} + \frac{61}{8\cdot 2}\int \frac{d\left(2x+\frac{9}{4}\right)}{\sqrt{\left(2x+\frac{9}{4}\right)^2-\frac{65}{16}}} =$$

$$= C' - \frac{5}{4}\sqrt{4x^2+9x+1} + \frac{61}{16}\ln\left|2x+\frac{9}{4}+\sqrt{4x^2+9x+1}\right| =$$

$$= C - \frac{5}{4}\sqrt{4x^2+9x+1} + \frac{61}{16}\ln\left|8x+9+4\sqrt{4x^2+9x+1}\right|.$$

278 (1953). $\displaystyle\int \frac{x\,dx}{\sqrt{3x^2-11x+2}} = \int \frac{\frac{1}{6}(6x-1)+\frac{11}{6}}{\sqrt{3x^2-11x+2}}\,dx =$

$$= \frac{1}{6}\int \frac{d(3x^2-11x+2)}{\sqrt{3x^2-11x+2}} + \frac{11}{6\sqrt{3}}\int \frac{dx}{\sqrt{\left(x-\frac{11}{6}\right)^2-\frac{117}{6}}} =$$

$$= \frac{1}{3}\sqrt{3x^2-11x+2} + \frac{11}{6\sqrt{3}}\ln\left|x-\frac{11}{6}\sqrt{3x^2-11x+2}\right| + C.$$

279 (1954). $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt{2x+3}} = \left[\frac{x}{2x+3}=z^2,\ x=\frac{3z^2}{1-2z^2},\ dx=\frac{6z}{(1-2z^2)^2}dz\right] =$

$$= \int \frac{6z^2dz}{(1-2z^2)^2} = \left[\begin{array}{ll} u=z, & du=dz, \\ dv=\dfrac{6z\,dz}{(1-2z^2)^2}, & v=-\dfrac{1}{4}\displaystyle\int\frac{d(1-2z^2)}{(1-2z^2)^2}=\frac{1}{4(1-2z^2)} \end{array}\right] =$$

$$=\frac{6}{4}\cdot\frac{z}{1-2z^2}-\frac{6}{4}\int\frac{dz}{1-2z^2}=\frac{3}{2}\cdot\frac{z}{1-2z^2}-\frac{3}{2}\cdot\frac{1}{2\sqrt{2}}\ln\frac{1+\sqrt{2}z}{1-\sqrt{2}z}+C=$$

$$=\frac{3}{2}\sqrt{\frac{x}{2x+3}}\cdot\frac{2x+3}{3}-\frac{3}{4\sqrt{2}}\ln\frac{1+\sqrt{2}\sqrt{\dfrac{x}{2x+3}}}{1-\sqrt{2}\sqrt{\dfrac{x}{2x+3}}}+C=\frac{1}{2}\sqrt{2x^2+3x}-$$

$$-\frac{3}{4\sqrt{2}}\ln\frac{1+2\sqrt{2}\sqrt{\dfrac{x}{2x+3}}+2\dfrac{x}{2x+3}}{\dfrac{3}{2x+3}}+C'=\frac{1}{2}\sqrt{2x^2+3x}-$$

$$-\frac{3}{4\sqrt{2}}\ln\frac{3+4x+4\sqrt{x^2+\dfrac{3}{2}x}}{3}+C'=$$

$$=\frac{1}{2}\sqrt{2x^2+3x}-\frac{3}{4\sqrt{2}}\ln\left|x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x}\right|+C\quad\left(C=C'+\ln\frac{4}{3}\right).$$

280 (1955). $$\int\sqrt{\frac{a-x}{x-b}}\,dx=\left[\frac{a-x}{x-b}=z^2,\ x=\frac{a+bz^2}{1+z^2},\ dx=\frac{2(b-a)z}{(1+z^2)^2}\,dz\right]=$$

$$=2(b-a)\int\frac{z^2dz}{(1+z^2)^2}=\left[\begin{array}{ll}u=z, & du=dz,\\ dv=\dfrac{z\,dz}{(1+z^2)^2}, & v=\dfrac{1}{2}\displaystyle\int\frac{d(z^2+1)}{(1+z^2)^2}=-\frac{1}{2(1+z^2)}\end{array}\right]=$$

$$=-(b-a)\frac{z}{1+z^2}+(b-a)\int\frac{dz}{1+z^2}=-(b-a)\frac{z}{1+z^2}+$$

$$+(b-a)\operatorname{arctg}z+C=(a-b)\sqrt{\frac{a-x}{x-b}}\cdot\frac{1}{1+\dfrac{a-x}{x-b}}+(b-a)\operatorname{arctg}\sqrt{\frac{a-x}{x-b}}+$$

$$+C=\sqrt{(a-x)(x-b)}+(b-a)\operatorname{arctg}\sqrt{\frac{a-x}{x-b}}+C.$$

281 (1956). $\int \operatorname{arctg} x\,dx = \left[\begin{array}{ll} u = \operatorname{arctg} x, & du = \dfrac{dx}{1+x^2}, \\ dv = dx, & v = x \end{array}\right] = x \operatorname{arctg} x -$

$$-\int \frac{x\,dx}{1+x^2} = \underline{\underline{x \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C}}.$$

282 (1957). $\int x \sin x \cdot \cos x\,dx =$

$$= \left[\begin{array}{ll} u = x, & du = dx, \\ dv = \sin x \cdot \cos x\,dx, & v = \int \sin x\, d(\sin x) = \dfrac{1}{2}\sin^2 x \end{array}\right] = \frac{x \sin^2 x}{2} -$$

$$-\frac{1}{2}\int \sin^2 x\,dx = \frac{x}{2}\sin^2 x - \frac{1}{4}\int (1 - \cos 2x)\,dx = \frac{x}{2}\sin^2 x - \frac{x}{4} +$$

$$+\frac{1}{8}\sin 2x + C = \frac{1}{8}\sin 2x - \frac{1}{4}x\,(1 - 2\sin^2 x) + C =$$

$$= \underline{\underline{\frac{1}{8}\sin 2x - \frac{1}{4}x \cos 2x + C}}.$$

283 (1958). $\int x^2 \cos \omega x\,dx = \left[\begin{array}{ll} u = x^2, & du = 2x\,dx, \\ dv = \cos \omega x\,dx, & v = \dfrac{\sin \omega x}{\omega} \end{array}\right] = \dfrac{x^2}{\omega}\sin \omega x -$

$$-\frac{2}{\omega}\int x \sin \omega x\,dx = \left[\begin{array}{ll} u = x, & du = dx, \\ dv = \sin \omega x\,dx, & v = -\dfrac{\cos \omega x}{\omega} \end{array}\right] = \frac{x^2}{\omega}\sin \omega x +$$

$$+\frac{2x}{\omega^2}\cos \omega x - \frac{2}{\omega^2}\int \cos \omega x\,dx = \underline{\underline{\frac{1}{\omega^3}((\omega^2 x^2 - 2)\sin \omega x + 2\omega x \cos \omega x) + C}}.$$

284 (1959). $\int e^{2x} x^3\,dx = \left[\begin{array}{ll} u = x^3, & du = 3x^2 dx, \\ dv = e^{2x} dx, & v = \dfrac{1}{2}e^{2x} \end{array}\right] = \dfrac{x^3}{2}e^{2x} -$

$$-\frac{3}{2}\int x^2e^{2x}dx=\begin{bmatrix} u=x^2, & du=2x\,dx, \\ dv=e^{2x}dx, & v=\frac{1}{2}e^{2x} \end{bmatrix}=\frac{x^3}{2}e^{2x}-\frac{3}{4}x^2e^{2x}+$$

$$+\frac{3}{2}\int xe^{2x}dx=\begin{bmatrix} u=x, & du=dx, \\ dv=e^{2x}dx, & v=\frac{1}{2}e^{2x} \end{bmatrix}=\frac{x^3}{2}e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}xe^{2x}-$$

$$-\frac{3}{4}\int e^{2x}dx=\underline{\underline{e^{2x}\left(\frac{x^3}{2}-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right)}}+C.$$

285 (1960). $\displaystyle\int\frac{\ln\cos x}{\cos^2 x}dx=\begin{bmatrix} u=\ln\cos x, & du=-\dfrac{\sin x}{\cos x}dx, \\ dv=\dfrac{dx}{\cos^2 x}, & v=\operatorname{tg}x \end{bmatrix}=\operatorname{tg}x\ln\cos x+$

$$+\int\frac{\sin^2 x}{\cos^2 x}dx=\underline{\underline{\operatorname{tg}x\ln\cos x+\operatorname{tg}x-x+C}}.$$

286 (1961). $\displaystyle\int\frac{\operatorname{ctg}x}{\ln\sin x}dx=\int\frac{\cos x\,dx}{\sin x\ln\sin x}=\int\frac{d(\ln\sin x)}{\ln\sin x}=\underline{\underline{\ln\left|\ln\sin x\right|+C}}.$

287 (1962). $\displaystyle\int\frac{x^7dx}{(1+x^4)^2}=\begin{bmatrix} u=x^4, & du=4x^3dx, \\ dv=\dfrac{x^3dx}{(1+x^4)^2}, & v=-\dfrac{1}{4(1+x^4)} \end{bmatrix}=-\frac{x^4}{4(1+x^4)}+$

$$+\frac{1}{4}\int\frac{4x^3dx}{1+x^4}=\frac{1}{4}\left(\ln(1+x^4)-\frac{x^4}{1+x^4}\right)+C'=\frac{1}{4}\left(\ln(1+x^4)-\frac{(1+x^4)-1}{1+x^4}\right)+$$

$$+C'=\underline{\underline{\frac{1}{4}\left(\ln(1+x^4)+\frac{1}{1+x^4}\right)+C}}.$$

288 (1963). $\displaystyle\int\frac{\cos^2 3x}{\sin 3x}dx=\int\frac{1-\sin^2 3x}{\sin 3x}dx=\int\frac{dx}{\sin 3x}-\int\sin 3x\,dx=$

$$=\int\frac{dx}{2\sin\frac{3x}{2}\cos\frac{3x}{2}}+\frac{1}{3}\cos 3x+C=\frac{1}{2}\int\frac{dx}{\operatorname{tg}\frac{3x}{2}\cos^2\frac{3x}{2}}+\frac{1}{3}\cos 3x+C=$$

$$=\frac{1}{3}\int\frac{d\left(\operatorname{tg}\frac{3x}{2}\right)}{\operatorname{tg}\frac{3x}{2}}+\frac{1}{3}\cos 3x+C=\underline{\underline{\frac{1}{3}\left(\ln\left|\operatorname{tg}\frac{3x}{2}\right|+\cos 3x\right)+C}}.$$

289 (1964). $\displaystyle\int\frac{dx}{1-\sin 3x}=\int\frac{dx}{1+\cos\left(\frac{\pi}{2}+3x\right)}=\int\frac{dx}{2\cos^2\left(\frac{\pi}{4}+\frac{3x}{2}\right)}=$

$$=\frac{1}{3}\int\frac{d\left(\frac{\pi}{4}+\frac{3x}{2}\right)}{\cos^2\left(\frac{\pi}{4}+\frac{3x}{2}\right)}=\underline{\underline{\frac{1}{3}\operatorname{tg}\left(\frac{\pi}{4}+\frac{3x}{2}\right)+C}}.$$

290 (1965). $\displaystyle\int\frac{\sin 2x\,dx}{4-\cos^2 2x}=\left[\begin{array}{c}\cos 2x=z,\\ -2\sin 2x\,dx=dz\end{array}\right]=-\frac{1}{2}\int\frac{dz}{4-z^2}=$

$$=-\frac{1}{8}\ln\frac{2+z}{2-z}+C=\underline{\underline{C-\frac{1}{8}\ln\frac{2+\cos 2x}{2-\cos 2x}}}.$$

291 (1966). $\displaystyle\int\frac{dx}{e^x+1}=\int\frac{(e^x+1)-e^x}{e^x+1}\,dx=\int dx-\int\frac{e^x dx}{e^x+1}=$

$$=x-\int\frac{d(e^x+1)}{e^x+1}=x-\ln(e^x+1)+C=\underline{\underline{\ln\frac{e^x}{e^x+1}+C}}$$

или $\displaystyle\int\frac{dx}{e^x+1}=\left[e^x+1=z,\ x=\ln(z-1),\ dx=\frac{dz}{z-1}\right]=\int\frac{dz}{z(z-1)}=$

$$=\int\frac{dz}{z-1}-\int\frac{dz}{z}=\ln\frac{z-1}{z}+C=\underline{\underline{\ln\frac{e^x}{e^x+1}+C}}.$$

292 (1967). $\int \frac{e^x - 1}{e^x + 1}\,dx = \int \frac{(e^x+1)-2}{e^x+1}\,dx = \int dx - 2\int \frac{dx}{e^x+1} =$

$$= x - 2\ln\frac{e^x}{e^x+1} + C = 2\ln e^{x/2} - 2\ln\frac{e^x}{e^x+1} + C =$$

$$= 2\ln\frac{e^{x/2}(e^x+1)}{e^x} + C = \underline{\underline{2\ln(e^{x/2} + e^{-x/2}) + C}}.$$

293 (1968). $\int e^{e^x + x}dx = \int e^{e^x} e^x\,dx = \int e^{e^x}\,d\,(e^x) = \underline{\underline{e^{e^x} + C}}.$

294 (1969). $\int e^{2x^2+\ln x}\,dx = \int e^{2x^2} e^{\ln x}\,dx = \int e^{2x^2} x\,dx = \frac{1}{4}\int e^{2x^2}\,d\,(2x^2) =$

$$= \underline{\underline{\frac{1}{4}e^{2x^2} + C}}.$$

295 (1970). $\int \frac{3+x^3}{\sqrt{2+2x^2}}\,dx = \frac{3}{\sqrt{2}}\int \frac{dx}{\sqrt{1+x^2}} + \frac{1}{\sqrt{2}}\int \frac{x^3dx}{\sqrt{1+x^2}} =$

$$= \frac{3}{\sqrt{2}}\ln\left|x+\sqrt{1+x^2}\right| + \frac{1}{\sqrt{2}}\int \frac{x^3dx}{\sqrt{1+x^2}} = \left[\begin{array}{ll} u = x^2, & du = 2x\,dx, \\ dv = \dfrac{x\,dx}{\sqrt{1+x^2}}, & v = \sqrt{1+x^2} \end{array}\right] =$$

$$= \frac{3}{\sqrt{2}}\ln\left|x+\sqrt{1+x^2}\right| + \frac{1}{\sqrt{2}}\left(x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,dx\right) =$$

$$= \frac{3}{\sqrt{2}}\ln\left|x+\sqrt{1+x^2}\right| + \frac{1}{\sqrt{2}}\left(x^2 - \frac{2}{3} - \frac{2}{3}x^2\right)\sqrt{1+x^2} + C =$$

$$= \underline{\underline{\frac{1}{\sqrt{2}}\left(3\ln\left|x+\sqrt{1+x^2}\right| + \frac{1}{3}(x^2-2)\sqrt{1+x^2}\right) + C}}.$$

296 (1971). $\displaystyle\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} u=\arcsin x, & du=\dfrac{dx}{\sqrt{1-x^2}}, \\ dv=\dfrac{x\,dx}{\sqrt{1-x^2}}, & v=-\sqrt{1-x^2} \end{bmatrix} =$

$$= -\sqrt{1-x^2}\arcsin x + \int dx = \underline{\underline{x-\sqrt{1-x^2}\arcsin x + C}}.$$

297 (1972). $\displaystyle\int \frac{x\cos x}{\sin^3 x}\,dx = \begin{bmatrix} u=x, & du=dx, \\ dv=\dfrac{\cos x}{\sin^3 x}\,dx, & v=\displaystyle\int\dfrac{d(\sin x)}{\sin^3 x}=-\dfrac{1}{2\sin^2 x} \end{bmatrix} =$

$$= -\frac{x}{2\sin^2 x}+\frac{1}{2}\int\frac{dx}{\sin^2 x} = \underline{\underline{C-\frac{1}{2}\left(\frac{x}{\sin^2 x}+\operatorname{ctg} x\right)}}.$$

298 (1973). $\displaystyle\int e^x\sin^2 x\,dx = \frac{1}{2}\int e^x(1-\cos 2x)\,dx =$

$$= \frac{1}{2}\left(e^x-\int e^x\cos 2x\,dx\right) = \frac{1}{2}(e^x - I)\text{, где}$$

$$I=\int e^x\cos 2x\,dx = \begin{bmatrix} u=e^x, & du=e^x dx, \\ dv=\cos 2x\,dx, & v=\dfrac{1}{2}\sin 2x \end{bmatrix} = \frac{e^x}{2}\sin 2x -$$

$$-\frac{1}{2}\int e^x\sin 2x\,dx = \begin{bmatrix} u=e^x, & du=e^x dx, \\ dv=\sin 2x\,dx, & v=-\dfrac{1}{2}\cos 2x \end{bmatrix} = \frac{e^x}{2}\sin 2x +$$

$$+\frac{e^x}{4}\cos 2x-\frac{1}{4}\int e^x\cos 2x\,dx = \frac{e^x}{2}\sin 2x+\frac{e^x}{4}\cos 2x-\frac{1}{4}I.$$

Таким образом, $\displaystyle I=\int e^x\cos 2x\,dx = \frac{2}{5}e^x\left(\sin 2x+\frac{1}{2}\cos 2x\right).$

$$\int e^x \sin^2 x\,dx = \frac{e^x}{2} - \frac{e^x}{5}\left(\sin 2x + \frac{1}{2}\cos 2x\right) + C =$$

$$= \underline{\underline{\frac{e^x}{2}\left(1 - \frac{2\sin 2x + \cos 2x}{5}\right) + C}}.$$

299 (1974). $\int \frac{(1+\operatorname{tg} x)\,dx}{\sin 2x} = \int \frac{dx}{2\sin x \cdot \cos x} + \int \frac{dx}{2\cos^2 x} = \int \frac{dx}{2\operatorname{tg} x \cdot \cos^2 x} +$

$$+ \int \frac{dx}{2\cos^2 x} = \underline{\underline{\frac{1}{2}(\ln|\operatorname{tg} x| + \operatorname{tg} x) + C}}.$$

300 (1975). $\int \frac{1-\operatorname{tg} x}{1+\operatorname{tg} x}\,dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx = \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} =$

$$= \underline{\underline{\ln|\sin x + \cos x| + C}}.$$

301 (1976). $\int \frac{d\varphi}{\sqrt{3}\cos\varphi + \sin\varphi} = \frac{1}{2}\int \frac{d\varphi}{\frac{\sqrt{3}}{2}\cos\varphi + \frac{1}{2}\sin\varphi} = \frac{1}{2}\int \frac{d\varphi}{\sin\left(\frac{\pi}{3} + \varphi\right)} =$

$$= \underline{\underline{\frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\varphi}{2} + \frac{\pi}{6}\right)\right| + C}}.$$

302 (1977). $\int \frac{\sin x\,dx}{1+\sin x} = \int \frac{(1+\sin x) - 1}{1+\sin x}\,dx = \int dx - \int \frac{dx}{1+\sin x} = x -$

$$-\int \frac{dx}{\sin 90^\circ + \sin x} = x - \int \frac{dx}{2\sin\left(45^\circ + \frac{x}{2}\right)\cos\left(45^\circ - \frac{x}{2}\right)} = x -$$

$$-\frac{1}{2}\int \frac{dx}{\cos^2\left(45^\circ - \frac{x}{2}\right)} = x - \operatorname{tg}\left(45^\circ - \frac{x}{2}\right) + C = x - \frac{\operatorname{tg} 45^\circ - \operatorname{tg}\frac{x}{2}}{1 + \operatorname{tg} 45^\circ \cdot \operatorname{tg}\frac{x}{2}} + C =$$

$$= x - \frac{1 - \operatorname{tg}\frac{x}{2}}{1 + \operatorname{tg}\frac{x}{2}} + C = x - \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} + C =$$

$$= x - \frac{\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) - \sin x}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} + C = x - \frac{1 - \sin x}{\cos x} + C =$$

$$= \underline{\underline{x - \sec x + \operatorname{tg} x + C}}.$$

303 (1978). $\displaystyle\int \frac{\sin^2 x \cdot \cos x}{(1 + \sin^2 x)}\,dx = \int \frac{(\sin^2 x + 1) - 1}{1 + \sin^2 x}\,d(\sin x) = \int d(\sin x) -$

$$-\int \frac{d(\sin x)}{1 + \sin^2 x} = \underline{\underline{\sin x - \operatorname{arctg}(\sin x) + C}}.$$

304 (1979). $\displaystyle\int \frac{\sqrt{1 + \cos x}}{\sin x}\,dx = \sqrt{2}\int \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2} \cdot \cos\frac{x}{2}}\,dx = \sqrt{2}\int \frac{dx}{\sin\frac{x}{2}} =$

$$= \underline{\underline{\sqrt{2}\ln \operatorname{tg}\left|\frac{x}{4}\right| + C}}.$$

305 (1980). $\displaystyle\int \frac{\ln\ln x}{x}\,dx = \int \ln\ln x\, d(\ln x) = \begin{bmatrix} u = \ln\ln x, & du = \dfrac{dx}{x\ln x}, \\ dv = d(\ln x), & v = \ln x \end{bmatrix} =$

$$= \ln x \ln\ln x - \int \frac{dx}{x} = \underline{\underline{\ln x(\ln\ln x - 1) + C}}.$$

306 (1981). $\displaystyle\int x^3 e^{x^2}\,dx = \begin{bmatrix} u = x^2, & du = 2x\,dx, \\ dv = xe^{x^2}\,dx, & v = \dfrac{1}{2}e^{x^2} \end{bmatrix} = \frac{x^2}{2}e^{x^2} - \int xe^{x^2}\,dx =$

$$= \frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C = \underline{\underline{\frac{e^{x^2}(x^2 - 1)}{2} + C}}.$$

307 (1982). $\int e^{-x^2}x^5dx=\left[\begin{array}{ll} u=x^4, & du=4x^3dx, \\ dv=xe^{-x^2}dx, & v=-\frac{1}{2}\int e^{-x^2}d(-x^2)=-\frac{1}{2}e^{-x^2}\end{array}\right]=$

$$=-\frac{x^4}{2}e^{-x^2}+2\int x^3e^{-x^2}dx=\left[\begin{array}{ll} u=x^2, & du=2x\,dx, \\ dv=xe^{-x^2}dx, & v=-\frac{1}{2}e^{-x^2}\end{array}\right]=$$

$$=-\frac{x^4}{2}e^{-x^2}-x^2e^{-x^2}+2\int xe^{-x^2}dx=-\frac{x^4}{2}e^{-x^2}-x^2e^{-x^2}-e^{-x^2}+C=$$

$$=\underline{\underline{C-\frac{1}{2}e^{x^2}(x^4+2x^2+2).}}$$

308 (1983). $\int\frac{x^3dx}{\sqrt{1+2x^2}}=\left[\begin{array}{l} x=\frac{\operatorname{tg} z}{\sqrt{2}},\ dx=\frac{dz}{\sqrt{2}\cos^2 z}, \\ 1+2x^2=1+\operatorname{tg}^2 z=\frac{1}{\cos^2 z}\end{array}\right]=$

$$=\int\frac{\operatorname{tg}^3 z\cdot\cos z\,dz}{2\sqrt{2}\cdot\sqrt{2}\cos^2 z}=\frac{1}{4}\int\frac{\sin^3 z}{\cos^4 z}dz=\frac{1}{4}\int\frac{(1-\cos^2 z)\sin z\,dz}{\cos^4 z}=$$

$$=\frac{1}{4}\left(-\int\frac{d(\cos z)}{\cos^4 z}+\int\frac{d(\cos z)}{\cos^2 z}\right)=\frac{1}{4}\left(\frac{1}{\cos^3 z}-\frac{1}{\cos z}\right)+C=$$

$$=\frac{1}{4}\left(\frac{1}{3}(1+2x^2)\sqrt{1+2x^2}-\sqrt{1+2x^2}\right)+C=\underline{\underline{\frac{1}{6}(x^2-1)\sqrt{1+2x^2}+C.}}$$

309 (1984). $\int\frac{x^4dx}{\sqrt{(1-x^2)^3}}=\left[\begin{array}{l} x=\sin z, \\ dx=\cos z\,dz\end{array}\right]=\int\frac{\sin^4 z\cdot\cos z}{\cos^3 z}dz=$

$$=\int\frac{\sin^4 z}{\cos^2 z}dz=\int\frac{(1-\cos^2 z)^2}{\cos^2 z}dz=\int\frac{1-2\cos^2 z+\cos^4 z}{\cos^2 z}dz=$$

$$=\int\frac{dz}{\cos^2 z}-2\int dz+\int\cos^2 z\,dz=\operatorname{tg} z-2z+\int\frac{1+\cos 2z}{2}dz=$$

$$= \operatorname{tg} z - \frac{3}{2}z + \frac{1}{4}\sin 2z + C = \frac{x}{\sqrt{1-x^2}} - \frac{3}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C =$$

$$= C - \frac{x(x^2-3)}{2\sqrt{1-x^2}} - \frac{3}{2}\arcsin x.$$

310 (1985). $$\int \frac{\sqrt{(x^2-a^2)^5}}{x}dx = \begin{bmatrix} x = \dfrac{a}{\cos z},\ dx = a\dfrac{\sin z}{\cos^2 z}dz, \\ x^2 - a^2 = a^2\left(\dfrac{1}{\cos^2 z} - 1\right) = \dfrac{a^2\sin^2 z}{\cos^2 z} \end{bmatrix} =$$

$$= a^5 \int \frac{\sin^5 z \cdot a\sin z \cdot \cos z}{\cos^5 z \cdot \cos^2 z \cdot a}dz = a^5 \int \frac{\sin^6 z}{\cos^6 z}dz = a^5 \int \operatorname{tg}^6 z\,dz =$$

$$= \begin{bmatrix} \operatorname{tg} z = t,\ z = \operatorname{arctg} t, \\ dz = \dfrac{dt}{1+t^2} \end{bmatrix} = a^5 \int \frac{t^6 dt}{1+t^2} = a^5 \int \frac{((t^2)^3+1)-1}{1+t^2}dt =$$

$$= a^5 \int \frac{(t^2+1)(t^4-t^2+1)}{1+t^2}dt - a^5 \int \frac{dt}{1+t^2} = a^5\left(\frac{t^5}{5} - \frac{t^3}{3} + t - \operatorname{arctg} t\right) +$$

$$+ C = a^5\left(\frac{\operatorname{tg}^5 z}{5} - \frac{\operatorname{tg}^3 z}{3} + \operatorname{tg} z - \operatorname{arctg}(\operatorname{tg} z)\right) + C = a^5\left(\frac{1}{5}\operatorname{tg}^5\left(\arccos\frac{a}{x}\right) -\right.$$

$$\left. - \frac{1}{3}\operatorname{tg}^3\left(\arccos\frac{a}{x}\right) + \operatorname{tg}\left(\arccos\frac{a}{x}\right) - z\right) + C = a^5\left(\frac{1}{5}\left(\frac{\sqrt{1-\frac{a^2}{x^2}}}{\frac{a}{x}}\right)^5 - \right.$$

$$\left. - \frac{1}{3}\left(\frac{\sqrt{1-\frac{a^2}{x^2}}}{\frac{a}{x}}\right)^3 + \frac{\sqrt{1-\frac{a^2}{x^2}}}{\frac{a}{x}} - z\right) + C = \frac{1}{5}\sqrt{(x^2-a^2)^5} - \frac{a^2}{3}\sqrt{(x^2-a^2)^3} +$$

$$+a^4\sqrt{x^2-a^2}-a^5\arccos\frac{a}{x}+C=\left[\begin{array}{c}\arccos\frac{a}{x}=\frac{\pi}{2}-\arcsin\frac{a}{x},\\ C_1=C-\frac{a^5\pi}{2}\end{array}\right]=$$

$$=\frac{1}{5}\sqrt{(x^2-a^2)^5}-\frac{a^2}{3}\sqrt{(x^2-a^2)^3}+a^4\sqrt{x^2-a^2}+a^5\arcsin\frac{a}{x}+C_1.$$

311 (1986). $\displaystyle\int\frac{dx}{x^4\sqrt{x^2+4}}=\left[\begin{array}{c}x=2\operatorname{tg}z,\\ dx=\frac{2dz}{\cos^2 z}\end{array}\right]=\int\frac{2dz}{\cos^2 z\cdot 4^2\frac{\sin^4 z}{\cos^4 z}\cdot 2\frac{1}{\cos z}}=$

$$=\frac{1}{4}\int\frac{\cos^3 z\,dz}{\sin^4 z}=\frac{1}{4}\int\frac{(1-\sin^2 z)\cos z\,dz}{\sin^4 z}=\frac{1}{4}\left(\int\frac{d(\sin z)}{\sin^4 z}-\int\frac{d(\sin z)}{\sin^2 z}\right)=$$

$$=\left(-\frac{1}{3\sin^3 z}+\frac{1}{\sin z}\right)+C=\frac{1}{4}\left(-\frac{1}{3}\frac{\sqrt{(1+\operatorname{tg}^2 z)^3}}{\operatorname{tg}^3 z}+\frac{\sqrt{1+\operatorname{tg}^2 z}}{\operatorname{tg} z}\right)=$$

$$=\frac{1}{4}\frac{\sqrt{1+\operatorname{tg}^2 z}}{\operatorname{tg} z}\left(-\frac{1}{3}\cdot\frac{1+\operatorname{tg}^2 z}{\operatorname{tg}^2 z}+1\right)+C=\frac{1}{4}\cdot\frac{\sqrt{1+\operatorname{tg}^2 z}}{\operatorname{tg} z}\cdot\frac{2\operatorname{tg}^2 z-1}{3\operatorname{tg}^2 z}+C=$$

$$=\frac{1}{4}\cdot\frac{\sqrt{1+x^2}}{4x}\cdot\frac{(x^2-2)4}{2\cdot 3\cdot x^2}+C=\frac{\sqrt{1+x^2}\,(x^2-2)}{24x^3}+C.$$

312 (1987). $\displaystyle\int\frac{\sqrt{x^2-8}}{x^4}\,dx=$

$$=\left[\begin{array}{l}x=\sqrt{8}\sec z,\ dz=\frac{\sqrt{8}\sin z}{\cos^2 z}\,dz,\ \cos z=\frac{\sqrt{8}}{x},\\ x^2-8=\frac{8\sin^2 z}{\cos^2 z},\ \sin z=\sqrt{1-\frac{8}{x^2}}=\frac{\sqrt{x^2-8}}{x}\end{array}\right]=$$

$$=\int \frac{\sqrt{8}\sin z\cdot\sqrt{8}\sin z\cdot\cos^4 z}{\cos z\cdot\cos^2 z\cdot 8^2}\,dz=\frac{1}{8}\int\sin^2 z\,d(\sin z)=\frac{1}{8}\cdot\frac{\sin^3 z}{3}+C=$$

$$=\underline{\underline{\frac{\sqrt{(x^2-8)^3}}{24x^3}}}+C.$$

313 (1988). $\displaystyle\int\frac{\sqrt{4+x^2}}{x^6}\,dx=\left[x=2\,\mathrm{tg}\,z,\ dx=\frac{2dz}{\cos^2 z},\ 4+x^2=\frac{4}{\cos^2 z}\right]=$

$$=\int\frac{2\cos^6 z\cdot 2dz}{\cos z\cdot 64\sin^6 z\cdot\cos^2 z}=\frac{1}{16}\int\frac{\cos^3 z}{\sin^6 z}dz=\frac{1}{16}\int\frac{(1-\sin^2 z)d(\sin z)}{\sin^6 z}=$$

$$=\frac{1}{16}\left(\frac{d(\sin z)}{\sin^6 z}-\int\frac{d(\sin z)}{\sin^4 z}\right)=\frac{1}{16}\left(-\frac{1}{5\sin^5 z}+\frac{1}{3\sin^3 z}\right)+C=$$

$$=\frac{1}{16}\left(-\frac{\sqrt{(1+\mathrm{tg}^2 z)^5}}{5\,\mathrm{tg}^5 z}+\frac{\sqrt{1+\mathrm{tg}^2 z}}{3\,\mathrm{tg}^3 z}\right)+C=$$

$$=\frac{1}{16}\cdot\frac{\sqrt{(4+x^2)^3}}{x^3}\left(\frac{1}{3}-\frac{1}{5}\cdot\frac{(4+x^2)}{x^2}\right)+C=\underline{\underline{\frac{\sqrt{(4+x^2)^3}\,(x^2-6)}{120x^5}}}+C.$$

314 (1989). $\displaystyle\int\frac{dx}{x^4\sqrt{x^2-3}}=\left[\begin{array}{c}x=\dfrac{\sqrt{3}}{\cos z},\ dx=\dfrac{\sqrt{3}\sin z\,dz}{\cos^2 z},\\ x^2-3=3\dfrac{\sin^2 z}{\cos^2 z}\end{array}\right]=$

$$=\int\frac{\sqrt{3}\sin z\cdot\cos^4 z\cdot\cos z}{\cos^2 z\cdot 9\cdot\sqrt{3}\sin z}\,dz=\frac{1}{9}\int\cos^3 z\,dz=\frac{1}{9}\int(1-\sin^2 z)\,d(\sin z)=$$

$$=\frac{1}{9}\left(\sin z-\frac{\sin^3 z}{3}\right)+C=\frac{1}{9}\cdot\frac{\sin z}{3}(3-\sin^2 z)+C=$$

$$=\frac{1}{27}\cdot\frac{\sqrt{x^2-3}}{x}\left(3-\frac{x^2-3}{x^2}\right)+C=\underline{\underline{\frac{\sqrt{x^2-3}}{27x^3}(2x^2+3)}}+C.$$

315 (1990). $\int \frac{\sqrt{x}\,dx}{\sqrt[4]{x^3}+1} = \left[\begin{array}{c} x = z^4, \\ dx = 4z^3\,dz \end{array}\right] = 4\int \frac{z^2 \cdot z^3}{z^3+1}\,dz = 4\int z^2 dz -$

$-4\int \frac{z^2}{z^3+1}\,dz = 4\left(\frac{z^3}{3} - \frac{1}{3}\ln\left|z^3+1\right|\right) = \underline{\underline{\frac{4}{3}\left(\sqrt[4]{x^3} - \ln\left|\sqrt[4]{x^3}+1\right|\right) + C}}.$

316 (1991). $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx = \left[\sqrt{x+1} = z,\ x+1 = z^2,\ dx = 2z\,dz\right] =$

$= 2\int z\frac{z+1}{z-1}\,dz = 2\int \frac{(z^2-1)+(z-1)+2}{z-1}\,dz = 2\left(\int (z+1)\,dz + \right.$

$\left. + \int dz + 3\int \frac{dz}{z-1}\right) + C = 2\frac{(z+1)^2}{2} + 2z + 4\ln|z-1| + C = (x+1) +$

$+ 2\sqrt{x+1} + 1 + 2\sqrt{x+1} + 4\ln\left|\sqrt{x+1}-1\right| + C_1 =$

$= \underline{\underline{x + 4\sqrt{x+1} + 4\ln\left|\sqrt{x+1}-1\right| + C}}.$

317 (1992). $\int \frac{dx}{(2+x)\sqrt{1+x}} = \left[\begin{array}{c} 1+x = z^2, \\ dx = 2z\,dz \end{array}\right] = \int \frac{2z\,dz}{(1+z^2)\,z} = 2\int \frac{dz}{1+z^2} =$

$= 2\,\mathrm{arctg}\, z + C = \underline{\underline{2\,\mathrm{arctg}\sqrt{1+x} + C}}.$

318 (1993). $\int \frac{\sqrt[3]{x}\,dx}{x\left(\sqrt{x}+\sqrt[3]{x}\right)} = \left[x = z^6,\ dx = 6z^5 dz\right] = 6\left(\int \frac{dz}{z} - \int \frac{dz}{1+z}\right) =$

$= 6\ln\frac{z}{1+z} + C = \underline{\underline{\ln\frac{x}{\left(\sqrt[6]{x}+1\right)^6} + C}}.$

319 (1994). $\int \frac{\sqrt{x^2+2x}}{x}\,dx = \left[\begin{array}{c} \sqrt{x^2+2x} = \sqrt{(x+1)^2-1}, \\ x+1 = z,\ dx = dz \end{array}\right] = \int \frac{\sqrt{z^2-1}}{z-1}\,dz =$

$$=\left[z=\frac{1}{\cos z},\ dz=\frac{\sin t\,dt}{\cos^2 t},\ \ \sqrt{z^2-1}=\frac{\sin t}{\cos t}\right]=\int\frac{\sin t\cdot\sin t\cdot\cos t\,dt}{\cos t\cdot\cos^2 t(1-\cos t)}=$$

$$=\int\frac{(1-\cos t)\,(1+\cos t)}{\cos^2 t\,(1-\cos t)}\,dt=\int\frac{dt}{\cos^2 t}+\int\frac{dt}{\cos t}=\operatorname{tg} t+\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{t}{2}\right)\right|=$$

$$=\frac{\sin\left(\arccos\frac{1}{z}\right)}{\cos\left(\arccos\frac{1}{z}\right)}+\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{1}{2}\arccos\frac{1}{z}\right)\right|+C=\frac{\sqrt{1-\frac{1}{z^2}}}{\frac{1}{z}}+$$

$$+\ln\left|\frac{\operatorname{tg}\frac{\pi}{4}+\dfrac{\sin\left(\frac{1}{2}\arccos\frac{1}{z}\right)}{\cos\left(\frac{1}{2}\arccos\frac{1}{z}\right)}}{1-\operatorname{tg}\frac{\pi}{4}\cdot\dfrac{\sin\left(\frac{1}{2}\arccos\frac{1}{z}\right)}{\cos\left(\frac{1}{2}\arccos\frac{1}{z}\right)}}\right|+C=\sqrt{x^2+2x}+\ln\left|\frac{1+\sqrt{\dfrac{1-\frac{1}{z}}{1+\frac{1}{z}}}}{1-\sqrt{\dfrac{1-\frac{1}{z}}{1+\frac{1}{z}}}}\right|+$$

$$+C=\sqrt{x^2+2x}+\ln\left|\frac{\sqrt{z+1}+\sqrt{z-1}}{\sqrt{z+1}-\sqrt{z-1}}\right|+C=\sqrt{x^2+2x}+$$

$$+\ln\left|z+\sqrt{z^2-1}\right|+C=\underline{\underline{\sqrt{x^2+2x}+\ln\left|x+1+\sqrt{x^2+2x}\right|+C}}.$$

320 (1995). $\displaystyle\int\frac{x^7dx}{(1-x^2)^5}=\left[\begin{matrix}x=\sin z,\\ dx=\cos z\,dz\end{matrix}\right]=\int\frac{\sin^7 z\cos z}{\cos^{10} z}\,dz=\int\frac{\sin^7 z\,dz}{\cos^9 z}=$

$$=-\int\frac{(1-\cos^2 z)^3\,d(\cos z)}{\cos^9 z}=\int\frac{-1+3\cos^2 z-3\cos^4 z+\cos^6 z}{\cos^9 z}\,d(\cos z)=$$

$$=-\int\frac{d(\cos z)}{\cos^9 z}+3\int\frac{d(\cos z)}{\cos^7 z}-3\int\frac{d(\cos z)}{\cos^5 z}+\int\frac{d(\cos z)}{\cos^3 z}=\frac{1}{8\cos^8 z}-$$

$$-\frac{1}{2\cos^6 z}+\frac{3}{4\cos^2 z}-\frac{1}{2\cos^2 z}+C=\frac{1-4\cos^2 z+6\cos^4 z-4\cos^6 z}{8\cos^8 z}+$$

$$+C=\underline{\underline{\frac{1-4(1-x^2)+6(1-x^2)^2-4(1-x^2)^3}{8(1-x^2)^8}+C}}.$$

321 (1996). $\displaystyle\int\frac{dx}{(ax+b)\sqrt{x}}=\left[\sqrt{x}=z,\ dx=2z\,dz\right]=2\int\frac{z\,dz}{(az^2+b)\,z}=$

$$=2\int\frac{dz}{az^2+b}=\frac{2}{\sqrt{a}}\int\frac{d\left(\sqrt{a}z\right)}{\left(\sqrt{a}z\right)^2+\left(\sqrt{b}\right)^2}=\frac{2}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\operatorname{arctg}\frac{\sqrt{a}z}{\sqrt{b}}+C=$$

$$=\underline{\underline{\frac{2}{\sqrt{ab}}\operatorname{arctg}\sqrt{\frac{ax}{b}}+C}}.$$

322 (1997). $\displaystyle\int\frac{\sqrt{1+x^8}}{x^{13}}\,dx=\left[\begin{matrix}\sqrt{1+x^8}=z,\ 1+x^8=z^2,\\ x=\sqrt[8]{z^2-1},\ dx=\dfrac{1}{8}\cdot\dfrac{2z\,dz}{\sqrt[8]{(z^2-1)^7}}\end{matrix}\right]=$

$$=\frac{1}{4}\int\frac{z^2dz}{\sqrt[8]{(z^2-1)^{13}}\cdot\sqrt[8]{(z^2-1)^7}}=\frac{1}{4}\int\frac{z^2\,dz}{\sqrt{(z^2-1)^5}}=$$

$$=\left[\begin{matrix}z=\sec t,\ dz=\dfrac{\sin t}{\cos^2 t}\,dt,\\ \cos t=\dfrac{1}{z},\ \sin t=\dfrac{\sqrt{z^2-1}}{z}\end{matrix}\right]=\frac{1}{4}\int\frac{\sin t\cdot\cos^5 t\,dt}{\cos^2 t\cdot\cos^2 t\cdot\sin^5 t}=\frac{1}{4}\int\frac{d(\sin t)}{\sin^4 t}=$$

$$=-\frac{1}{4}\cdot\frac{1}{3\sin^3 t}+C=C-\frac{z^3}{12\sqrt{(z^2-1)^3}}=\underline{\underline{C-\frac{(1+x^8)^{3/2}}{12x^{12}}}}.$$

323 (1998). $$\int \frac{x\,dx}{(1-x^4)^{3/2}}=\left[\begin{array}{c}\sqrt{1-x^4}=z,\ 1-x^4=z^2,\\ x=\sqrt[4]{1-z^2},\ dx=\dfrac{-2z\,dz}{4\sqrt[4]{(1-z^2)^3}}\end{array}\right]=$$

$$=-\int \frac{\sqrt[4]{1-z^2}\cdot 2z\,dz}{4\sqrt[4]{(1-z^2)^3}\cdot z^3}=-\frac{1}{2}\int \frac{dz}{z^2\cdot\sqrt{1-z^2}}=\left[\begin{array}{c}z=\sin t,\\ dz=\cos t\,dt\end{array}\right]=$$

$$=-\frac{1}{2}\int \frac{\cos t\,dt}{\cos t\cdot\sin^2 t}=\frac{1}{2}\operatorname{ctg} t+C=\frac{1}{2}\cdot\frac{\cos t}{\sin t}+C=\frac{1}{2}\cdot\frac{\sqrt{1-z^2}}{z}+C=$$

$$=\frac{1}{2}\cdot\frac{x^2}{\sqrt{1-x^4}}+C.$$

324 (1999). $$\int \frac{x^5\,dx}{\sqrt{x^4+4}}=\left[\begin{array}{c}\sqrt{x^4+4}=z,\ x^4+4=z^2,\\ x=\sqrt[4]{z^2-4},\ dx=\dfrac{2z\,dz}{4\cdot\sqrt[4]{(z^2-4)^3}}\end{array}\right]=$$

$$=\int \frac{2\sqrt[4]{(z^2-4)^5}\,z}{4\cdot\sqrt[4]{(z^2-4)^3}}dz=\frac{1}{2}\int\sqrt{z^2-4}\;dz=\frac{1}{2}\int\frac{z^2-4}{\sqrt{z^2-4}}\,dz=$$

$$=\frac{1}{2}\left(\int\frac{z\cdot z\,dz}{\sqrt{z^2-4}}-4\int\frac{dz}{\sqrt{z^2-4}}\right)=$$

$$=\left[\begin{array}{cc}u=z, & du=dz,\\ dv=\dfrac{z\,dz}{\sqrt{z^2-4}}, & v=\dfrac{1}{2}\displaystyle\int\frac{d(z^2-4)}{\sqrt{z^2-4}}=\sqrt{z^2-4}\end{array}\right]=\frac{1}{2}z\sqrt{z^2-4}-$$

$$-\frac{1}{2}\int\sqrt{z^2-4}dz-2\ln\left|z+\sqrt{z^2-4}\right|+C,\ \text{откуда}$$

$$\int\sqrt{z^2-4}\;dz=\frac{1}{2}z\sqrt{z^2-4}-2\ln\left|z+\sqrt{z^2-4}\right|+C=\frac{1}{2}x^2\sqrt{x^4+4}-$$

$-2\ln\left|x^2+\sqrt{x^4+4}\right|+C$; тогда

$$\int\frac{x^5\,dx}{\sqrt{x^4+4}}=\frac{1}{2}\int\sqrt{z^2-4}\,dz=\underline{\underline{\frac{1}{4}x^2\sqrt{x^4+4}-\ln\left|x^2+\sqrt{x^4+4}\right|+C}}.$$

325 (2000). $\displaystyle\int\frac{dx}{\sqrt{x}(x-1)}=\left[\begin{array}{c}\sqrt{x}=z,\ x=z^2,\\ dx=2z\,dz\end{array}\right]=\int\frac{2z\,dz}{z(z^2-1)}=2\int\frac{dz}{z^2-1}=$

$$=\ln\frac{z-1}{z+1}+C=\underline{\underline{\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|+C}}.$$

326 (2001). $\displaystyle\int\frac{\sqrt{1-x^3}}{x^2\sqrt{x}}=\left[\sqrt{x^3}=z,\ x=\sqrt[3]{z^2},\ dx=\frac{2}{3\sqrt[3]{z}}\,dz\right]=$

$$=\frac{2}{3}\int\frac{\sqrt{1-z^2}\;dz}{\sqrt[3]{z}\cdot\sqrt[3]{z^4}\cdot\sqrt[3]{z}}=\int\frac{\sqrt{1-z^2}}{z^2}\,dz=\left[z=\sin t,\ dz=\cos t\,dt\right]=$$

$$=\frac{2}{3}\int\frac{\cos^2 t}{\sin^2 t}\,dt=\frac{2}{3}\int\frac{dt}{\sin^2 t}-\frac{2}{3}\int dt=-\frac{2}{3}\operatorname{ctg} t-\frac{2}{3}t+C=$$

$$=\underline{\underline{C-\frac{2}{3}\sqrt{\frac{1-x^3}{x^3}}-\frac{2}{3}\arcsin\sqrt{x^3}}}.$$

327 (2002). $\displaystyle\int\frac{x^4\,dx}{(1+x^2)^3}=\left[\begin{array}{cc}u=x^3, & du=3x^2\,dx,\\ dv=\dfrac{x\,dx}{(1+x^2)^3}, & v=-\dfrac{1}{4(1+x^2)^2}\end{array}\right]=-\frac{x^3}{4(1+x^2)}+$

$$+\frac{3}{4}\int\frac{x^2\,dx}{(1+x^2)^2}=\left[\begin{array}{cc}u=x, & du=dx,\\ dv=\dfrac{x\,dx}{(1+x^2)^3}, & v=-\dfrac{1}{2(1+x^2)^2}\end{array}\right]=-\frac{x^3}{4(1+x^2)}-$$

$$-\frac{3}{8}\frac{x}{(1+x^2)}+\frac{3}{8}\int\frac{dx}{1+x^2}=\underline{\underline{-\frac{x^3}{4(1+x^2)}-\frac{3}{8}\cdot\frac{x}{1+x^2}+\frac{3}{8}\operatorname{arctg} x+C}}.$$

328 (2003). $\displaystyle\int \frac{3x^2-1}{2x\sqrt{x}}\operatorname{arctg} x\,dx = \left[\begin{array}{ll} u=\operatorname{arctg} x, & du=\dfrac{dx}{1+x^2}, \\ dv=\dfrac{3x^2-1}{2x\sqrt{x}}\,dx, & v=\dfrac{x^2+1}{\sqrt{x}} \end{array}\right] =$

$$= \frac{(x^2+1)\operatorname{arctg} x}{\sqrt{x}} - \int\frac{dx}{\sqrt{x}} = \underline{\underline{\frac{(x^2+1)\operatorname{arctg} x}{\sqrt{x}} - 2\sqrt{x} + C}}.$$

329 (2004). $\displaystyle\int \frac{e^x(1+e^x)}{\sqrt{1-e^{2x}}}\,dx = \int\frac{e^x dx}{\sqrt{1-e^{2x}}} + \int\frac{e^{2x}dx}{\sqrt{1-e^{2x}}} = \int\frac{d(e^x)}{\sqrt{1-(e^x)^2}} -$

$$-\frac{1}{2}\int\frac{d(1-e^{2x})}{\sqrt{1-e^{2x}}} = \underline{\underline{\arcsin e^x - \frac{1}{2}\cdot 2\sqrt{1-e^{2x}} + C}}.$$

330 (2005). $\displaystyle\int\sqrt{e^x-1}\,dx = \int\frac{e^x-1}{\sqrt{e^x-1}}\,dx = \int\frac{d(e^x-1)}{\sqrt{e^x-1}} - \int\frac{dx}{\sqrt{e^x-1}} =$

$$= 2\sqrt{e^x-1} - 2\int d\left(\operatorname{arctg}\sqrt{e^x-1}\right) = \underline{\underline{2\sqrt{e^x-1} - 2\operatorname{arctg}\sqrt{e^x-1} + C}}.$$

331 (2006). $\displaystyle\int\frac{\ln(x+1)-\ln x}{x(x+1)}\,dx = \int\frac{\ln\left(1+\frac{1}{x}\right)}{x(x+1)}\,dx = \int\frac{\ln\left(1+\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x}\right)}\,dx =$

$$= \left[z=1+\frac{1}{x},\ \ dz=-\frac{dx}{x^2}\right] = -\int\frac{\ln z\,dz}{z} = -\int\ln z\,d(\ln z) = -\frac{1}{2}\ln^2 z + C =$$

$$= \underline{\underline{-\frac{1}{2}\ln^2\left(1+\frac{1}{x}\right) + C}}.$$

332 (2007). $\displaystyle\int\frac{dx}{x^6+x^4} = \int\frac{dx}{x^4(1+x^2)} = \int\left(\frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{1+x^2}\right)dx =$

$$= \underline{\underline{-\frac{1}{3x^2} + \frac{1}{x} + \operatorname{arctg} x + C}}.$$

333 (2008). $\int \arccos\sqrt{\frac{x}{x+1}}\,dx =$

$$= \begin{bmatrix} u=\arccos\sqrt{\frac{x}{x+1}}, & du=-\frac{(x+1-x)\sqrt{x+1}\,dx}{\sqrt{1-\frac{x}{x+1}}\cdot 2(x+1)^2\sqrt{x}}=-\frac{1}{2\sqrt{x}(x+1)}\,dx, \\ dv=dx, & v=x \end{bmatrix} =$$

$$= x\arccos\sqrt{\frac{x}{x+1}}+\int\frac{x}{2\sqrt{x}(x+1)}\,dx = x\arccos\sqrt{\frac{x}{x+1}}+\frac{1}{2}\left(\int\frac{dx}{\sqrt{x}}-\right.$$

$$\left.-\int\frac{dx}{\sqrt{x}(x+1)}\right)= x\arccos\sqrt{\frac{x}{x+1}}+\int\frac{dx}{2\sqrt{x}}-\int\frac{d\left(\sqrt{x}\right)}{1+\left(\sqrt{x}\right)^2}=$$

$$= \underline{\underline{x\arccos\sqrt{\frac{x}{x+1}}+\sqrt{x}-\operatorname{arctg}\sqrt{x}+C.}}$$

334 (2009). $\int \ln\left(x+\sqrt{1+x^2}\right)dx = \begin{bmatrix} u=\ln\left(x+\sqrt{1+x^2}\right), & du=\frac{dx}{\sqrt{1+x^2}}, \\ dv=dx, & v=x \end{bmatrix} =$

$$= x\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x\,dx}{\sqrt{1+x^2}} = \underline{\underline{x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C.}}$$

335 (2010). $\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}}\,dx = \begin{bmatrix} \operatorname{tg} x=z,\ \cos^2 x=\frac{1}{1+z^2}, \\ \sin^2 x=\frac{z^2}{1+z^2},\ dx=\frac{dz}{1+z^2} \end{bmatrix} =$

$$= \int\sqrt[3]{\frac{z^2(1+z^2)^7}{1+z^2}}\cdot\frac{dz}{1+z^2} = \int(1+z^2)\,z^{2/3}dz = \frac{3}{5}z^{5/3}+\frac{3}{11}z^{11/3}+C=$$

$$= \frac{3}{55}z^{5/3}(5z^2+11)+C = \underline{\underline{\frac{3}{55}\sqrt[3]{\operatorname{tg}^5 x}\,(5\operatorname{tg}^2 x+11)+C.}}$$

336 (2011). $\displaystyle\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}} = \int \frac{dx}{\cos^3 x\sqrt{2\sin x\cdot\cos x}} =$

$$= \left[\cos x = \frac{1}{\sqrt{1+\operatorname{tg}^2 x}},\ \sin x = \frac{\operatorname{tg} x}{\sqrt{1+\operatorname{tg}^2 x}} \right] = \int \frac{(1+\operatorname{tg}^2 x)^2\,dx}{\sqrt{2\operatorname{tg} x}} =$$

$$= \left[\operatorname{tg} x = t,\ dx = \cos^2 x\,dt = \frac{1}{1+\operatorname{tg}^2 x}\,dt = \frac{dt}{1+\mathrm{t}^2} \right] = \frac{1}{\sqrt{2}}\int \frac{(1+t^2)^2\,dt}{\sqrt{t}\,(1+t^2)} =$$

$$= \frac{1}{\sqrt{2}}\left(\int \frac{dt}{\sqrt{t}} + \int t^{3/2}dt \right) = \frac{1}{\sqrt{2}}\left(2\sqrt{t} + \frac{2}{5}t^{5/2} \right) + C = \frac{\sqrt{2}}{5}(t^2+5)\sqrt{t} + C =$$

$$= \underline{\underline{\frac{\sqrt{2}}{5}(\operatorname{tg}^2 x+5)\sqrt{\operatorname{tg} x} + C}}.$$

§ 3. ОСНОВНЫЕ КЛАССЫ ИНТЕГРИРУЕМЫХ ФУНКЦИЙ

3.1. Дробно-рациональные функции

В задачах **337** (2012)–**392** (2067) найти интегралы.

1. Знаменатель имеет только действительные различные корни.

337 (2012). $\displaystyle\int \frac{x\,dx}{(x+1)(2x+1)} = I.$

$$\frac{x}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1},$$

$$x = A(2x+1) + B(x+1),$$

$$x = -\frac{1}{2}:\quad -\frac{1}{2} = \frac{1}{2}B,\quad B = -1,$$
$$x = -1:\quad -1 = -A,\quad A = 1.$$

$$I = \int \frac{dx}{x+1} - \int \frac{dx}{2x+1} = \ln|x+1| - \frac{1}{2}\ln|2x+1| + C = \underline{\underline{\ln\frac{|x+1|}{\sqrt{2x+1}} + C}}.$$

338 (2013). $\displaystyle\int \frac{x\,dx}{2x^2-3x-2} = \left[\begin{array}{c} 2x^2-3x-2=0, \\ x_{1,2}=\dfrac{3\pm\sqrt{9+16}}{4},\ x_1=2,\ x_2=-\dfrac{1}{2} \end{array}\right] =$

$$=\int \frac{x\,dx}{(x-2)(2x+1)} = I.$$

$$\frac{x}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1},$$

$$x = A(2x+1) + B(x-2),$$

$$x=2:\qquad 2=5A,\qquad A=\frac{2}{5},$$

$$x=-\frac{1}{2}:\quad -\frac{1}{2}=-\frac{5}{2}B,\quad B=\frac{1}{5}.$$

$$I=\frac{2}{5}\int\frac{dx}{x-2}+\frac{1}{5}\int\frac{dx}{2x+1}=\frac{2}{5}\ln\left|x-2\right|+\frac{1}{10}\ln\left|2x+1\right|+C=$$

$$=\underline{\underline{\frac{1}{5}\ln\left|(x-2)^2\sqrt{2x+1}\right|+C}}\,.$$

339 (2014). $\displaystyle\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}\,dx = I.$

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1}+\frac{B}{x+3}+\frac{C}{x-4},$$

$$2x^2+41x-91 = A(x+3)(x-4)+B(x-1)(x-4)+C(x-1)(x+3),$$

$$x=-3:\quad -196=28B,\qquad B=-7,$$

$$x=4:\qquad 105=21C,\qquad C=5,$$

$$x=1:\qquad -48=-12A,\quad A=4.$$

$$I = 4\int \frac{dx}{x-1} - 7\int \frac{dx}{x+3} + 5\int \frac{dx}{x-4} = 4\ln|x-1| - 7\ln|x+3| +$$

$$+ 5\ln|x-4| + C = \underline{\underline{\ln\left|\frac{(x-1)^4 (x-4)^5}{(x+3)^7}\right| + C}}.$$

340 (2015). $\displaystyle\int \frac{dx}{6x^3 - 7x^2 - 3x} = I.$

$$6x^3 - 7x^2 - 3x = x(6x^2 - 7x - 3) = x(2x-3)(3x+1),$$

$$6x^2 - 7x - 3 = 0,$$

$$x_{1,2} = \frac{7 \pm \sqrt{49+72}}{12}, \quad x_1 = -\frac{1}{3}, \quad x_2 = \frac{3}{2},$$

$$\frac{1}{6x^3 - 7x^2 - 3x} = \frac{A}{x} + \frac{B}{2x-3} + \frac{C}{3x+1},$$

$$1 = A(2x-3)(3x+1) + Bx(3x+1) + Cx(2x-3),$$

$$x = 0: \quad 1 = -3A, \quad A = -\frac{1}{3},$$

$$x = -\frac{1}{3}: \quad 1 = \frac{11}{9}C, \quad C = \frac{9}{11},$$

$$x = \frac{3}{2}: \quad 1 = \frac{33}{4}B, \quad B = \frac{4}{33}.$$

$$I = -\frac{1}{3}\int \frac{dx}{x} + \frac{4}{33}\int \frac{dx}{2x-3} + \frac{9}{11}\int \frac{dx}{3x+1} =$$

$$= \underline{\underline{-\frac{1}{3}\ln|x| + \frac{2}{33}\ln|2x-3| + \frac{3}{11}\ln|3x+1| + C}}.$$

341 (2016). $\displaystyle\int \frac{x^5+x^4-8}{x^3-4x}\,dx = I.$

$$
\begin{array}{r|l}
x^5 + x^4 - 8 & x^3-4x \\ \cline{2-2}
-\ \underline{x^5-4x^3} & x^2+x+4 \\
x^4+4x^3 & \\
-\ \underline{x^4-4x^2} & \\
4x^3+4x^2-8 & \\
-\ \underline{4x^3-16x} & \\
4x^2+16x-8 &
\end{array}
$$

$$I=\int (x^2+x+4)\,dx+4\int \frac{x^2+4x-2}{x(x-2)(x+2)}\,dx.$$

$$\frac{x^2+4x-2}{x(x-2)(x+2)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2},$$

$$x^2+4x-2=A(x-2)(x+2)+Bx(x+2)+Cx(x-2),$$

$$x=0:\quad -2=-4A,\quad A=\frac{1}{2},$$

$$x=2:\quad 10=8B,\quad B=\frac{5}{4},$$

$$x=-2:\quad -6=8C,\quad C=-\frac{3}{4}.$$

$$I=\frac{x^3}{3}+\frac{x^2}{4}+4x+4\left(\frac{1}{2}\ln|x|-\frac{3}{4}\ln|x+2|+\frac{5}{4}\ln|x-2|\right)=$$

$$=\frac{x^3}{3}+\frac{x^2}{4}+4x+\ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+C.$$

342 (2017). $\displaystyle\int \frac{x^3-1}{4x^3-x}\,dx = I.$

$$\begin{array}{r|l} x^3-1 & 4x^3-x \\ \cline{2-2} -\;x^3-\frac{1}{4}x & \frac{1}{4} \\ \cline{1-1} \frac{1}{4}x-1 & \end{array}$$

$$I=\frac{1}{4}\int dx+\frac{1}{4}\int \frac{x-4}{x(2x-1)(2x+1)}\,dx.$$

$$\frac{x-4}{x(2x-1)(2x+1)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{2x+1},$$

$$x-4=A(2x+1)(2x-1)+Bx(2x+1)+Cx(2x-1),$$

$$\begin{aligned} &x=0: && A=4,\\ &x=\frac{1}{2}: && B=-\frac{7}{2},\\ &x=-\frac{1}{2}: && C=-\frac{9}{2}. \end{aligned}$$

$$I=\frac{1}{4}x+\frac{1}{4}\left(4\int\frac{dx}{x}-\frac{7}{2}\int\frac{dx}{2x-1}-\frac{9}{2}\int\frac{dx}{2x+1}\right)=$$

$$=\underline{\underline{\frac{1}{4}x+\ln|x|-\frac{7}{16}\ln|2x-1|-\frac{9}{16}\ln|2x+1|+C}}.$$

343 (2018). $\displaystyle\int \frac{32x\,dx}{(2x-1)(4x^2-16x+15)} = I.$

$$4x^2-16x+15=0,\ x_{1,2}=\frac{8\pm\sqrt{64-60}}{4}=\frac{8\pm 2}{4},\ x_1=\frac{3}{2},\ x_2=\frac{5}{2},$$

$$4x^2-16x+15=4\left(x-\frac{3}{2}\right)\left(x-\frac{5}{2}\right)=(2x-3)(2x-5).$$

$$\frac{32x\,dx}{(2x-1)(4x^2-16x+15)}=\frac{A}{2x-1}+\frac{B}{2x-3}+\frac{C}{2x-5},$$

$$32x=A(2x-3)(2x-5)+B(2x-1)(2x-5)+C(2x-1)(2x-3),$$

$$x=\frac{1}{2}:\quad 16=8A,\quad A=2,$$
$$x=\frac{3}{2}:\quad 48=-4B,\quad B=-12,$$
$$x=-\frac{5}{2}:\quad 80=8C,\quad C=10.$$

$$I=\underline{\underline{\ln|2x-1|-6\ln|2x-3|+5\ln|2x-5|+C}}.$$

344 (2019). $\displaystyle\int\frac{x\,dx}{x^4-3x^2+2}=I.$

$$x^4-3x^2+2=0,$$

$$x_{1,2,3,4}=\pm\sqrt{\frac{3\pm\sqrt{9-8}}{2}}=\pm\sqrt{\frac{3\pm1}{2}},\ x_{1,2}=\pm1,\ x_{3,4}=\pm\sqrt{2},$$

$$\frac{x\,dx}{x^4-3x^2+2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-\sqrt{2}}+\frac{D}{x+\sqrt{2}},$$

$$x=A(x+1)\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)+B(x-1)\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)+$$
$$+C(x-1)(x+1)\left(x+\sqrt{2}\right)+D(x-1)(x+1)\left(x-\sqrt{2}\right),$$

$$x=1:\quad 1=-2A,\quad A=-\frac{1}{2},$$
$$x=-1:\quad -1=2B,\quad B=-\frac{1}{2},$$
$$x=\sqrt{2}:\quad \sqrt{2}=2\sqrt{2}C,\quad C=\frac{1}{2},$$
$$x=-\sqrt{2}:\quad -\sqrt{2}=-2\sqrt{2}D,\quad D=\frac{1}{2}.$$

$$I=-\frac{1}{2}\left(\ln\left|x-1\right|+\ln\left|x+1\right|+\ln\left|x-\sqrt{2}\right|+\ln\left|x+\sqrt{2}\right|\right)+C=$$

$$=\ln\sqrt{\frac{x^2-2}{x^2-1}}+C.$$

345 (2020). $\int\frac{(2x^2-5)\,dx}{x^4-5x^2+6}=I.$

$$x^4-5x^2+6=0,$$

$$x_{1,2,3,4}=\pm\sqrt{\frac{5\pm\sqrt{25-24}}{2}}=\pm\sqrt{\frac{5\pm1}{2}},\ x_{1,2}=\pm\sqrt{2},\ x_{3,4}=\pm\sqrt{3}.$$

$$\frac{2x^2-5}{\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)}=\frac{A}{x-\sqrt{2}}+\frac{B}{x+\sqrt{2}}+\frac{C}{x-\sqrt{3}}+\frac{D}{x+\sqrt{3}},$$

$$2x^2-5=A\left(x+\sqrt{2}\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)+B\left(x-\sqrt{2}\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)+$$
$$+C\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\left(x+\sqrt{3}\right)+D\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{3}\right).$$

$$x=\sqrt{2}:\quad -1=-2\sqrt{2}A,\quad A=\frac{1}{2\sqrt{2}},$$
$$x=-\sqrt{2}:\quad -1=2\sqrt{2}B,\quad B=-\frac{1}{2\sqrt{2}},$$
$$x=\sqrt{3}:\quad 1=2\sqrt{3}C,\quad C=\frac{1}{2\sqrt{3}},$$
$$x=-\sqrt{3}:\quad 1=-2\sqrt{3}D,\quad D=-\frac{1}{2\sqrt{3}}.$$

$$I=\frac{1}{2\sqrt{2}}\left(\ln\left|x-\sqrt{2}\right|-\ln\left|x+\sqrt{2}\right|\right)-\frac{1}{2\sqrt{3}}\left(\ln\left|x-\sqrt{3}\right|-\ln\left|x+\sqrt{3}\right|\right)+C=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|+\frac{1}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C.$$

346 (2021). $\displaystyle\int \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}\,dx = I.$

$$
\begin{array}{r|l}
-\,x^6-2x^4+\ 3x^3-9x^2+4 & x^5-5x^3+4x \\
\cline{2-2}
\underline{x^6-5x^4+0\cdot x^3+4x^2} & x \\
3x^4+\ 3x^3-13x^2+4 &
\end{array}
$$

$$x^4-5x^2+4=0,$$

$$x_{1,2,3,4}=\pm\sqrt{\frac{5\pm\sqrt{25-16}}{2}}=\pm\sqrt{\frac{5\pm3}{2}},\ \ x_{1,2}=\pm1,\ \ x_{3,4}=\pm2,$$

$$x\,(x^4-5x^2+4)=x(x-1)\,(x+1)(x-2)\,(x+2).$$

$$I=\int x\,dx+\int\frac{3x^4+3x^3-13x^2+4}{x(x-1)\,(x+1)(x-2)\,(x+2)}\,dx,$$

$$\frac{3x^4+3x^3-13x^2+4}{x(x-1)\,(x+1)(x-2)\,(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}+\frac{D}{x-2}+\frac{E}{x+2},$$

$$3x^4+3x^3-13x^2+4=A(x-1)\,(x+1)(x-2)\,(x+2)+$$

$$+B(x+1)(x-2)\,(x+2)x+C(x+1)(x-2)\,(x+2)x+$$

$$+D(x-1)\,(x+1)(x+2)x+E(x-1)\,(x+1)(x+2)x,$$

$$
\begin{array}{lll}
x=0: & 4=4A, & A=1, \\
x=1: & -3=-6B, & B=\dfrac{1}{2}, \\
x=-1: & -9=-6C, & C=\dfrac{3}{2}, \\
x=2: & 24=24D, & D=1, \\
x=-2: & -24=24E, & E=-1.
\end{array}
$$

$$I=\frac{x^2}{2}+\ln\left|\frac{x(x-2)\sqrt{(x-1)(x+1)^3}}{x+2}\right|+C.$$

2. Знаменатель имеет только действительные корни, некоторые из них – кратные.

347 (2022). $\int \frac{x^2+3x+2}{x(x^2+2x+1)}dx = I.$

$$\frac{x^2+3x+2}{x(x^2+2x+1)} = \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x+1},$$

$$x^2-3x+2 = A(x+1)^2 + Bx + Cx(x+1),$$

$$\begin{array}{lll} x=0: & 2=A, & A=2, \\ x=-1: & 6=-B, & B=-6, \\ x=1: & 0=4A+B+2C, & C=-1. \end{array}$$

$$I = 2\int\frac{dx}{x} - 6\int\frac{dx}{(x+1)^2} - \int\frac{dx}{x+1} = \underline{\underline{\ln\left|\frac{x^2}{x+1}\right| + \frac{6}{x+1} + C}}.$$

348 (2023). $\int \left(\frac{x+2}{x-1}\right)^2 \frac{dx}{x} = I.$

$$\frac{x^2+4x+4}{(x-1)^2 x} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1},$$

$$x^2+4x+4 = A(x-1)^2 + Bx + Cx(x-1).$$

$$\begin{array}{lll} x=0: & 4=A, & \\ x=1: & 9=B, & \\ x=-1: & 1=4A-B+2C, & C=-3. \end{array}$$

$$\underline{\underline{I = 4\ln|x| - 3\ln|x-1| - \frac{9}{x-1} + C}}.$$

349 (2024). $\int \frac{x^2dx}{x^3+5x^2+8x+4} = I.$

$$\frac{x^2}{x^3+5x^2+8x+4}=\frac{x^2}{(x+1)(x+2)^2}=\frac{A}{x+1}+\frac{B}{(x+2)^2}+\frac{C}{x+2},$$

$$\begin{array}{r|l} -\,x^3+5x^2+8x+4 & x+1 \\ \underline{x^3+\ x^2\qquad\qquad} & \overline{x^2+4x+4} \\ -\,4x^2+8x\qquad & \\ \underline{4x^2+4x}\qquad & \\ -\,4x+4 & \\ \underline{4x+4} & \\ 0\ \ & \end{array}$$

$$x^2=A(x+2)^2+B(x+1)+C(x+1)(x+2).$$

$$\begin{array}{lll} x=-1: & 1=A, & \\ x=-2: & 4=-B, & B=-4, \\ x=2: & 4=16A+3B+12C, & C=0. \end{array}$$

$$I=\int\frac{dx}{x+1}-4\int\frac{dx}{(x+2)^2}=\underline{\underline{\ln|x+1|+\frac{4}{x+2}+C}}.$$

350 (2025). $\displaystyle\int\frac{x^3+1}{x^3-x^2}=I.$

$$\begin{array}{r|l} -\,x^3+1 & x^3-x^2 \\ \underline{x^3-x^2} & \overline{1} \\ x^2+1 & \end{array}$$

$$I=\int dx+\int\frac{x^2+1}{x^2(x-1)}\,dx,$$

$$\frac{x^2}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1},$$

$$x^2+1=A(x-1)+Bx(x-1)+Cx^2.$$

$$x=0: \quad 1=-A, \; A=-1,$$
$$x=1: \quad 2=C,$$
$$x=-1: \quad 2=-2A+2B+C, \quad B=-1.$$

$$I=x+\frac{1}{x}-\ln|x|+2\ln|x-1|+x+C=\underline{\underline{x+\frac{1}{x}+\ln\frac{(x-1)^2}{|x|}+C}}.$$

351 (2026). $\displaystyle\int\frac{x^3-6x^2+11x-5}{(x-2)^4}dx=I.$

$$\frac{x^3-6x^2+11x-5}{(x-2)^4}=\frac{A}{(x-2)^4}+\frac{B}{(x-2)^3}+\frac{C}{(x-2)^2}+\frac{D}{x-2},$$

$$x^3-6x^2+11x-5=A+Bx-2B+Cx^2-4Cx+4C+Dx^3-$$
$$-6Dx^2+12Dx-8D.$$

$$\left.\begin{array}{l|l} x^3 & D=1, \\ x^2 & C-6D=-6, \\ x^1 & B-4C+12D=11, \\ x^0 & A-2B-8D+4C=-5 \end{array}\right\} \Rightarrow D=1,\; C=0,\; B=-1,\; A=1.$$

$$I=\int\frac{dx}{(x-2)^4}-\int\frac{dx}{(x-2)^3}+\int\frac{dx}{x-2}=$$

$$=\underline{\underline{\frac{1}{3(x-2)^3}-\frac{1}{2(x-2)^2}+\ln|x-2|+C}}.$$

352 (2027). $\displaystyle\int\frac{dx}{x^4-x^2}=I.$

$$\frac{1}{x^2(x-1)(x+1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}+\frac{D}{x+1},$$

$$1 = Ax^2 - A + Bx^3 - Bx + Cx^3 + Cx^2 + Dx^3 - Dx^2.$$

$$\left.\begin{array}{l|l} x^3 & B + C + D = 0, \\ x^2 & A + C - D = 0, \\ x^1 & B = 0, \\ x^0 & A = -1 \end{array}\right\} \Rightarrow \begin{array}{l} A = -1, \\ B = 0, \end{array} \begin{cases} C + D = 0 \\ C - D = 1 \end{cases} \Rightarrow C = \frac{1}{2},\ D = -\frac{1}{2}.$$

$$I = \frac{1}{x} + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$$

353 (2028). $\displaystyle\int \frac{x^2 dx}{(x+2)^2 (x+4)^2} = I.$

$$x^2 = A(x+4)^2 + B(x+2)(x+4)^2 + C(x+2)^2 + D(x+4)(x+2)^2.$$

$$\begin{array}{lll} x = -2: & 4 = 4A, & A = 1, \\ x = -4: & 16 = 4C, & C = 4, \\ x = 0: & 0 = 16A + 32B + 4C + 16D, & \\ x = 1: & 1 = 25A + 75B + 9C + 45D, & \end{array}$$

$$\begin{cases} 16A + 32B + 4C + 16D = 0 \\ 25A + 75B + 9C + 45D = -1 \end{cases} \Rightarrow \begin{cases} 8B + 4D = -8 \\ 75B + 45D = -60 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 2B + D = -2 \\ 15B + 9D = -12 \end{cases} \Rightarrow \begin{array}{l} D = -2 - 2B, \\ 15B - 18 - 18B = -12,\ -7B = 6, \end{array} \Rightarrow$$

$$\Rightarrow B = -2,\ D = 2.$$

$$I = \int \frac{dx}{(x+2)^2} - 2\int \frac{dx}{x+2} + 4\int \frac{dx}{(x+4)^2} + 2\int \frac{dx}{x+4} = 2\ln\left|\frac{x+4}{x-2}\right| -$$

$$-\frac{1}{x+2} - \frac{4}{x+4} + C = 2\ln\left|\frac{x+4}{x-2}\right| - \frac{5x+12}{x^2+6x+8} + C.$$

354 (2029). $\displaystyle\int \frac{x^3 - 6x^2 + 9x + 7}{(x-2)^3 (x-5)} dx = I.$

$$\frac{x^3-6x^2+9x+7}{(x-2)^3(x-5)}=\frac{A}{(x-2)^3}+\frac{B}{(x-2)^2}+\frac{C}{x-2}+\frac{D}{x-5},$$

$$x^3-6x^2+9x+7=Ax-5A+Bx^2-7Bx+10B+Cx^3-4Cx^2+$$

$$+4Cx-5Cx^2+20Cx-20C+Dx^3-6Dx^2+12Dx-8D.$$

$$\left.\begin{array}{l|l} x^3 & C+D=1, \\ x^2 & B-9C-6D=-6, \\ x^1 & A-7B+24C+12D=9, \\ x^0 & -5A+10B-20C-8D=7 \end{array}\right\} \Rightarrow \begin{array}{l} A=-3, \\ C=0, \\ D=1, \\ B=0. \end{array}$$

$$I=-3\int\frac{dx}{(x-3)^3}+\int\frac{dx}{x-5}=\underline{\underline{\frac{3}{2(x-2)^2}+\ln|x-5|+C}}.$$

355 (2030). $\displaystyle\int\frac{1}{8}\left(\frac{x-1}{x+1}\right)^4 dx=I.$

$$\left(\frac{x-1}{x+1}\right)^4=\frac{((x+1)-2)^4}{(x+1)^4}=$$

$$=\frac{(x+1)^4-8(x+1)^3+24(x+1)^2-32(x+1)+16}{(x+1)^4}=$$

$$=1-\frac{8}{x+1}+\frac{24}{(x+1)^2}-\frac{32}{(x+1)^3}+\frac{16}{(x+1)^4},$$

$$I=\frac{1}{8}\int dx-\int\frac{dx}{x+1}+3\int\frac{dx}{(x+1)^2}-4\int\frac{dx}{(x+1)^3}+2\int\frac{dx}{(x+1)^4}=$$

$$=\frac{1}{8}x-\ln|x+1|-\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{2}{3(x+1)^3}+C=$$

$$=\underline{\underline{\frac{1}{8}x-\ln|x+1|-\frac{9x^2+12x+5}{3(x+1)^3}+C}}.$$

356 (2031). $\displaystyle\int \frac{x^5\,dx}{(x-1)^2(x^2-1)} = I.$

$$(x-1)^2(x^2-1) = x^4-2x^3+2x-1,$$

$$\begin{array}{r|l} -\,x^5 & x^4-2x^3+2x-1 \\ \cline{2-2} \underline{x^5-2x^4+2x^2-x} & x+2 \\ -\,2x^4-2x^2+x & \\ \underline{2x^4-4x^3+4x-2} & \\ 4x^3-2x^2-3x+2 & \end{array}$$

$$I = \int (x+2)\,dx + \int \frac{4x^3-2x^2-3x+2}{(x-1)^2(x^2-1)}\,dx.$$

$$\frac{4x^3-2x^2-3x+2}{(x-1)^2(x-1)(x+1)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+1},$$

$$4x^3-2x^2-3x+2 = A(x+1) + B(x-1)(x+1) + C(x-1)^2(x+1) +$$
$$+ D(x+1)^3.$$

$$x=1: \quad 1 = 2A,\ A = \frac{1}{2},$$
$$x=-1: \quad -1 = -8D,\ D = \frac{1}{8},$$
$$x=0: \quad 2 = A - B + C - D,$$
$$x=2: \quad 20 = 3A + 3B + 3C + D,$$

$$\begin{cases} \dfrac{1}{2} - B + C - \dfrac{1}{8} = 2, \\ \dfrac{3}{2} + 3B + 3C + \dfrac{1}{8} = 20 \end{cases} \Rightarrow \begin{cases} -B + C = \dfrac{13}{8}, \\ 3B + 3C = \dfrac{147}{8} \end{cases} \Rightarrow C = \frac{31}{8},\ B = \frac{9}{4}.$$

$$I = \frac{(x+2)^2}{2} - \frac{1}{(x-1)^2} - \frac{9}{4(x-1)} + \frac{31}{8}\ln|x-1| + \frac{1}{8}\ln|x+1| + C.$$

357 (2032). $\displaystyle\int \frac{x^2-2x+3}{(x-1)(x^3-4x^2+3x)}=I.$

$$(x-1)(x^3-4x^2+3x)=x(x-1)^2(x-3),$$

$$\frac{(x^2-2x+3)\,dx}{x(x-1)^2(x-3)}=\frac{A}{x}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{D}{x-3},$$

$$x^2-2x+3=A(x-1)^2(x-3)+Bx(x-3)+Cx(x-1)(x-3)+$$
$$+D(x-1)^2x.$$

$$\begin{array}{lll} x=0: & 3=-3A, & A=-1, \\ x=1: & 2=-2B, & B=-1, \\ x=3: & 6=12D, & D=\dfrac{1}{2}, \\ x=2: & 3=-A-2B-2C+2D, & C=\dfrac{1}{2}. \end{array}$$

$$I=-\int\frac{dx}{x}-\int\frac{dx}{(x-1)^2}+\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{x-3}=-\ln|x|+\frac{1}{x-1}+$$
$$+\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x-3|+C=\underline{\underline{\frac{1}{x-1}+\ln\frac{\sqrt{(x-1)(x-3)}}{|x|}+C}}.$$

358 (2033). $\displaystyle\int \frac{(7x^3-9)\,dx}{x^4-5x^3+6x^2}=I.$

$$x^2(x^2-5x+6)=x^2(x-3)(x-2),$$

$$\frac{7x^3-9}{x^2(x-3)(x-2)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-3}+\frac{D}{x-2},$$

$$7x^2-9=A(x-3)(x-2)+Bx(x-3)(x-2)+Cx^2(x-2)+$$
$$+D(x-3)x^2,$$

$$x=0:\quad -9=6A,\qquad A=-\frac{3}{2},$$
$$x=2:\quad 47=-4D,\qquad D=-\frac{47}{4},$$
$$x=3:\quad 180=9C,\qquad C=20,$$
$$x=1:\quad -2=2A+2B+C-2D,\quad B=-\frac{5}{4}.$$

$$\underline{\underline{I=\frac{3}{2x}-\frac{5}{4}\ln|x|+20\ln|x-3|-\frac{47}{4}\ln|x-2|+C.}}$$

359 (2034). $\int\frac{x^3-2x^2+4}{x^3(x-2)^2}\,dx=I.$

$$\frac{x^3-2x^2+4}{x^3(x-2)^2}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{D}{(x-2)^2}+\frac{E}{x-2},$$
$$x^3-2x^2+4=A(x-2)^2+Bx(x-2)^2+Cx^2(x-2)^2+Dx^3+$$
$$+Ex^3(x-2).$$

$$x=0:\quad 4=4A,\qquad A=1,$$
$$x=1:\quad 3=A+B+C+D-E,$$
$$x=2:\quad 4=8D,\qquad D=\frac{1}{2},$$
$$x=3:\quad 13=A+3B+9C+27D+27E,$$
$$x=4:\quad 36=4A+16B+64C+64D+128E,$$

$$\begin{cases}A+B+C+D-E=3,\\ A+3B+9C+27D+27E=13,\\ 4A+16B+64C+64D+128E=36\end{cases}\Rightarrow\begin{cases}B+C-E=\frac{3}{2},\\ 3B+9C+27E=-\frac{3}{2},\\ 4B+16C+32E=0\end{cases}$$

$$B=1,\ C=\frac{1}{4},\ E=-\frac{1}{4}.$$

$$I=\frac{1}{4}\ln\left|\frac{x}{x-2}\right|-\frac{1}{x}\left(1+\frac{1}{2x}\right)-\frac{1}{2(x-2)}+C.$$

360 (2035). $\displaystyle\int\frac{3x^2+1}{(x^2-1)^3}\,dx=I.$

$$\frac{3x^2+1}{(x-1)^3(x+1)^3}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{D}{(x+1)^3}+$$

$$+\frac{E}{(x+1)^2}+\frac{L}{x+1},$$

$$3x^2+1=A(x+1)^3+B(x-1)(x+1)^3+C(x-1)^2(x+1)^3+$$

$$+D(x-1)^3+E(x+1)(x-1)^3+L(x+1)^2(x-1)^3.$$

$$x=-1:\quad 4=-8D,\quad D=-\frac{1}{2},$$

$$x=1:\quad 4=8A,\quad A=\frac{1}{2}.$$

$$x=0:\quad 1=A-B+C-D-E-L,$$

$$x=2:\quad 13=27A+27B+27C+D+3E+9L,$$

$$x=-2:\quad 13=-A+3B-9C-27D+27E-27L,$$

$$x=3:\quad 28=64A+128B+256C+8D+32E+128L,$$

$$\begin{cases}-B+C-E-L=0,\\ 9B+9C+E+3L=0,\\ B-3C+9E-9L=0,\\ 4B+8C+E+4L=0\end{cases}\Rightarrow\begin{vmatrix}-1 & 1 & -1 & -1\\ 9 & 9 & 1 & 3\\ 1 & -3 & 1 & -9\\ 4 & 8 & 1 & 4\end{vmatrix}\neq 0.$$

Значит, однородная система уравнений имеет только нулевое решение: $B=C=E=L=0$.

$$I=\frac{1}{2}\int\frac{dx}{(x-1)^3}-\frac{1}{2}\int\frac{dx}{(x+1)^3}=-\frac{1}{2}\left(-\frac{1}{2(x-1)^2}+\frac{1}{2(x+1)^2}\right)=-\frac{x}{(x^2-1)^2}.$$

3. Знаменатель имеет комплексные различные корни.

361 (2036). $\int \dfrac{dx}{x(x^2+1)} = I.$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

$$1 = ax^2 + A + Bx^2 + Cx.$$

$$\left.\begin{array}{l|l} x^2 & A+B=0, \\ x & C=0, \\ x^0 & A=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=1, \\ B=-1, \\ C=0, \end{array}$$

$$I = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} = \ln|x| - \frac{1}{2}\ln(x^2+1) + C = \underline{\underline{\ln\frac{|x|}{\sqrt{x^2+1}} + C}}.$$

362 (2037). $\int \dfrac{dx}{1+x^3} = I.$

$$\frac{1}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2},$$

$$1 = A - Ax + Ax^2 + Bx + C + Bx^2 + Cx.$$

$$\left.\begin{array}{l|l} x^2 & A+B=0, \\ x & -A+B+C=0, \\ x^0 & A+C=1 \end{array}\right\} \Rightarrow \begin{cases} B=-A, \\ -2A+C=0, \\ A+C=1 \end{cases} \Rightarrow \begin{array}{l} A=\dfrac{1}{3}, \\ B=-\dfrac{1}{3}, \\ C=\dfrac{2}{3}. \end{array}$$

$$I = \frac{1}{3}\int \frac{dx}{1+x} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx = \frac{1}{3}\ln|x+1| -$$

$$-\frac{1}{3}\int\frac{\frac{1}{2}(2x-1)+\left(-2+\frac{1}{2}\right)}{x^2-x+1}dx=\frac{1}{3}\ln|x+1|-\frac{1}{6}\int\frac{d(x^2-x+1)}{x^2-x+1}+$$

$$+\frac{1}{2}\int\frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{3}\ln|x+1|-\frac{1}{3}\ln\left|x^2-x+1\right|+$$

$$+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+C=\underline{\underline{\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+C.}}$$

363 (2038). $\displaystyle\int\frac{x\,dx}{x^3-1}=I.$

$$\frac{x}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1},$$

$$x=Ax^2+Ax+A+Bx^2+Cx-Bx-C.$$

$$\begin{array}{l|l} x^2 & A+B=0, \\ x & A-B+C=1, \\ x^0 & A+C=0 \end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\}\Rightarrow\begin{cases}A=C,\\ B+C=0,\\ -B+2C=1\end{cases}\Rightarrow\quad \begin{array}{l} A=\frac{1}{3}, \\ B=-\frac{1}{3}, \\ C=\frac{1}{3}. \end{array}$$

$$I=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x-1}{x^2+x+1}dx=\frac{1}{3}\ln|x-1|-$$

$$-\frac{1}{3}\int\frac{\frac{1}{2}(2x+1)+\left(-1-\frac{1}{2}\right)}{x^2+x+1}dx=\frac{1}{3}\ln|x-1|-\frac{1}{6}\int\frac{d(x^2+x+1)}{x^2+x+1}+$$

$$+\frac{1}{2}\int\frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln\left|x^2+x+1\right|+$$

$$+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C=\underline{\underline{\frac{1}{3}\ln\frac{|x-1|}{\sqrt{x^2+x+1}}+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C}}\,.$$

364 (2039). $\displaystyle\int\frac{(2x^2-3x-3)}{(x-1)(x^2-2x+5)}\,dx=I.$

$$\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}=\frac{A}{x-1}+\frac{Bx+C}{x^2-2x+5},$$

$$2x^2-3x-3=Ax^2-2Ax+5A+Bx^2-Bx+Cx-C.$$

$$\left.\begin{array}{l|l} x^2 & A+B=2, \\ x & -2A-B+C=-3, \\ x^0 & 5A-C=-3 \end{array}\right\}\Rightarrow\begin{array}{l} A=-1, \\ B=3, \\ C=-2. \end{array}$$

$$I=-\int\frac{dx}{x-1}+\int\frac{3x-2}{x^2-2x+5}\,dx=-\ln|x-1|+\int\frac{(2x-2)\frac{3}{2}+1}{x^2-2x+5}\,dx=$$

$$=-\ln|x-1|+\frac{3}{2}\int\frac{d\,(x^2-2x+5)}{x^2-2x+5}+\int\frac{dx}{(x-1)^2+2^2}=-\ln|x-1|+$$

$$+\frac{3}{2}\ln\left|x^2-2x+5\right|+\frac{1}{2}\operatorname{arctg}\frac{x-1}{2}+C=$$

$$=\underline{\underline{\ln\frac{\sqrt{(x^2-2x+5)^3}}{|x-1|}+\frac{1}{2}\operatorname{arctg}\frac{x-1}{2}+C}}\,.$$

365 (2040). $\displaystyle\int \frac{(x^4+1)\,dx}{x^3-x^2+x-1}=\int (x+1)\,dx+2\int \frac{dx}{(x-1)(x^2+1)}=I.$

$$\begin{array}{r|l} -\; x^4+1 & x^3-x^2+x-1 \\ \cline{2-2} \underline{x^4-x^3+x^2-x} & x+1 \\ -\; x^3-x^2+x+1 & \\ \underline{x^3-x^2+x-1} & \\ 2 & \end{array}$$

$$\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}.$$

$$\begin{array}{l|l} x^2 & A+B=0, \\ x & C-B=0, \\ x^0 & A-C=2 \end{array}\Bigg\}\Rightarrow \begin{array}{l} A=1, \\ B=-1, \\ C=-1. \end{array}$$

$$I=\frac{(x+1)^2}{2}+\ln|x-1|-\int\frac{x+1}{x-1}\,dx=\frac{(x+1)^2}{2}+\ln|x-1|-$$

$$-\int\frac{\left((2x)\frac{1}{2}+1\right)dx}{x^2+1}=\frac{(x+1)^2}{2}+\ln|x-1|-\frac{1}{2}\ln(x^2+1)-\operatorname{arctg}x+C=$$

$$=\underline{\underline{\frac{(x+1)^2}{2}+\ln\frac{x-1}{\sqrt{x^2+1}}-\operatorname{arctg}x+C}}.$$

366 (2041). $\displaystyle\int\frac{x^2dx}{1-x^4}=I.$

$$\frac{x^2}{1-x^4}=\frac{x^2}{(1-x)(1+x)(1+x^2)}=\frac{A}{1-x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2},$$

$$x^2 = A + Ax + Ax^2 + Ax^3 + B + Bx^2 - Bx - Bx^3 + Cx + D - Cx^3 - Dx^2.$$

$$\begin{array}{l|l} x^3 & A - B - C = 0, \\ x^2 & A + B - D = 1, \\ x^1 & A - B + C = 0, \\ x^0 & A + B + D = 0 \end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\\1\end{array}}\right\} \Rightarrow \begin{array}{l} A = \frac{1}{4}, \\ B = \frac{1}{4}, \\ C = 0, \\ D = -\frac{1}{2}. \end{array}$$

$$\underline{\underline{I = \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\operatorname{arctg} x + C.}}$$

367 (2042). $\displaystyle\int \frac{dx}{(x^2+1)(x^2+x)} = I.$

$$\frac{1}{(x^2+1)x(x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1},$$

$$1 = Ax^3 + Ax + Ax^2 + A + Bx^3 + Bx + Cx^3 + Dx^2 + Cx^2 + Dx.$$

$$\begin{array}{l|l} x^3 & A + B + C = 0, \\ x^2 & A + D + C = 0, \\ x^1 & A + B + D = 0, \\ x^0 & A = 1 \end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\\1\end{array}}\right\} \Rightarrow \begin{cases} C - D = 0, \\ C - B = 0 \end{cases} \Rightarrow \begin{cases} C = D, \\ C = B \end{cases} \Rightarrow C = B = D = -\frac{1}{2}.$$

$$I = \int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) -$$

$$-\frac{1}{2}\operatorname{arctg} x + C = \underline{\underline{\frac{1}{4}\ln\frac{x^4}{(x+1)^2(x^2+1)} - \frac{1}{2}\operatorname{arctg} x + C.}}$$

368 (2043). $\displaystyle\int \frac{dx}{(x+1)^2(x^2+1)} = I.$

$$\frac{1}{(x+1)^2(x+1)}=\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{Dx+K}{x^2+1},$$

$$1=Ax^2+A+Bx^3+Bx+Bx^2+B+Dx^3+2Dx^2+Dx+Kx^2+2Kx+K.$$

$$\left.\begin{array}{l|l} x^3 & B+D=0, \\ x^2 & A+B+2D+K=0, \\ x^1 & B+D+2K=0, \\ x^0 & A+B+K=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=\dfrac{1}{2}, \\ B=\dfrac{1}{2}, \\ D=-\dfrac{1}{2}, \\ K=0. \end{array}$$

$$I=\frac{1}{2}\int\frac{dx}{(x+1)^2}+\frac{1}{2}\int\frac{dx}{x+1}-\frac{1}{4}\int\frac{2x\,dx}{x^2+1}=$$

$$=-\frac{1}{2(x+1)}+\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)+C.$$

369 (2044). $\displaystyle\int\frac{(3x^2+x+3)\,dx}{(x-1)^3(x^2+1)}=I.$

$$\frac{3x^2+x+3}{(x-1)^3(x^2+1)}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{Dx+K}{x^2+1},$$

$$3x^2+x+3=Ax^2+A+Bx^3+Bx-Bx^2-B+Cx^4-2Cx^3+2Cx^2-$$

$$-2Cx+C+Dx^4-3Dx^3+3Dx^2-Dx+Kx^3-3Kx^2+3Kx-K.$$

$$\begin{cases} C+D=0, \\ B-2C-3D+K=0, \\ A-B+2C+3D-3K=3, \\ B-2C-D+3K=1, \\ A-B+C-K=3 \end{cases} \Rightarrow \begin{cases} C=-D, \\ B-D+K=0, \\ B+D+3K=1 \end{cases}$$

$$\begin{cases} A-2K=3, \\ A+2D=4, \\ 2B+4K=1 \end{cases} \Rightarrow\ K=\frac{A-3}{2},\ D=\frac{4-A}{2},\ B=\frac{7-2A}{2},$$

$$A-\frac{7-2A}{2}-\frac{4-A}{2}-\frac{A-3}{2}=3,\ 2A-7+2A-4+A-A+3=6,$$

$$A=\frac{7}{2}\ \Rightarrow\ K=\frac{1}{4},\ D=\frac{1}{4},\ B=0,\ C=-\frac{1}{4}.$$

$$I=\frac{7}{2}\int\frac{dx}{(x-1)^3}-\frac{1}{4}\int\frac{dx}{x-1}+\frac{1}{4}\int\frac{x+1}{x^2+1}\,dx=-\frac{1}{4(x-1)^2}-$$

$$-\frac{1}{4}\ln|x-1|+\frac{1}{8}\ln(x^2+1)+\frac{1}{4}\operatorname{arctg}x+C=$$

$$=\frac{1}{4}\left(\ln\frac{\sqrt{x^2+1}}{|x-1|}+\operatorname{arctg}x-\frac{1}{2(x-1)^2}\right)+C.$$

370 (2045). $$\int\frac{x^5+2x^3+4x+4}{x^2(x^2+2x+2)}\,dx=\int(x-2)\,dx+$$

$$+4\int\frac{x^3+x^2+x+1}{x^2(x^2+2x+2)}\,dx=I.$$

$$\frac{x^3+x^2+x+1}{x^2(x^2+2x+2)}=\frac{A}{x^2}+\frac{B}{x}+\frac{Cx+D}{x^2+2x+2},$$

$$x^3+x^2+x+1=Ax^2+2Ax+2A+Bx^3+2Bx^2+2Bx+Cx^3+Dx^2.$$

$$\left.\begin{array}{l|l} x^3 & B+C=1, \\ x^2 & A+2B+D=1, \\ x^1 & 2A+2B=1, \\ x^0 & 2A=1 \end{array}\right\}\Rightarrow\ \begin{array}{l} A=\frac{1}{2}, \\ B=0, \\ C=1, \\ D=\frac{1}{2}. \end{array}$$

$$I=\frac{x^2}{2}-2x+2\int\frac{dx}{x^2}+2\int\frac{2x+1}{x^2+2x+2}\,dx=\frac{x^2}{2}-2x-$$

$$-\frac{2}{x}+2\int\frac{(2x+2)\,dx}{x^2+2x+2}-2\int\frac{d\,(x+1)}{(x+1)^2+1}=$$

$$=\frac{x^2}{2}-2x-\frac{2}{x}+2\ln\left|x^2+2x+2\right|-2\,\mathrm{arctg}\,(x+1)+C.$$

371 (2046). $\displaystyle\int\frac{(x^3-6)\,dx}{x^4+6x^2+8}=I.$

$$\frac{x^3-6}{(x^2+2)(x^2+4)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+4},$$

$$x^3-6=Ax^3+Bx^2+4Ax+4B+Cx^3+2Cx+2D+Dx^2.$$

$$\left.\begin{array}{l|l} x^3 & A+C=1, \\ x^2 & B+D=0, \\ x^1 & 4A+2C=0, \\ x^0 & 4B+2D=-6 \end{array}\right\}\Rightarrow \begin{array}{l} A=-1, \\ B=-3, \\ C=2, \\ D=3. \end{array}$$

$$I=-\int\frac{x+3}{x^2+2}\,dx+\int\frac{2x+3}{x^2+4}\,dx=-\frac{1}{2}\int\frac{d\,(x^2+2)}{x^2+2}+\int\frac{d\,(x^2+4)}{x^2+4}-$$

$$-3\int\frac{dx}{x^2+\left(\sqrt{3}\right)^2}+\int\frac{dx}{x^2+2^2}=$$

$$=-\frac{1}{2}\ln(x^2+2)+\ln(x^2+4)-\frac{3}{\sqrt{2}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}}+\frac{3}{2}\,\mathrm{arctg}\,\frac{x}{2}+C.$$

372 (2047). $\displaystyle\int\frac{dx}{1+x^4}=I.$

$$\frac{1}{1+x^4}=\frac{1}{1+x^4+2x^2-2x^2}=\frac{1}{\left(x^2-x\sqrt{2}+1\right)\left(x^2+x\sqrt{2}+1\right)}=$$

$$=\frac{Ax+B}{x^2-x\sqrt{2}+1}+\frac{Cx+D}{x^2+x\sqrt{2}+1},$$

$$1=Ax^3+Bx^2+A\sqrt{2}x^2+B\sqrt{2}x+Ax+B+Cx^3-C\sqrt{2}x^2+Cx+$$

$$+Dx^2-D\sqrt{2}x+D.$$

$$\left.\begin{array}{l|l} x^3 & A+C=0, \\ x^2 & B+A\sqrt{2}-C\sqrt{2}+D=0, \\ x^1 & B\sqrt{2}+A+C-D\sqrt{2}=0, \\ x^0 & B+D=1 \end{array}\right\} \Rightarrow \begin{cases}(A-C)\sqrt{2}=-1, \\ (B-D)\sqrt{2}=0\end{cases} \Rightarrow B=D,$$

$$B=D=\frac{1}{2},\ A=-C,\ A=-\frac{\sqrt{2}}{4},\ C=\frac{\sqrt{2}}{4}.$$

$$I=-\frac{1}{4}\int\frac{\sqrt{2}x-2}{x^2-x\sqrt{2}+1}\,dx+\frac{1}{4}\int\frac{\sqrt{2}x+2}{x^2+x\sqrt{2}+1}\,dx=$$

$$=-\frac{1}{4}\int\frac{\frac{\sqrt{2}}{2}\left(2x-\sqrt{2}\right)+(-2+1)}{x^2-x\sqrt{2}+1}\,dx+-\frac{1}{4}\int\frac{\frac{\sqrt{2}}{2}\left(2x+\sqrt{2}\right)+(2-1)}{x^2+x\sqrt{2}+1}\,dx=$$

$$=-\frac{\sqrt{2}}{8}\int\frac{d\left(x^2-x\sqrt{2}+1\right)}{x^2-x\sqrt{2}+1}+\frac{1}{4}\int\frac{dx}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}+$$

$$+\frac{\sqrt{2}}{8}\int\frac{d\left(x^2+x\sqrt{2}+1\right)}{x^2-x\sqrt{2}+1}+\frac{1}{4}\int\frac{dx}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=$$

$$=\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\operatorname{arctg}\frac{2x-\sqrt{2}}{\sqrt{2}}+\frac{\sqrt{2}}{4}\operatorname{arctg}\frac{2x+\sqrt{2}}{\sqrt{2}}+C=$$

$$=\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\frac{2x-\sqrt{2}}{\sqrt{2}}+\frac{2x+\sqrt{2}}{\sqrt{2}}}{1-\frac{4x^2-2}{2}}+C=$$

$$=\underline{\underline{\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt{2}x}{1-x^2}+C}}.$$

4. Знаменатель имеет комплексные кратные корни.

373 (2048). $\displaystyle\int\frac{x^3+x-1}{(x^2+2)^2}dx=I.$

$$\frac{x^3+x-1}{(x^2+2)^2}=\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2},$$

$$x^3+x-1=Ax+B+Cx^3+Dx^2+2Cx+2D.$$

$$\left.\begin{array}{l|l} x^3 & C=1, \\ x^2 & D=0, \\ x^1 & A+2C=1, \\ x^0 & B+2D=-1 \end{array}\right\}\Rightarrow \begin{array}{l} A=-1, \\ B=-1, \\ C=1, \\ D=0. \end{array}$$

$$I=-\int\frac{x+1}{(x^2+2)^2}dx+\int\frac{x\,dx}{x^2+2}=-\frac{1}{2}\int\frac{d(x^2+2)}{(x^2+2)^2}+$$

$$+\frac{1}{2}\int\frac{d(x^2+2)}{x^2+2}-\int\frac{dx}{(x^2+2)^2},$$

$$\int\frac{dx}{(x^2+2)^2}=\frac{1}{2}\int\frac{(x^2+2)-x^2}{(x^2+2)^2}dx=\frac{1}{2}\int\frac{dx}{x^2+2}-\frac{1}{2}\int\frac{x^2dx}{(x^2+2)^2}=$$

$$=\left[\begin{array}{ll} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(x^2+2)^2}, & v=\dfrac{1}{2}\displaystyle\int\frac{d(x^2+2)}{(x^2+2)^2}=-\frac{1}{2(x^2+2)} \end{array}\right]=\frac{1}{2}\int\frac{dx}{x^2+2}+$$

$$+\frac{x}{4(x^2+2)}-\frac{1}{4}\int\frac{dx}{x^2+2}=\frac{x}{4(x^2+2)}+\frac{1}{4}\int\frac{dx}{x^2+2}=\frac{x}{4(x^2+2)}+$$

$$+\frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}.$$

$$I=\frac{1}{2(x^2+2)}+\frac{1}{2}\ln(x^2+2)-\frac{x}{4(x^2+2)}-\frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}+C=$$

$$=\frac{2-x}{4(x^2+2)}+\frac{1}{2}\ln(x^2+2)-\frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}+C.$$

374 (2049). $\displaystyle\int\frac{dx}{x(4+x^2)^2(1+x^2)}=I.$

$$\frac{1}{x(4+x^2)^2(1+x^2)}=\frac{A}{x}+\frac{Bx+C}{(4+x^2)^2}+\frac{Kx+N}{4+x^2}+\frac{Mx+P}{1+x^2},$$

$$1=A(4+x^2)^2(1+x^2)+(Bx+C)x(1+x^2)+$$

$$+(Kx+N)x(4+x^2)(1+x^2)+(Mx+P)x(4+x^2)^2.$$

$$x=0:\quad 1=16A,\quad A=\frac{1}{16}.$$

$$\left.\begin{aligned}x=i:&\quad 1=(Mi+P)i\cdot 9\\ x=-i:&\quad 1=-(Mi+P)i\cdot 9\end{aligned}\right\}\Rightarrow\ \begin{aligned}M&=-\frac{1}{9},\\ P&=0,\end{aligned}$$

$$\left.\begin{aligned}x=2i:&\quad (2Bi+C)2i(-3)=1\\ x=-2i:&\quad (-2Bi+C)(-2i)(-3)=1\end{aligned}\right\}\Rightarrow\ \begin{aligned}B&=\frac{1}{12},\\ C&=0,\end{aligned}$$

$$\left.\begin{aligned}x=1:&\quad 1=\frac{25}{8}+\frac{1}{6}+10K+10N-\frac{25}{9},\\ x=-1:&\quad 1=\frac{25}{8}+\frac{1}{6}+10K-10N-\frac{25}{9}\end{aligned}\right\}\Rightarrow\begin{cases}10K+10N=\dfrac{35}{72},\\ 10K-10N=\dfrac{35}{72}\end{cases}\Rightarrow$$

$$\Rightarrow\ K=\frac{7}{144},\ \ N=0.$$

$$I=\frac{1}{16}\int\frac{dx}{x}+\frac{1}{24}\int\frac{d(x^2+4)}{(x^2+4)^2}+\frac{7}{288}\int\frac{d(x^2+4)}{x^2+4}-\frac{1}{18}\int\frac{d(x^2+1)}{x^2+1}=$$

$$=\frac{1}{16}\ln|x|-\frac{1}{24(x^2+4)}+\frac{7}{288}\ln(x^2+4)-\frac{1}{18}\ln(x^2+1)+C.$$

375 (2050). $\displaystyle\int\frac{(5x^2-12)\,dx}{(x^2-6x+13)^2}=I.$

$$\frac{5x^2-12}{(x^2-6x+13)^2}=\frac{Ax+B}{(x^2-6x+13)^2}+\frac{Cx+D}{x^2-6x+13},$$

$$5x^2-12=Ax+B+Cx^3-6Cx^2+13Cx+Dx^2-6Dx+13D.$$

$$\begin{array}{l|l}x^3 & C=0,\\ x^2 & -6C+D=5,\\ x^1 & A+13C-6D=0,\\ x^0 & B+13D=-12\end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\}\Rightarrow\begin{array}{l}C=0,\\ D=5,\\ A=30,\\ B=-77.\end{array}$$

$$I=\int\frac{30x-77}{((x-3)^2+2^2)}dx+5\int\frac{dx}{(x-3)^2+2^2}=I_1+I_2,$$

$$I_1=\int\frac{30x-77}{((x-3)^2+2^2)}\,dx=\begin{bmatrix}x-3=z, & x=z+3,\\ & dx=dz\end{bmatrix}=\int\frac{30z+13}{(z^2+2^2)}\,dz=$$

$$=15\int\frac{d\,(z^2+4)}{(z^2+4)^2}+13\int\frac{dz}{(z^2+4)^2},$$

$$\int\frac{dz}{(z^2+4)^2}=\frac{1}{4}\int\frac{(z^2+4)-z^2}{(z^2+4)^2}\,dz=\frac{1}{4}\left(\int\frac{dz}{z^2+4}-\int\frac{z^2dz}{(z^2+4)^2}\right)=$$

$$=\begin{bmatrix}u=z, & du=dz,\\ dv=\dfrac{z\,dz}{(z^2+4)^2}, & v=-\dfrac{1}{2(z^2+4)}\end{bmatrix}=\frac{1}{4}\left(\frac{1}{2}\operatorname{arctg}\frac{z}{2}+\frac{z}{2(z^2+4)}-\right.$$

$$-\frac{1}{2}\int\frac{dz}{z^2+4}\Bigg)=\frac{1}{4}\left(\frac{1}{2}\operatorname{arctg}\frac{z}{2}+\frac{z}{2(z^2+4)}-\frac{1}{4}\operatorname{arctg}\frac{z}{2}\right)=$$

$$=\frac{1}{16}\operatorname{arctg}\frac{x-3}{2}+\frac{x-3}{8(x^2-6x+13)},$$

$$I_1=-\frac{15}{x^2-6x+13}+\frac{13}{16}\operatorname{arctg}\frac{x-3}{2}+\frac{13(x-3)}{8(x^2-6x+13)},$$

$$I_2=\frac{5}{2}\operatorname{arctg}\frac{x-3}{2},$$

$$\underline{\underline{I=\frac{13x-159}{x^2-6x+13}+\frac{53}{16}\operatorname{arctg}\frac{x-3}{2}+C.}}$$

376 (2051). $\displaystyle\int\frac{(x+1)^4\,dx}{(x^2+2x+2)^3}=I.$

$$\frac{(x+1)^4}{(x^2+2x+2)^3}=\frac{(x+1)^4}{((x+1)^2+1)^3}=\frac{z^4}{(z^2+1)^3}=\frac{Az+B}{(z^2+1)^3}+\frac{Cz+D}{(z^2+1)^2}+$$

$$+\frac{Kz+N}{z^2+1},\text{ где } z=x+1,$$

$$z^4=(Az+B)+(Cz+D)(z^2+1)+(Kz+N)(z^2+1)^2.$$

$z=0:\quad 0=B+D+N,$

$z=1:\quad 1=A+B+2C+2D+4K+4N,$

$z=-1:\quad 1=-A+B+4C+2D+8K+4N,$

$z=i:\quad 1=Ai+B,$

$z=-i:\quad 1=-Ai+B,$

$z=2:\quad 16=2A+B+10C+5D+50K+25N,$

$B=1,\ A=0,$

$$\begin{cases} 2C + D + 10K + 5N = 3, \\ 2C + D + 4K + 2N = 0, \\ 2C + 2D + 4K + 4N = 0, \\ D + N = -1 \end{cases} \Rightarrow \begin{matrix} N = 1, \\ D = -2, \\ K = 0, \\ C = 0. \end{matrix}$$

$$\frac{z^4}{(z^2+1)^3} = \frac{1}{(z^2+1)^3} - \frac{2}{(z^2+1)^2} + \frac{1}{z^2+1}.$$

$I = I_1 + I_2 + I_3$, где

$$I_1 = \int \frac{dz}{(z^2+1)^3} = \int \frac{(z^2+1) - z^2}{(z^2+1)^3} dz = \int \frac{dz}{(z^2+1)^2} - \int \frac{z^2 dz}{(z^2+1)^3} =$$

$$= \left[\begin{matrix} u = z, & du = dz, \\ dv = \dfrac{z\,dz}{(z^2+1)^2}, & v = \dfrac{1}{2}\displaystyle\int \dfrac{d(z^2+1)}{(z^2+1)^3} = -\dfrac{1}{4(z^2+1)^2} \end{matrix} \right] = \int \frac{dz}{(z^2+1)^2} -$$

$$-\left(-\frac{z}{4(z^2+1)^2} + \frac{1}{4}\int \frac{dz}{(z^2+1)^2} \right) = \frac{3}{4}\int \frac{dz}{(z^2+1)^2} + \frac{z}{4(z^2+1)^2},$$

$$I_2 = -2\int \frac{dz}{(z^2+1)^2},$$

$$I_3 = \int \frac{dz}{z^2+1}.$$

$$I_1 + I_2 = -\frac{5}{4}\int \frac{dz}{(z^2+1)^2} + \frac{z}{4(z^2+1)^2},$$

$$\int \frac{dz}{(z^2+1)^2} = \int \frac{(z^2+1) - z^2}{(z^2+1)^2} dz = \int \frac{dz}{z^2+1} - \int \frac{z^2 dz}{(z^2+1)^2} =$$

$$= \left[\begin{matrix} u = z, & du = dz, \\ dv = \dfrac{z\,dz}{(z^2+1)^2}, & v = \dfrac{1}{2}\displaystyle\int \dfrac{d(z^2+1)}{(z^2+1)^2} = -\dfrac{1}{2(z^2+1)} \end{matrix} \right] = \int \frac{dz}{z^2+1} +$$

$$+\frac{z}{2(z^2+1)}-\frac{1}{2}\int\frac{dz}{z^2+1}=\frac{1}{2}\int\frac{dz}{z^2+1}+\frac{z}{2(z^2+1)},$$

$$I_1+I_2=-\frac{5}{8}\int\frac{dz}{z^2+1}-\frac{5z}{8(z^2+1)}+\frac{z}{4(z^2+1)^2}=-\frac{5}{8}\int\frac{dz}{z^2+1}-$$

$$-\frac{5z}{8(z^2+1)}+\frac{z}{4(z^2+1)^2}.$$

$$I=\frac{3}{8}\int\frac{dz}{z^2+1}-\frac{5}{8}\cdot\frac{z}{z^2+1}+\frac{z}{4(z^2+1)^2}=\frac{3}{8}\operatorname{arctg} z-\frac{5z^3+3z}{8(z^2+1)}+C=$$

$$=\underline{\underline{\frac{3}{8}\operatorname{arctg}(x-1)-\frac{5x^3+15x^2+18x+8}{8(x^2+2x+2)^2}+C}}.$$

377 (2052). $\displaystyle\int\frac{dx}{(x^2+9)^3}=\frac{1}{9}\int\frac{(x^2+9)-x^2}{(x^2+9)^3}\,dx=\frac{1}{9}\int\frac{dx}{(x^2+9)^2}-$

$$-\frac{1}{9}\int\frac{x^2dx}{(x^2+9)^3}=\frac{1}{9}I_2-\frac{1}{9}I_1=I,\ \text{где}$$

$$I_1=\int\frac{x^2dx}{(x^2+9)^3}=$$

$$=\left[\begin{array}{ll} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(x^2+9)^3}, & v=\dfrac{1}{2}\displaystyle\int\frac{d\,(x^2+9)}{(x^2+9)^3}=-\frac{1}{4}\cdot\frac{1}{(x^2+9)^2}\end{array}\right]=$$

$$=-\frac{x}{4(x^2+9)^2}+\frac{1}{4}\int\frac{dx}{(x^2+9)^2},$$

$$I=\frac{1}{9}\int\frac{dx}{(x^2+9)^2}+\frac{1}{36}\cdot\frac{x}{(x^2+9)^2}-\frac{1}{36}\int\frac{dx}{(x^2+9)^2}=$$

$$=\frac{1}{12}\int\frac{dx}{(x^2+9)^2}+\frac{1}{36}\cdot\frac{x}{(x^2+9)^2}=\frac{1}{12}I_2+\frac{1}{36}\cdot\frac{x}{(x^2+9)^2},$$

$$I_2=\int\frac{dx}{(x^2+9)^2}=\frac{1}{9}\int\frac{(x^2+9)-x^2}{(x^2+9)^2}dx=\frac{1}{9}\int\frac{dx}{x^2+9}-\frac{1}{9}\int\frac{x^2dx}{(x^2+9)^2},$$

$$\int\frac{x^2dx}{(x^2+9)^2}=\left[\begin{array}{ll} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(x^2+9)^2}, & v=\dfrac{1}{2}\displaystyle\int\frac{d\,(x^2+9)}{(x^2+9)^2}=-\dfrac{1}{2(x^2+9)}\end{array}\right]=$$

$$=-\frac{x}{2(x^2+9)}+\frac{1}{2}\int\frac{dx}{x^2+9}=-\frac{x}{2(x^2+9)}+\frac{1}{6}\operatorname{arctg}\frac{x}{3},$$

$$I_2=\frac{1}{9\cdot 3}\operatorname{arctg}\frac{x}{3}+\frac{x}{18(x^2+9)}-\frac{1}{54}\operatorname{arctg}\frac{x}{3}=\frac{1}{54}\operatorname{arctg}\frac{x}{3}+\frac{x}{18(x^2+9)}.$$

$$I=\frac{1}{648}\operatorname{arctg}\frac{x}{3}+\frac{1}{216}\cdot\frac{x}{x^2+9}+\frac{1}{36}\cdot\frac{x}{(x^2+9)^9}+C.$$

378 (2053). $\displaystyle\int\frac{2x\,dx}{(1+x)(1+x^2)^2}=I.$

$$\frac{2x}{(1+x)(1+x^2)^2}=\frac{A}{1+x}+\frac{Bx+C}{(1+x^2)^2}+\frac{Dx+E}{1+x^2},$$

$$2x=A(1+x^2)^2+(Bx+C)(1+x)+(Dx+E)(1+x)(1+x^2).$$

$$\left.\begin{array}{ll} x=0: & 0=A+C+E, \\ x=1: & 2=4A+2B+2C+4D+4E, \\ x=-1: & -2=4A, \\ x=i: & 2i=B(i-1)+C(i+i) \end{array}\right\}\Rightarrow\ A=-\frac{1}{2},\ 2i=(B+C)i+$$

$$+(C-B)\ \Rightarrow\ \begin{cases} B+C=2, \\ -B+C=0 \end{cases}\Rightarrow\ \begin{array}{l} C=1, \\ B=1, \end{array}$$

$$\begin{cases} E=-\dfrac{1}{2}, \\ D+E=0 \end{cases}\ \Rightarrow\ D=\frac{1}{2}.$$

$$I=-\frac{1}{2}\int\frac{dx}{x+1}+\int\frac{x+1}{(1+x^2)^2}\,dx+\frac{1}{2}\int\frac{x-1}{1+x^2}\,dx,$$

$$\int\frac{x-1}{1+x^2}\,dx=\frac{1}{2}\int\frac{d(1+x^2)}{(1+x^2)^2}+\int\frac{dx}{(1+x^2)^2}=-\frac{1}{2(1+x^2)}+$$

$$+\int\frac{(1+x^2)-x^2}{(1+x^2)^2}dx=-\frac{1}{2(1+x^2)}+\int\frac{dx}{1+x^2}-\int\frac{x^2dx}{(1+x^2)^2}=$$

$$=\left[\begin{matrix} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(1+x^2)^2}, & v=\dfrac{1}{2}\displaystyle\int\frac{d(1+x^2)}{(1+x^2)^2}=-\dfrac{1}{2(1+x^2)}\end{matrix}\right]=-\frac{1}{2(1+x^2)}+$$

$$+\operatorname{arctg}x+\frac{x}{2(1+x^2)}-\frac{1}{2}\int\frac{dx}{1+x^2}=-\frac{1}{2(1+x^2)}+\frac{1}{2}\operatorname{arctg}x+\frac{x}{2(1+x^2)},$$

$$\int\frac{x-1}{1+x^2}\,dx=\frac{1}{2}\int\frac{d\,(1+x^2)}{1+x^2}-\int\frac{dx}{1+x^2}=\frac{1}{2}\ln(1+x^2)-\operatorname{arcrg}x,$$

$$I=-\frac{1}{2}\ln|x+1|-\frac{1}{2(1+x^2)}+\frac{1}{2}\operatorname{arctg}x+\frac{x}{2(1+x^2)}+\frac{1}{4}\ln(1+x^2)-$$

$$-\frac{1}{2}\operatorname{arctg}x+C=\underline{\underline{-\frac{1}{2}\ln|x+1|+\frac{x-1}{2(1+x^2)}+\frac{1}{4}\ln(1+x^2)+\frac{1}{4}\ln(1+x^2)+C}}.$$

379 (2054). $\displaystyle\int\frac{dx}{(1+x^2)^4}=\int\frac{(1+x^2)-x^2}{(1+x^2)^4}dx=\int\frac{dx}{(1+x^2)^3}-$

$$-\int\frac{x^2dx}{(1+x^2)^4}=\left[\begin{matrix} u=x, & du=dx, \\ \dfrac{x\,dx}{(1+x^2)^4}=dv, & v=-\dfrac{1}{6}\cdot\dfrac{1}{(1+x^2)^3}\end{matrix}\right]=\int\frac{dx}{(1+x^2)^3}+$$

$$+\frac{x}{6(1+x^2)^3}-\frac{1}{6}\int\frac{dx}{(1+x^2)^3}=\frac{5}{6}\int\frac{dx}{(1+x^2)^2}-\frac{5}{6}\int\frac{x^2dx}{(1+x^2)^3}=$$

$$=\frac{5}{6}\int\frac{(1+x^2)-x^2}{(1+x^2)^3}dx+\frac{x}{6(1+x^2)^3}=\frac{5}{6}\int\frac{dx}{(1+x^2)^2}-$$

$$-\frac{5}{6}\int\frac{x^2}{(1+x^2)^3}dx+\frac{x}{6(1+x^2)^3}=\begin{bmatrix} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(1+x^2)^3}, & v=-\dfrac{1}{4(1+x^2)^2}\end{bmatrix}=$$

$$=\frac{5}{6}\int\frac{dx}{(1+x^2)^2}-\frac{5}{6}\left(-\frac{x}{4(1+x^2)^2}+\frac{1}{4}\int\frac{dx}{(1+x^2)^2}\right)+\frac{x}{6(1+x^2)^3}=$$

$$=\frac{5}{6}\int\frac{dx}{(1+x^2)^2}+\frac{5x}{24(1+x^2)^2}-\frac{5}{24}\int\frac{dx}{(1+x^2)^2}+\frac{x}{6(1+x^2)^3}=$$

$$=\frac{15}{24}\int\frac{dx}{(1+x^2)^2}+\frac{5x}{24(1+x^2)^2}-\frac{x}{6(1+x^2)^3}=\frac{5}{8}\int\frac{(1+x^2)-x^2}{(1+x^2)^2}dx+$$

$$+\frac{5x}{24(1+x^2)^2}+\frac{x}{6(1+x^2)^3}=\frac{5}{8}\int\frac{dx}{1+x^2}-\frac{5}{8}\int\frac{x^2dx}{(1+x^2)^2}+$$

$$+\frac{5x}{24(1+x^2)^2}+\frac{x}{6(1+x^2)^3}=\begin{bmatrix} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(1+x^2)}, & v=-\dfrac{1}{2(1+x^2)}\end{bmatrix}=$$

$$=\frac{5}{8}\operatorname{arctg}x-\frac{5}{8}\left(-\frac{x}{2(1+x^2)}+\frac{1}{2}\int\frac{dx}{1+x^2}\right)+\frac{5x}{24(1+x^2)^2}+\frac{x}{6(1+x^2)^3}=$$

$$=\frac{5}{8}\operatorname{arctg}x+\frac{5x}{16(1+x^2)}-\frac{5}{16}\operatorname{arctg}x+\frac{5x}{24(1+x^2)^2}+\frac{x}{6(1+x^2)^3}+C=$$

$$=\frac{5}{16}\operatorname{arctg}x+\frac{15x^5+40x^3+33x}{48(1+x^2)^3}+C.$$

380 (2055). $\int \frac{x^9 dx}{(x^4-1)^2} = I.$

$$\begin{array}{r|l} -\,x^9 & x^8-2x^4+1 \\ \cline{2-2} \underline{x^9-2x^5+x} & x \\ 2x^5-x & \end{array}$$

$$I = \int x\,dx + \int \frac{2x^5-x}{(x^4-1)^2}\,dx = \int x\,dx + \int \frac{2x(x^4-1)+x}{(x^4-1)^2}\,dx = \frac{x^2}{2} +$$

$$+\int \frac{d(x^2)}{(x^2)^2-1} + \frac{1}{2}\int \frac{dx^2}{((x^2)^2-1)^2} = [x^2 = z] = \frac{x^2}{2} + \int \frac{dz}{z^2-1} +$$

$$+\frac{1}{2}\int \frac{dz}{(z^2-1)^2} = \frac{x^2}{2} + \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| - \frac{1}{2}\int \frac{(z^2-1)-z^2}{(z^2-1)^2}\,dz = \frac{x^2}{2} +$$

$$+\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| - \frac{1}{2}\int \frac{dz}{z^2-1} + \frac{1}{2}\int \frac{z^2 dz}{(z^2-1)^2} =$$

$$= \left[\begin{array}{ll} u = z, & du = dz, \\ dv = \dfrac{z\,dz}{(z^2-1)^2}, & v = -\dfrac{1}{2(z^2-1)} \end{array}\right] = \frac{x^2}{2} + \frac{1}{4}\ln\left|\frac{z-1}{z+1}\right| -$$

$$-\frac{1}{4(z^2-1)} + \frac{1}{4}\int \frac{dz}{z^2-1} = \frac{x^2}{2} + \frac{3}{8}\ln\left|\frac{z-1}{z+1}\right| - \frac{z}{4(z^2-1)} + C = \frac{x^2}{2} +$$

$$+\frac{3}{8}\ln\left|\frac{x^2-1}{x^2+1}\right| - \frac{x^2}{4(x^4-1)} = \underline{\underline{\frac{1}{4}\left(\frac{2x^6+3x^2}{x^4-1} + \frac{3}{2}\ln\left|\frac{x^2-1}{x^2+1}\right|\right) + C}}.$$

5. Метод Остроградского.

Метод Остроградского позволяет вычислить интеграл от рациональной функции в случае наличия *кратных* корней у знаменателя. Этот способ позволяет, не производя разложения дроби на простейшие, выделить рациональную часть интеграла, а потом интегрировать рациональную дробь, знаменатель которой имеет только *простые* корни.

В основе метода лежит следующая формула Остроградского:

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\,dx, \tag{1}$$

где $\dfrac{P(x)}{Q(x)}$ – правильная рациональная дробь,

$$Q(x) = (x-a_1)^{\alpha_1}\cdot\ldots\cdot(x-a_r)^{\alpha_r}(x^2+p_1x+q_1)^{\beta_1}\cdot\ldots\cdot(x^2+p_sx+q_s)^{\beta_s},$$

а разложение $Q_1(x)$ и $Q_2(x)$ имеет вид

$$Q_1(x) = (x-a_1)^{\alpha_1-1}\ldots(x-a_r)^{\alpha_r-1}(x^2+p_1x+q_1)^{\beta_1-1}\ldots(x^2+p_sx+q_s)^{\beta_s-1},$$

$$Q_2(x) = (x-a_1)\cdot\ldots\cdot(x-a_r)(x^2+p_1x+q_1)\cdot\ldots\cdot(x^2+p_sx+q_s);$$

$P_1(x)$ и $P_2(x)$ – многочлены с неопределенными коэффициентами, степени которых соответственно на единицу меньше степеней $Q_1(x)$ и $Q_2(x)$. Применение метода Остроградского покажем на примерах решения задач.

381 (2056). $\displaystyle\int \frac{x^7+2}{(x^2+x+1)^2}\,dx = I.$

$$\begin{array}{l|l}
-\;x^7+2 & x^4+2x^3+3x^2+2x+1+1 \\ \cline{2-2}
\underline{\;\;x^7+2x^6+3x^5+2x^4+x^3} & x^3-2x^2+x+2 \\
\qquad -\;-2x^6-3x^5-2x^4-x^3+2 & \\
\qquad \underline{\;\;\;-2x^6-4x^5-6x^4-4x^3-2x^2} & \\
\qquad\qquad\qquad -\;x^5+4x^4+3x^3+2x^2+2 & \\
\qquad\qquad\qquad \underline{\;\;\;x^5+2x^4+3x^3+2x^2+x} & \\
\qquad\qquad\qquad\qquad -\;2x^4-x+2 & \\
\qquad\qquad\qquad\qquad \underline{\;\;\;2x^4+4x^3+6x^2+4x+2} & \\
\qquad\qquad\qquad\qquad\qquad -4x^3-6x^2-5x &
\end{array}$$

$$I = \int (x^3-2x^2+x+2)\,dx - \int \frac{4x^3+6x^2+5x}{(x^2+x+1)^2}\,dx.$$

По формуле (1) имеем:

$$-\int \frac{4x^3+6x^2+5x}{(x^2+x+1)^2}\,dx=\frac{Ax+B}{x^2+x+1}+\int \frac{Cx+D}{x^2+x+1}\,dx. \qquad (2)$$

Дифференцируя обе части равенства (2), получаем

$$-\frac{4x^3+6x^2+5x}{(x^2+x+1)^2}=\frac{A(x^2+x+1)-(2x+1)(Ax+B)}{(x^2+x+1)^2}+\frac{Cx+D}{x^2+x+1},$$

$$-\frac{4x^3+6x^2+5x}{(x^2+x+1)^2}=\frac{-Ax^2-2Bx+A-B}{(x^2+x+1)^2}+\frac{Cx+D}{x^2+x+1}.$$

Освободившись от знаменателя, будем иметь:

$$-(4x^3+6x^2+6x)=-Ax^2-2Bx+A-B+Cx^3+Cx^2+Cx+$$

$$+Dx^2+Dx+D.$$

Приравняв коэффициенты при одинаковых степенях x, получим систему уравнений

$$\left.\begin{array}{l|l} x^3 & C=-4, \\ x^2 & -A+C+D=-6, \\ x^1 & -2B+C+D=-5, \\ x^0 & A-B+D=0 \end{array}\right\} \Rightarrow \begin{array}{l} C=-4, \\ D=-1, \\ B=0, \\ A=1. \end{array}$$

$$I=\frac{x^4}{4}-\frac{2}{3}x^3+\frac{x^2}{2}+2x+\frac{x}{x^2+x+1}-4\int\frac{\left(x+\frac{1}{4}\right)dx}{x^2+x+1}=\frac{x^4}{4}-\frac{2}{3}x^3+\frac{x^2}{2}+$$

$$+2x+\frac{x}{x^2+x+1}-\int\frac{d\,(x^2+x+1)}{x^2+x+1}+\int\frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=$$

$$=\frac{x^4}{4}-\frac{2}{3}x^3+\frac{x^2}{2}+2x+\frac{x}{x^2+x+1}-2\ln\left|x^2+x+1\right|+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C.$$

382 (2057). $\displaystyle\int \frac{(4x^2-8x)dx}{(x-1)^2(x^2+1)^2} = \frac{Ax^2+Bx+C}{(x-1)(x^2+1)} + \int \frac{Dx^2+Lx+N}{(x-1)(x^2+1)}\,dx = I.$

Дифференцируя обе части равенства, получим

$$\frac{4x^2-8x}{(x-1)^2(x^2+1)^2} =$$

$$= \frac{(2Ax+B)(x-1)(x^2+1)-(Ax^2+Bx+C)(x^2+1+2x(x-1))}{(x-1)^2(x^2+1)^2} +$$

$$+\frac{Dx^2+Lx+N}{(x-1)(x^2+1)} \Rightarrow 4x^2-8x = (2Ax+B)(x-1)(x^2+1)-$$

$$-(Ax^2+Bx+C)(x^2+1+2x(x-1))+(Dx^2+Lx+N)(x-1)(x^2+1).$$

$$x=0:\quad 0=-B+C-N,$$

$$x=1:\quad -4=-2(A+B+C),$$

$$x=i:\quad -4-8i=-2((A-C+B)+(A-B-C)i),$$

$$\begin{cases} -B+C-N=0, \\ A+B+C=2, \\ A-C+B=2, \\ A-B-C=4 \end{cases} \Rightarrow \begin{array}{l} A=3, \\ B=-1, \\ C=0, \\ N=1. \end{array}$$

$$4x^2-8x=(6x-1)(x-1)(x^2+1)-(3x^2-x)(3x^2-2x+1)+$$

$$+(Dx^2+Lx+1)(x-1)(x^2+1),$$

$$x=-1:\quad 12=28-24-4(D-L+1),$$

$$x=2:\quad 0=55-99+5(4D+2L+1),$$

$$\begin{cases} -D+L=3, \\ 2D+L=3 \end{cases} \Rightarrow \begin{array}{l} D=0, \\ L=3, \end{array}$$

$$I=\frac{3x^2-x}{(x-1)(x^2+1)}+\int\frac{3x+1}{(x-1)(x^2+1)}\,dx.$$

$$\frac{3x+1}{(x-1)(x^2+1)}=\frac{P}{x-1}+\frac{Qx+G}{x^2+1},$$

$$3x+1=P(x^2+1)+(Qx+G)(x-1).$$

$$\left.\begin{array}{ll} x=0: & 1=P-G, \\ x=1: & 4=2P, \\ x=2: & 7=5P+2Q+G \end{array}\right\} \Rightarrow \begin{array}{l} P=2, \\ Q=-2, \\ G=1. \end{array}$$

$$I=\frac{3x^2-x}{(x-1)(x^2+1)}+2\int\frac{dx}{x-1}+\int\frac{-2x+1}{x^2+1}\,dx=\frac{3x^2-x}{(x-1)(x^2+1)}+$$

$$+2\ln|x-1|-\int\frac{d(x^2+1)}{x^2+1}+\int\frac{dx}{x^2+1}=\frac{3x^2-x}{(x-1)(x^2+1)}+2\ln|x-1|-$$

$$-\ln(x^2+1)+\operatorname{arctg}x+C=\underline{\underline{\frac{3x^2-x}{(x-1)(x^2+1)}+\ln\frac{(x-1)^2}{x^2+1}+\operatorname{arctg}x+C}}.$$

383 (2058). $\displaystyle\int\frac{(x^2+x+1)\,dx}{x^5-2x^4+x^3}=I,$

$$I=\int\frac{(x^2+x+1)\,dx}{x^3(x-1)^2}=\frac{Ax^2+Bx+C}{x^2(x-1)}+\int\frac{Dx+L}{x(x-1)}\,dx.$$

Дифференцируя обе части равенства, получим

$$\frac{x^2+x+1}{x^5-2x^4+x^3}=\frac{(2Ax+B)x^2(x-1)-(Ax^2+Bx+C)(3x^2-2x)}{x^4(x-1)^2}+$$

$$+\frac{Dx+L}{x(x-1)}\Rightarrow(x^2+x+1)x=(2Ax+B)x^2(x-1)-$$

$$-(Ax^2+Bx+C)(3x^2-2x)+(Dx+L)x^3(x-1).$$

$$\left.\begin{array}{ll} x=1: & -3=A+B+C, \\ x=-1: & -1=-A+3B-5C-2D+2L, \\ x=2: & 14=-16A-12B-8C+16D+8L, \\ x=-2: & -6=-16A+20B-16C-48D+24L, \\ x=3: & 39=-81A-45B-21C+162D+54L \end{array}\right\} \Rightarrow \begin{array}{l} A=-6, \\ B=\dfrac{5}{2}, \\ C=\dfrac{1}{2}, \\ D=0, \\ L=-6. \end{array}$$

$$I=-\frac{12x^2-5x-1}{2x^2(x-1)}-6\int\frac{dx}{x(x-1)}=-\frac{12x^2-5x-1}{2x^2(x-1)}-6\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx=$$

$$=-\frac{12x^2-5x-1}{2x^2(x-1)}-6\ln\left|\frac{x-1}{x}\right|+C.$$

384 (2059). $\displaystyle\int\frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}dx=\frac{Ax^3+Bx^2+Cx+D}{x^2(x^2+1)}+$

$$+\int\frac{Mx^2+Nx+L}{x(x^2+1)}dx=I.$$

Дифференцируя обе части равенства, получим

$$\frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}=\frac{-Ax^5-2Bx^4+Ax^3-Cx^3-Cx-4Dx^2-2D}{x^3(x^2+1)^2}+$$

$$+\frac{Mx^2+Nx+L}{x(x^2+1)}.$$

Приравнивая коэффициенты при одинаковых степенях x, имеем систему

$$\begin{cases} M=1,\\ -A+N=0,\\ -2B+M+L=1,\\ A-C+N=4,\\ -4D+L=-4,\\ -C=0,\\ -2D=-2 \end{cases} \Rightarrow \quad \begin{array}{c} A=B=C=L=N=0,\\ M=D=1. \end{array}$$

$$I=\frac{1}{x^2(x^2+1)}+\int\frac{x^2dx}{x(x^2+1)}=\frac{1}{x^2(x^2+1)}+\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}=$$

$$=\frac{1}{x^2(x^2+1)}+\ln\sqrt{x^2+1}+C.$$

385 (2060). $\int \frac{(x^2-1)^2\,dx}{(1+x)(1+x^2)^3} = \frac{Ax^3+Bx^2+Cx+D}{(1+x^2)^2} +$

$+\int \frac{Mx^2+Nx+L}{(1+x^2)(1+x)}\,dx = I.$

Дифференцируя обе части равенства, получим

$$\frac{(x^2-1)^2}{(1+x)(1+x^2)^3} =$$

$$= \frac{(3Ax^2+2Bx+C)(1+x^2)-4x(Ax^3+Bx^2+Cx+D)}{(1+x^2)^3} +$$

$$+\frac{Mx^2+Nx+L}{(1+x^2)(1+x)},$$

$$x^4-2x^2+1 = 3Ax^2+2Bx+C+3Ax^3+2Bx^2+Cx+3Ax^4+$$

$$+2Bx^3+Cx^2+3Ax^5+2Bx^4+Cx^4+Cx^3-4Ax^4-4Bx^3-$$

$$-4Cx^2-4Dx-4Ax^5-4Bx^4-4Cx^3-4Dx^2+Mx^2+Nx+L+$$

$$+Mx^4+2Nx^3+Mx^6+Nx^5+2Lx^2+Lx^4.$$

$$\left.\begin{array}{l|l}
x^6 & M=0, \\
x^5 & -A+N=0, \\
x^4 & -A-2B+2M+L=1, \\
x^3 & 3A-2B-3C+2N=0, \\
x^2 & 3A+2B-3C-4D+M+2L=-2, \\
x^1 & 2B+C-4D+N=0, \\
x^0 & L+C=1
\end{array}\right\} \Rightarrow
\begin{array}{ll}
 & A=\frac{1}{4}, \\
M=0, & N=\frac{1}{4}, \\
A=N, & \\
L=1-C, & C=\frac{3}{4}, \\
C=3A, & B=-\frac{1}{2}, \\
D=0, & \\
 & L=\frac{1}{4}.
\end{array}$$

$$I=\frac{1}{4}\cdot\frac{x^3-2x^2+3x}{(x^2+1)^2}+\frac{1}{4}\int\frac{x+1}{(x^2+1)(x+1)}\,dx=$$

$$=\frac{1}{4}\cdot\frac{(2+2x)+(x^3-2x^2+x-2)}{(x^2+1)^2}+\frac{1}{4}\operatorname{arctg}x+C=\frac{1}{2}\cdot\frac{1+x}{(x^2+1)^2}+$$

$$+\frac{1}{4}\cdot\frac{x^2(x-2)+(x-2)}{(x^2+1)^2}+\frac{1}{4}\operatorname{arctg}x+C=$$

$$=\frac{1}{2}\cdot\frac{1+x}{(x^2+1)^2}+\frac{1}{4}\cdot\frac{x-2}{x^2+1}+\frac{1}{4}\operatorname{arctg}x+C.$$

386 (2061). $\displaystyle\int\frac{dx}{x^4(x^3+1)^2}=I.$

$$\frac{1}{x^4(x^3+1)^2}=\frac{A}{x^4}+\frac{B}{x^3}+\frac{C}{x^2}+\frac{D}{x}+\frac{Ex^2+Fx+Q}{(x^3+1)^2}+\frac{Mx^2+Nx+P}{x^3+1},$$

$$1=A(x^6+2x^3+1)+B(x^7+2x^4+x)+C(x^8+2x^5+x^2)+$$

$$+D(x^9+2x^6+x^3)+Ex^6+Fx^5+Qx^4+Mx^9+Nx^8+Px^7+$$

$$+Mx^6+Nx^5+Px^4.$$

$$\left.\begin{array}{l|l}
x^9 & M+D=0,\\
x^8 & C+N=0,\\
x^7 & B+P=0,\\
x^6 & A+2D+E+M=0,\\
x^5 & 2C+F+N=0,\\
x^4 & 2B+Q+P=0,\\
x^3 & 2A+D=0,\\
x^2 & C=0,\\
x^1 & B=0,\\
x^0 & A=1
\end{array}\right\}\Rightarrow\quad
\begin{array}{l}
A=1,\\ C=0,\\ B=0,\\ P=0,\\ N=0,\\ F=0,\\ Q=0,\\ D=-2,\\ E=1,\\ M=2.
\end{array}$$

$$I=\int\frac{dx}{x^4}-2\int\frac{dx}{x}+\int\frac{x^2dx}{(x^3+1)^2}+2\int\frac{x^2dx}{x^3+1}=-\frac{1}{3x^3}-2\ln|x|+$$

$$+\frac{1}{3}\int\frac{d(x^3+1)}{(x^3+1)^2}+\frac{2}{3}\int\frac{d(x^3+1)}{x^3+1}=-\frac{1}{3x^2}-3\cdot\frac{2}{3}\ln|x|-\frac{1}{3}\cdot\frac{1}{x^3+1}+$$

$$+\frac{2}{3}\ln\left|x^3+1\right|+C=\underline{\underline{\frac{2}{3}\ln\left|\frac{x^3+1}{x^3}\right|-\frac{1}{3x^3}-\frac{1}{3(x^3+1)}}}+C.$$

387 (2062). $\displaystyle\int\frac{dx}{(x^2+2x+10)^3}=\frac{Ax^3+Bx^2+Cx+D}{(x^2+2x+10)^2}+\int\frac{Mx+N}{x^2+2x+10}=I.$

Дифференцируя обе части равенства, получим

$$\frac{1}{(x^2+2x+10)^3}=$$

$$=\frac{(3Ax^2+2Bx+C)(x^2+2x+10)-2(2x+2)(Ax^3+Bx^2+Cx+D)}{(x^2+2x+10)^3}+$$

$$+\frac{Mx+N}{x^2+2x+10},$$

$$1=3Ax^4+2Bx^3+Cx^2+6Ax^3+4Bx^2+2Cx+30Ax+20Bx+10C-$$
$$-4Ax^4-4Bx^3-4Cx^2-4Dx-4Ax^3-4Bx^2-4Cx-4D+Mx^5+$$
$$+Nx^4+4Mx^3+4Nx^2+100Mx+100N+4Mx^4+4Nx^3+20Mx^3+$$
$$+20Nx^2+40Mx^2+40Nx.$$

$$\left.\begin{array}{l|l}
x^5 & M=0,\\
x^4 & -A+N+4M=0,\\
x^3 & 2A-2B+24M+4N=0,\\
x^2 & 30A-3C+24N+40M=0,\\
x^1 & 20B-4D-2C+100M+40N=0,\\
x^0 & -4D+10C+100N=1
\end{array}\right\}\Rightarrow\begin{cases}M=0,\\ A=N,\\ B=3A,\end{cases}\Rightarrow\quad \begin{array}{l} N=A=\dfrac{1}{216},\\ B=\dfrac{3}{216},\\ C=\dfrac{18}{216},\\ D=\dfrac{16}{216}.\end{array}$$

$$I=\frac{1}{216}\cdot\frac{x^3+3x^2+18x+16}{(x^2+2x+10)^2}+\frac{1}{216}\int\frac{dx}{x^2+2x+10}=$$

$$=\frac{1}{216}\cdot\frac{(x^3+2x^2+10x)+(x^2+2x+10)+6x+6}{(x^2+2x+10)^2}+\frac{1}{216}\cdot\frac{1}{3}\operatorname{arctg}\frac{x+1}{3}+$$

$$+C=\frac{1}{216}\left(\frac{x+1}{x^2+2x+10}+\frac{6(x+1)}{(x^2+2x+10)^2}\right)+\frac{1}{648}\operatorname{arctg}\frac{x+1}{3}+C=$$

$$=\underline{\underline{\frac{1}{648}\left(\operatorname{arctg}\frac{x+1}{3}+\frac{3(x+1)}{x^2+2x+10}+\frac{18(x+1)}{(x^2+2x+10)^2}\right)+C.}}$$

388 (2063). $\displaystyle\int\frac{(x+2)\,dx}{(x^2+2x+2)^3}=\frac{Ax^3+Bx^2+Cx+D}{(x^2+2x+2)^3}+\int\frac{Mx+N}{x^2+2x+2}\,dx=I.$

Дифференцируя обе части равенства, получим

$$\frac{x+2}{(x^2+2x+2)^3}=$$

$$=\frac{(3Ax^2+2Bx+C)\,(x^2+2x+2)-(4x+4)\,(Ax^3+Bx^2+Cx+D)}{(x^2+2x+2)^3}+$$

$$+\frac{Mx+N}{x^2+2x+2},$$

$$x+2=3Ax^4+6Ax^3+6Ax^2+2Bx^3+4Bx^2+4Bx+Cx^2+$$

$$+2Cx+2C+4Ax^4-4Ax^3-4Bx^3-4Bx^2-4Cx^2-4Cx-4Dx-$$

$$-4D+Mx^5+4Mx^4+8Mx^3+8Mx^2+4Mx+Nx^4+4Nx^3+$$

$$+8Nx^2+8Nx+4N.$$

$$\left.\begin{array}{l|l} x^5 & M=0, \\ x^4 & -A+4M+N=0, \\ x^3 & 2A-2B+8M+4N=0, \\ x^2 & 6A-3C+8M+8N=0, \\ x^1 & 4B-2C-4D+4M+8N=1, \\ x^0 & 2C-4D+4N=2 \end{array}\right\} \Rightarrow \begin{array}{ll} M=0, & B=\dfrac{9}{8}, \\ A=N, & \\ C=\dfrac{14}{3}A, & C=\dfrac{7}{4}, \\ A=N=\dfrac{3}{8}, & D=\dfrac{3}{4}. \end{array}$$

$$I=\frac{\frac{3}{8}x^3+\frac{9}{8}x^2+\frac{7}{4}x+\frac{3}{4}}{(x^2+2x+2)^2}+\frac{3}{8}\int\frac{dx}{(x+1)^2+1}=\frac{3}{8}\operatorname{arctg}(x+1)+$$

$$+\frac{3}{8}\cdot\frac{x(x^2+2x+2)+(x^2+2x+2)+\frac{2}{3}x}{(x^2+2x+2)^2}+C=$$

$$=\underline{\underline{\frac{3}{8}\operatorname{arctg}(x+1)+\frac{3}{8}\cdot\frac{x+1}{x^2+2x+2}+\frac{1}{4}\cdot\frac{x}{(x^2+2x+2)}+C.}}$$

389 (2064). $\displaystyle\int\frac{x^5-x^4-26x^2-24x-25}{(x^2+4x+5)^5(x^2+4)^2}\,dx=\frac{Ax+B}{x^2+4x+5}+\frac{Cx+D}{x^2+4}+$

$$+\int\frac{Mx+N}{x^2+4x+5}\,dx+\int\frac{Lx+Q}{x^2+4}\,dx=I.$$

Взяв производные от обеих частей равенств и приравняв затем коэффициенты при одинаковых степенях x, получим систему из восьми уравнений

$$\begin{cases} M+L=0, \\ -A-C+4M+8L+N+Q=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -64B+80A+100C+100Q+80C=-25. \end{cases}$$

Решив ее, будем иметь: $A=-1,\ B=-\dfrac{5}{2},\ C=Q=-\dfrac{1}{8},\ D=M=L=0,$

$N=-1.$

$$I = C - \frac{x}{8(x^2+4)} - \frac{2x+5}{2(x^2+4x+5)} - \frac{1}{8}\int \frac{dx}{x^2+4} - \int \frac{dx}{(x+2)^2+1} =$$

$$= C - \frac{x}{8(x^2+4)} - \frac{2x+5}{2(x^2+4x+5)} - \frac{1}{16}\operatorname{arctg}\frac{x}{2} - \operatorname{arctg}(x+2).$$

390 (2065). $\displaystyle\int \frac{3x^4+4}{x^2(x^2+1)^3}dx = \frac{Ax^4+Bx^3+Cx^2+Dx+E}{x(x^2+1)^2} +$

$$+\int \frac{Fx^2+Lx+N}{x(x^2+1)}dx = I.$$

Дифференцируя обе части равенства, получим

$$\frac{3x^4+4}{x^2(x^2+1)^3} = \frac{(4Ax^3+3Bx^2+2Cx+D)(x^3+x)}{x^2(x^2+1)^3} -$$

$$-\frac{(5x^2+1)(Ax^4+Bx^3+Cx^2+Dx+E)}{x^2(x^2+1)^3} + \frac{Fx^2+Lx+N}{x(x^2+1)},$$

$$3x^4+4 = 4Ax^6+3Bx^5+2Cx^4+Dx^3+4Ax^4+3Bx^3+2Cx^2+$$

$$+Dx-5Ax^6-5Bx^5-5Cx^4-5Dx^3-5Ex^2-Ax^4-Bx^3-Cx^2-$$

$$-Dx-E+Fx^7+Lx^6+Nx^5+2Fx^5+2Lx^4+2Nx^3+Fx^3+$$

$$+Lx^2+Nx.$$

$$\left.\begin{array}{l|l}
x^7 & F=0, \\
x^6 & -A+L=0, \\
x^5 & -2B+N+2F=0, \\
x^4 & -3C+3A+2L=3, \\
x^3 & -4D+2B+2N+F=0, \\
x^2 & C-5E+L=0, \\
x^1 & N=0, \\
x^0 & -E=4
\end{array}\right\} \Rightarrow
\begin{array}{l}
E=-4, \\
A=L, \\
N=B=F=D=0, \\
A=-\dfrac{57}{8}, \\
C=-\dfrac{103}{8}.
\end{array}$$

$$I=-\frac{1}{8}\cdot\frac{57x^4+103x^2+32}{x(x^2+1)^2}-\frac{57}{8}\int\frac{x\,dx}{x(x^2+1)}=$$

$$=C-\frac{1}{8}\cdot\frac{57x^4+103x^2+32}{x(x^2+1)^2}-\frac{57}{8}\operatorname{arctg}x.$$

391 (2066). $\displaystyle\int\frac{5-3x+6x^2+5x^3-x^4}{x^5-x^4-2x^3+2x^2+x-1}\,dx=I.$

$$x^5-x^4-2x^3+2x^2+x-1=x^4(x-1)-2x^2(x-1)+(x-1)=$$

$$=(x-1)(x^4-2x^2+1)=(x-1)(x^2-1)^2=(x-1)^3(x+1)^2,$$

$$\frac{5-3x+6x^2+5x^3-x^4}{(x-1)^3(x+1)^2}=\frac{Ax^2+Bx+C}{(x-1)^2(x+1)}+\int\frac{Dx+E}{(x-1)(x+1)}\,dx.$$

$$\frac{5-3x+6x^2+5x^3-x^4}{x^5-x^4-2x^3+2x^2+x-1}=\frac{(2Ax+B)(x-1)^2(x+1)}{(x-1)^3(x+1)^2}-$$

$$-\frac{(2(x-1)(x+1)+(x-1)^2)(Ax^2+Bx+C)}{(x-1)^3(x+1)^2}+\frac{Dx+E}{(x-1)(x+1)},$$

$$5-3x+6x^2+5x^3-x^4=2Ax^3+Bx^2-2Ax-B-3Ax^3-3Bx^2-$$

$$-3Cx-Ax^2-Bx-C+Dx^4-Dx^3-Dx^2+D+Ex^3-Ex^2-Ex+E.$$

$$\left.\begin{array}{l|l}
x^4 & D=-1,\\
x^3 & -A-D+E=5,\\
x^2 & -A-2B-D-E=6,\\
x^1 & -2A-3C-B-E+D=-3,\\
x^0 & -B+E-C=5
\end{array}\right\}\Rightarrow
\begin{array}{l}
D=-1,\\
-A+E=4,\\
\begin{cases}2A+2B=-9,\\ 3A+B+3C=-2,\\ A-B-C=1.\end{cases}
\end{array}$$

$$\Delta=\begin{vmatrix}2&2&0\\3&1&3\\1&-1&-1\end{vmatrix}=16,\quad \Delta_A=\begin{vmatrix}-9&2&0\\-2&1&3\\1&-1&-1\end{vmatrix}=-16,$$

$$\Delta_B = \begin{vmatrix} 2 & -9 & 0 \\ 3 & -2 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -56, \quad \Delta_C = \begin{vmatrix} 2 & 2 & -9 \\ 3 & 1 & -2 \\ 1 & -1 & 1 \end{vmatrix} = 24,$$

$$A = -1, \; B = -\frac{7}{2}, \; C = \frac{3}{2}, \; E = 3.$$

$$I = \frac{3 - 7x - 2x^2}{2(x^3 - x^2 - x + 1)} + \int \frac{-x + 3}{(x-1)(x+1)}\,dx.$$

$$\frac{-x+3}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{2}{x+1},$$

$$\underline{\underline{I = \frac{3 - 7x - 2x^2}{2(x^3 - x^2 - x + 1)} + \ln\frac{|x-1|}{(x+1)^2} + C.}}$$

392 (2067). $\displaystyle\int \frac{9\,dx}{5x^2(3-2x^2)^3} = \frac{9}{5}\int \frac{dx}{x^2(3-2x^2)^3} = I.$

$$\int \frac{dx}{x^2(3-2x^2)^3} = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{x(3-2x^2)^2} + \int \frac{Mx^2 + Nx + L}{x(3-2x^2)}\,dx,$$

$$1 = (4Ax^3 - 3Bx^2 + 2Cx + D)(3x - 2x^3) - (3 - 10x^2)(Ax^4 + Bx^3 + Cx^2 +$$

$$+ Dx + E) + (Mx^2 + Nx + L)(9x - 12x^3 + 4x^5).$$

$$\left.\begin{array}{l|l} x^7 & 4M = 0, \\ x^6 & 2A + 4N = 0, \\ x^5 & 10B - 12M + 4L = 0, \\ x^4 & 9A + 6C - 12N = 0, \\ x^3 & -12B + 10D + 9M + 12L = 0, \\ x^2 & 3C + 10E + 9N = 0, \\ x^1 & 9L = 0, \\ x^0 & -3E = 1 \end{array}\right\} \Rightarrow \begin{array}{l} M = L = B = D = 0, \\ A = -2N, \\ C = 5N, \\ A = -\dfrac{5}{18}, \\ C = \dfrac{25}{36}, \\ N = \dfrac{5}{36}. \end{array}$$

$$I=\left(-\frac{1}{2}x^4+\frac{5}{4}x^2-\frac{3}{5}\right)\frac{1}{x(3-2x^2)^2}+\frac{1}{4}\int\frac{dx}{3-2x^2}=$$

$$=\left(-\frac{1}{2}x^4+\frac{5}{4}x^2-\frac{3}{5}\right)\frac{1}{x(3-2x^2)^2}+\frac{1}{8\sqrt{6}}\ln\left|\frac{\sqrt{3}+x\sqrt{2}}{\sqrt{3}-x\sqrt{2}}\right|+C.$$

3.2. Некоторые иррациональные функции

1. *Функции вида* $R\left(x, \sqrt[m]{\frac{ax+b}{a_1x+b_1}}, \sqrt[p]{\frac{ax+b}{a_1x+b_1}} \ldots\right)$.

Интегралы берутся подстановкой $\frac{ax+b}{a_1x+b_1}=t^n$, где n – наименьшее общее кратное (НОК) чисел m, p ...

393 (2068). $\displaystyle\int\frac{dx}{x\left(\sqrt{x}+\sqrt[5]{x^2}\right)}=\begin{bmatrix} x=t^{10},\ dx=10t^9dt, \\ t=\sqrt[10]{x}\end{bmatrix}=10\int\frac{t^9dt}{t^{10}(t^5+t^4)}=$

$$=10\int\frac{dt}{t^5(t+1)}=I.$$

$$\frac{1}{t^5(t+1)}=\frac{A}{t^5}+\frac{B}{t^4}+\frac{C}{t^3}+\frac{D}{t^2}+\frac{K}{t}+\frac{M}{t+1},$$

$$1=A(t+1)+Bt(t+1)+Ct^2(t+1)+Dt^3(t+1)+Kt^4(t+1)+Mt^5.$$

$$\left.\begin{array}{ll} t=0: & A=1, \\ t=-1: & -M=1, \\ t=1: & 2A+2B+2C+2D+2K+M=1, \\ t=2: & 3A+6B+12C+24D+48K+32M=1, \\ t=-2: & -A+2B-4C+8D-16K-32M=1, \\ t=3: & 4A+12B++36C+108D+324K+243M=1 \end{array}\right\}\Rightarrow$$

$$\Rightarrow \begin{cases} B+C+D+K=0, \\ B+2C+4D+8K=5, \\ B-2C+4D-8K=-15, \\ B+3C+9D+27K=20 \end{cases} \Rightarrow \begin{cases} C+3D+7K=5, \\ -3C+3D-9K=-15, \\ C+4D+13D=10 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} D+K=0, \\ D+6K=5 \end{cases} \Rightarrow K=1,\ D=-1,\ C=1,\ B=-1.$$

$$I=10\left(\int\frac{dt}{t^5}-\int\frac{dt}{t^4}+\int\frac{dt}{t^3}-\int\frac{dt}{t^2}+\int\frac{dt}{t}-\int\frac{dt}{t+1}\right)=$$

$$=-\frac{5}{2\sqrt[5]{x^2}}+\frac{10}{3\sqrt[10]{x^3}}-\frac{5}{\sqrt[5]{x}}+\frac{10}{\sqrt[10]{x}}+\ln\frac{x}{\left(1+\sqrt[10]{x}\right)^{10}}+C.$$

394 (2069). $\displaystyle\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}+2\sqrt[4]{x}}=\begin{bmatrix}\sqrt[12]{x}=t,\ x=t^{12}, \\ dx=12t^{11}dt\end{bmatrix}=\int\frac{12t^{11}dt}{t^6+t^4+2t^3}=$

$$=\int\frac{12t^{11}dt}{t^6+t^4+2t^3}=12\int\frac{t^8dt}{t^3+t+2}=12\left(\int(t^5\ \ -t^3-2t^2+t+4)\,dt+\right.$$

$$\left.+\int\frac{3t^2-6t-8}{t^3+t+2}\,dt\right)=12\left(\left(\frac{t^6}{6}-\frac{t^4}{4}-\frac{2}{3}t^3+\frac{t^2}{2}+4t\right)+\int\frac{3t^2-6t-8}{t^3+t+2}\,dt\right)=$$

$$=2t^6-3t^4-8t^3+6t^2+48t+12\int\frac{3t^2-6t-8}{(t+1)(t^2-t+2)}\,dt=I.$$

$$\frac{3t^2-6t-8}{(t+1)(t^2-t+2)}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+2},$$

$$3t^2-6t-8=A(t^2-t+2)+(Bt+C)(t+1).$$

$$\left.\begin{array}{ll} t=0: & -8=2A+C, \\ t=1: & -11=2A+2B+2C, \\ t=-1: & 1=4A \end{array}\right\} \Rightarrow\ A=\frac{1}{4},\ B=\frac{11}{4},\ C=-\frac{17}{2}.$$

$$I=2\sqrt{x}-3\sqrt[3]{x}-8\sqrt[4]{x}+6\sqrt[6]{x}+48\sqrt[12]{x}+3\int\frac{dt}{t+1}+3\int\frac{11t-34}{t^2-t+2}\,dt=$$

$$=2\sqrt{x}-3\sqrt[3]{x}-8\sqrt[4]{x}+6\sqrt[6]{x}+48\sqrt[12]{x}+3\ln\left|\sqrt[12]{x}+1\right|+\frac{33}{2}\ln\left(\sqrt[6]{x}-\sqrt[12]{x}+2\right)-$$

$$-\frac{171}{\sqrt{7}}\operatorname{arctg}\frac{2\sqrt[12]{x}-1}{\sqrt{7}}+C.$$

Замечание.

$$\int\frac{11t-34}{t^2-t+2}\,dt=\int\frac{(2t-1)\frac{11}{2}+\left(\frac{11}{2}-34\right)}{t^2-t+2}\,dt=\frac{11}{2}\int\frac{d\,(t^2-t+2)}{t^2-t+2}-$$

$$-\frac{57}{2}\int\frac{d\left(t-\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)^2+\left(\sqrt{\frac{7}{2^2}}\right)^2}=\frac{11}{2}\ln\left|t^2-t+2\right|-\frac{57}{\sqrt{7}}\operatorname{arctg}\frac{t-\frac{1}{2}}{\frac{\sqrt{7}}{2}}+C.$$

395 (2070). $\displaystyle\int\frac{x\,dx}{(x+1)^{1/2}+(x+1)^{1/3}}=\left[\begin{matrix}(x+1)^{1/6}=t,\ x+1=t^6,\\ dx=6t^5dt\end{matrix}\right]=$

$$=6\int\frac{(t^6-1)t^5}{t^3+t^2}\,dt=6\int\frac{(t^3-1)(t^3+1)\,t^5}{t^2(t+1)}\,dt=$$

$$=6\int\frac{(t^3-1)(t+1)(t^2-t+1)\,t^3}{t+1}\,dt=6\int(t^8-t^7+t^6-t^5+t^4-t^3)\,dt=$$

$$=6\left(\frac{t^9}{9}-\frac{t^8}{8}+\frac{t^7}{7}-\frac{t^6}{6}+\frac{t^5}{5}-\frac{t^4}{4}\right)+C=$$

$$=6\left(\frac{1}{9}(x+1)^{3/2}-\frac{1}{8}(x+1)^{4/3}+\frac{1}{7}(x+1)^{7/6}-\frac{1}{6}(x+1)+\frac{1}{5}(x+1)^{5/6}-\right.$$

$$\left.-\frac{1}{4}(x+1)^{2/3}\right)+C.$$

396 (2071). $\displaystyle\int\sqrt{\frac{1-x}{1+x}}\,\frac{dx}{x}=\left[\begin{array}{l}\sqrt{\dfrac{1-x}{1+x}}=t,\ \dfrac{1-x}{1+x}=t^2,\\ x=\dfrac{1-t^2}{1+t^2},\ dx=\dfrac{-4t}{(1+t^2)^2}\,dt\end{array}\right]=$

$$=-4\int t\cdot\frac{t}{(1+t^2)^2}\cdot\frac{1+t^2}{(1-t^2)}\,dt=-4\int\frac{t^2dt}{(1+t^2)(1-t^2)}=I.$$

$$\frac{t^2}{(1+t^2)(1-t^2)}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{Ct+D}{1+t^2}.$$

$$\left.\begin{array}{ll}t=0: & A+B+D=0,\\ t=-1: & 4B=1,\\ t=1: & 4A=1,\\ t=2: & 15A-5B-6C-3D=4\end{array}\right\}\Rightarrow\ \begin{array}{c}A=\dfrac{1}{4},\quad B=\dfrac{1}{4},\quad D=-\dfrac{1}{2}.\\ C=0.\end{array}$$

$$I=-\int\frac{dt}{1-t}-\int\frac{dt}{1+t}+2\int\frac{dt}{1+t^2}=\ln\left|\frac{t-1}{t+1}\right|+2\,\mathrm{arctg}\,t+C=$$

$$=\ln\left|\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right|+2\,\mathrm{arctg}\sqrt{\frac{1-x}{1+x}}+C.$$

397 (2072). $\displaystyle\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx=\left[\begin{array}{l}\sqrt{x}=\cos t,\ x=\cos^2 t,\\ dx=-2\cos t\sin t\,dt,\\ t=\arccos\sqrt{x}=\dfrac{\pi}{2}-\arcsin\sqrt{x}\end{array}\right]=$

$$=-2\int\sqrt{\frac{1-\cos t}{1+\cos t}}\cos t\sin t\ dt=-2\int\sqrt{\frac{\sin^2\frac{t}{2}}{\cos^2\frac{t}{2}}}\,2\cos t\cdot\sin\frac{t}{2}\cdot\cos\frac{t}{2}\,dt=$$

$$=-4\int\sin^2\frac{t}{2}\cos t\ dt=-2\int(1-\cos t)\cos t\ dt=-2\Big(\int\cos t\,dt-$$

$$-\frac{1}{2}\int (1+\cos 2t)\,dt \Bigg) = -2\sin t + t + \frac{1}{2}\sin 2t + C_0 = -2\sqrt{1-\cos^2 t} + t +$$

$$+\cos t\sqrt{1-\cos^2 t} + C_0 = -2\sqrt{1-x} + \frac{\pi}{2} - \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + C_0 =$$

$$= \underline{\underline{\left(\sqrt{x}-2\right)\sqrt{1-x} - \arcsin\sqrt{x} + C}} \qquad \left(C = C_0 + \frac{\pi}{2}\right).$$

398 (2073). $\displaystyle\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}\,dx = \begin{bmatrix} 1+x=t^6,\ \sqrt{1+x}=t^3, \\ \sqrt[3]{1+x}=t^2,\ dx=6t^5dt \end{bmatrix} =$

$$= 6\int \frac{t^{12}-2t^6+1+t^3}{t^2}t^5dt = 6\int (t^{15}-2t^9+t^6+t^3)\,dt =$$

$$= 6\left(\frac{t^{16}}{16} - \frac{t^{10}}{5} + \frac{t^7}{7} + \frac{t^4}{4}\right) + C = 6t^4\left(\frac{t^{12}}{16} - \frac{t^6}{5} + \frac{t^3}{7} + \frac{1}{4}\right) + C =$$

$$= \underline{\underline{6\sqrt[3]{(1+x)^2}\left(\frac{(1+x)^2}{16} - \frac{1+x}{5} + \frac{\sqrt{1+x}}{7} + \frac{1}{4}\right) + C}}.$$

399 (2074). $\displaystyle\int \sqrt[3]{\frac{1-x}{1+x}}\,\frac{dx}{x} = \begin{bmatrix} \dfrac{1-x}{1+x}=t^3,\ x=\dfrac{1-t^3}{1+t^3}, \\ dx=\dfrac{-6t^2}{(1+t^3)^2}dt \end{bmatrix} = -6\int \frac{t^3(1+t^3)\,dt}{(1+t^3)^2(1-t^3)} =$

$$= -3\int\left(\frac{-1}{1+t^3} + \frac{1}{1-t^3}\right)dt = I_1 + I_2.$$

$$I_1 = 3\int \frac{dt}{1+t^3} = 3\int \frac{dt}{(1+t)(1-t+t^2)},$$

$$\frac{1}{(1+t)(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{t^2-t+1},$$

$$1 = A(t^2 - t + 1) + (Bt + C)(1 + t).$$

$$\left.\begin{array}{ll} t = 0: & 1 = A + C, \\ t = 1: & 1 = A + 2B + 2C, \\ t = -1: & 1 = 3A \end{array}\right\} \Rightarrow\ A = \frac{1}{3},\ C = \frac{2}{3},\ B = -\frac{1}{3}.$$

$$I_1 = 3\left(\frac{1}{3}\int \frac{dt}{1+t} - \frac{1}{3}\int \frac{(t-2)\,dt}{t^2 - t + 1}\right) = \ln|1+t| - \frac{1}{2}\int \frac{(2t-1)\,dt}{t^2 - t + 1} +$$

$$+\frac{3}{2}\int \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \ln|1+t| - \frac{1}{2}\ln\left|t^2 - t + 1\right| + \sqrt{3}\,\mathrm{arctg}\,\frac{2t-1}{\sqrt{3}}.$$

$$I_2 = 3\int \frac{dt}{t^3 - 1} = 3\int \frac{dt}{(t-1)(t^2 + t + 1)}.$$

$$\frac{1}{(t-1)(t^2 + t + 1)} = \frac{M}{t-1} + \frac{Nt + L}{t^2 + t + 1},$$

$$1 = M(t^2 + t + 1) + (Nt + L)(t - 1).$$

$$\left.\begin{array}{ll} t = 0: & 1 = M - L, \\ t = 1: & 1 = 3M, \\ t = -1: & 1 = M + 2N - 2L \end{array}\right\} \Rightarrow\ M = \frac{1}{3},\ L = -\frac{2}{3},\ N = -\frac{1}{3}.$$

$$I_2 = \int \frac{dt}{t-1} - \int \frac{t+2}{t^2 + t + 1}\,dt = \ln|t-1| - \frac{1}{2}\int \frac{2t+1}{t^2 + t + 1}\,dt -$$

$$-\frac{3}{2}\int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \ln|t-1| - \frac{1}{2}\ln\left|t^2 + t + 1\right| - \sqrt{3}\,\mathrm{arctg}\,\frac{2t+1}{\sqrt{3}}.$$

$$I = \ln|t-1| + \ln|t+1| - \frac{1}{2}\left(\ln\left|t^2 + t + 1\right| + \ln\left|t^2 - t + 1\right|\right) - \sqrt{3}\left(\mathrm{arctg}\,\frac{2t+1}{\sqrt{3}} -\right.$$

$$-\operatorname{arctg}\frac{2t-1}{\sqrt{3}}\Bigg)+C=\ln\left|t^2-1\right|-\ln\sqrt{t^4+t^2+1}-\sqrt{3}\operatorname{arctg}\frac{\frac{2t+1}{\sqrt{3}}-\frac{2t-1}{\sqrt{3}}}{1+\frac{4t^2-1}{3}}+$$

$$+C=\ln\frac{\left|t^2-1\right|}{\sqrt{t^4+t^2+1}}-\sqrt{3}\operatorname{arctg}\frac{\sqrt{3}}{2t^2+1}+C.$$

400 (2075). $$\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}=\int\frac{\sqrt[4]{x-1}}{\sqrt[4]{(x-1)^4(x+2)^5}}dx=$$

$$=\int\sqrt[4]{\frac{x-1}{x+2}}\cdot\frac{dx}{(x-1)(x+2)}=\left[\begin{array}{c}\frac{x-1}{x+2}=t^4,\ x=\frac{1+2t^4}{1-t^4},\\ dx=\frac{12t^3dt}{(1-t^4)^2}\end{array}\right]=$$

$$=12\int\frac{t^4dt}{(1-t^4)^2\,\frac{3t^4}{1-t^4}\cdot\frac{3}{1-t^4}}=\frac{4}{3}\int dt=\frac{4}{3}t+C=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}+C.$$

2. *Дифференциальные биномы*

Выражение вида $x^m(a+bx^n)^p\,dx$, где m, n, p, a, b – постоянные числа, называется дифференциальным биномом.

Теорема. Интеграл от дифференциального бинома

$$\int x^m(a+bx^n)^p\,dx,$$

если m, n, p – рациональные числа, приводится к интегралу от рациональной функции и, следовательно, выражается через элементарные функции в следующих трех случаях:

1) p есть целое число (положительное, отрицательное или нуль);

2) $\frac{m+1}{n}$ есть целое число (положительное, отрицательное или нуль);

3) $\frac{m+1}{n}+p$ есть целое число (положительное, отрицательное или нуль).

401 (2076). $\int \sqrt{x}\left(1+\sqrt[3]{x}\right)^4 dx = I,$

$$m=\frac{1}{2};\ n=\frac{1}{3};\ p=4.$$

Имеет место первый случай.

Подстановка: $x=t^6,\ dx=6t^5dt,\ t=\sqrt[6]{x}.$

$$I=\int \sqrt{t^6}\left(1+\sqrt[3]{t^6}\right)^4 6t^5dt=6\int t^3(1+t^2)^4t^5dt=$$

$$=6\int t^8(1+4t^2+6t^4+4t^6+t^8)\,dt=6\int (t^8+4t^{10}+6t^{12}+4t^{14}+t^{16})\,dt=$$

$$=6\left(\frac{t^9}{9}+\frac{4}{11}t^{11}+\frac{6}{13}t^{13}+\frac{4}{15}t^{15}+\frac{1}{17}t^{17}\right)+C=$$

$$=\frac{2}{3}x\sqrt{x}+\frac{24}{11}x\sqrt[6]{x^5}+\frac{36}{13}x^2\sqrt[6]{x}+\frac{24}{15}x^2\sqrt{x}+\frac{6}{17}x^2\sqrt[6]{x^5}+C.$$

402 (2077). $\int x^{-1}(1+x^{1/3})^{-3}dx=I.$

$p=-3$ целое, $m=-1$ целое, $n=\dfrac{1}{3}.$

Имеет место первый случай. Подстановка: $x=t^3,\ dx=3t^2dt,\ t=\sqrt[3]{x}.$

$$I=\int \frac{1}{t^3}(1+t)^{-3}\cdot 3t^2dt=3\int \frac{dt}{t(1+t)^3}.$$

$$\frac{1}{t(1+t)^3}=\frac{A}{t}+\frac{B}{(1+t)^3}+\frac{C}{(1+t)^2}+\frac{D}{1+t},$$

$$1=A(1+t)^3+Bt+Ct(1+t)+Dt(1+t)^2.$$

$$\left.\begin{aligned} t=0:&\quad A=1,\\ t=1:&\quad 8A+B+2C+4D=1,\\ t=-1:&\quad -B=1,\\ t=2:&\quad 27A+2B+6C+18D=1\end{aligned}\right\}\Rightarrow \begin{aligned}&A=1,\\ &B=-1,\\ &\begin{cases}C+2D=-3,\\ C+3D=-4\end{cases}\Rightarrow \begin{aligned}D=-1,\\ C=-1.\end{aligned}\end{aligned}$$

$$I=3\left(\int\frac{dt}{t}-\int\frac{dt}{(1+t)^3}-\int\frac{dt}{(1+t)^2}-\int\frac{dt}{1+t}\right)=3\left(\ln\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}+\right.$$

$$\left.+\frac{1}{2\left(1+\sqrt[3]{x}\right)^2}+\frac{1}{1+\sqrt[3]{x}}\right)+C=\underline{\underline{3\left(\ln\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}+\frac{2\sqrt[3]{x}+3}{2\left(1+\sqrt[3]{x}\right)^2}\right)+C.}}$$

403 (2078). $\displaystyle\int\frac{dx}{x\sqrt[3]{x^2+1}}=\int x^{-1}(x^2+1)^{-1/3}\,dx=I.$

$$p=-\frac{1}{3},\ m=-1,\ n=2,\ \frac{m+1}{n}=0\ \text{ – второй случай.}$$

$$x^2+1=t^3,\ 2x\,dx=3t^2dt,\ dx=\frac{3t^2dt}{2x}.$$

$$I=\frac{3}{2}\int\frac{t^2dt}{(t^3-1)t}=\frac{3}{2}\int\frac{t\,dt}{t^3-1}.$$

$$\frac{t}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1},$$

$$t=A(t^2+t+1)+(t-1)(Bt+C).$$

$$\left.\begin{array}{ll} t=0: & A-C=0,\\ t=1: & 3A=1,\\ t=-1: & A+2B-2C=-1\end{array}\right\}\Rightarrow\ A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{1}{3}.$$

$$I=\frac{1}{2}\left(\int\frac{dt}{t-1}-\int\frac{t-1}{t^2+t+1}dt\right)=\frac{1}{2}\left(\int\frac{dt}{t-1}-\int\frac{(2t+1)\frac{1}{2}-\frac{3}{2}}{t^2+t+1}dt\right)=$$

$$=\frac{1}{2}\ln|t-1|-\frac{1}{4}\ln\left|t^2+t+1\right|+\frac{\sqrt{3}}{2}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+C=$$

$$=\frac{1}{2}\ln\left(\sqrt[3]{x^2+1}-1\right)-\frac{1}{4}\ln\left(\sqrt[3]{(x^2+1)^2}+\sqrt[3]{x^2+1}+1\right)+$$

$$+\frac{\sqrt{3}}{2}\operatorname{arctg}\frac{2\sqrt[3]{x^2+1}+1}{\sqrt{3}}+C.$$

404 (2079). $\int x^5\sqrt[3]{(1+x^3)^2}\,dx=\int x^5(1+x^3)^{2/3}\,dx=I.$

$$p=\frac{2}{3},\ m=5,\ n=3,\ \frac{m+1}{n}=\frac{6}{3}=2 \text{ – второй случай.}$$

$$1+x^3=t^3,\ 3x^2\,dx=3t^2dt.$$

$$I=\int x^2\cdot x^3(1+x^2)^{2/3}\,dx=\int (t^3-1)\,t^4dt=\frac{t^8}{8}-\frac{t^5}{5}+C=$$

$$=\frac{1}{8}\sqrt[3]{(1+x^3)^8}-\frac{1}{5}\sqrt{(1+x^3)^5}+C.$$

405 (2080). $\int\frac{dx}{\sqrt[3]{1+x^3}}=\left[\begin{array}{l}\dfrac{\sqrt[3]{1+x^3}}{x}=t,\ \dfrac{1+x^3}{x^3}=t^3,\\ x^3=\dfrac{1}{t^3-1},\ 3x^2dx=-\dfrac{3t^2dt}{(t^3-1)^2}\end{array}\right]=\int\frac{x^2dx}{x^3\,\dfrac{\sqrt[3]{1+x^3}}{x}}=$

$$=-\int\frac{t^2dt}{\dfrac{1}{t^3-1}(t^3-1)^2t}=-\int\frac{t\,dt}{t^3-1}=I.$$

$$\frac{t}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1},$$

$$t=A(t^2+t+1)+(t-1)(Bt+C).$$

$$\left.\begin{array}{ll} t=0: & A-C=0,\\ t=1: & 3A=1,\\ t=-1: & A+2B-2C=-1\end{array}\right\}\Rightarrow A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{1}{3}.$$

$$I=-\left(\frac{1}{3}\int\frac{dt}{t-1}-\frac{1}{3}\int\frac{t-1}{t^2+t+1}\,dt\right)=-\left(\frac{1}{3}\ln|t-1|-\frac{1}{3}\int\frac{(2t+1)\frac{1}{2}-\frac{3}{2}}{t^2+t+1}\,dt\right)=$$

$$=-\frac{1}{3}\ln|t-1|+\frac{1}{6}\ln\left|t^2+t+1\right|-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+C=\frac{1}{6}\ln\frac{t^2+t+1}{(t-1)^2}-$$

$$-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+C=$$

$$=\frac{1}{6}\ln\frac{\sqrt[3]{(1+x^3)^2}+x\sqrt[3]{1+x^3}+x^2}{\left(\sqrt[3]{1+x^3}-x\right)^2}-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{1+x^3}+x}{\sqrt{3x}}+C.$$

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406 (2081). $\displaystyle\int\frac{dx}{\sqrt[4]{1+x^4}}=\int\frac{x^3dx}{x^4\,\dfrac{\sqrt[4]{1+x^4}}{x}}=\left[\begin{array}{c}\dfrac{\sqrt[4]{1+x^4}}{x}=t,\ x^4=\dfrac{1}{t^4-1},\\ x^3dx=-\dfrac{t^3dt}{(t^4-1)^2}\end{array}\right]=$

$$=-\int\frac{t^3dt}{(t^4-1)^2\,\dfrac{t}{t^4-1}}=-\int\frac{t^2dt}{t^4-1}=I.$$

$$\frac{t^2}{(t-1)(t+1)(t^2+1)}=\frac{A}{t-1}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1},$$

$$t^2=A(t+1)(t^2+1)+B(t-1)(t^2+1)+(t^2-1)(Ct+D).$$

$$\left.\begin{array}{ll}t=0: & A-B-D=0,\\ t=1: & 4A=1,\\ t=-1: & -4B=1,\\ t=2: & 15A+5B+6C+3D=4\end{array}\right\}\Rightarrow\ A=\frac{1}{4},\ B=-\frac{1}{4},\ C=0,\ D=\frac{1}{2}.$$

$$I=-\frac{1}{4}\int\frac{dt}{t-1}+\frac{1}{4}\int\frac{dt}{t+1}-\frac{1}{2}\int\frac{dt}{t^2+1}=\frac{1}{4}\ln\left|\frac{t+1}{t-1}\right|+\frac{1}{2}-\frac{1}{2}\operatorname{arctg}t+C=$$

$$=\frac{1}{4}\ln\left|\frac{\sqrt[4]{1+x^4}+1}{\sqrt[4]{1+x^4}-1}\right|-\frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{1+x^4}}{x}+C.$$

407 (2082). $\int\frac{\sqrt{1-x^4}}{x^5}\,dx=I,$

$p=\frac{1}{2},\ m=-5,\ n=4,\ \frac{m+1}{n}=\frac{-5+1}{4}=-1$ – целое, имеет место второй случай.

$$\frac{\sqrt{1-x^4}}{x^2}=t,\ x^4=\frac{1}{t^2+1},\ x^3\,dx=-\frac{t\,dt}{2(t^2+1)^2}.$$

$$I=\int\frac{x^3\sqrt{1-x^4}dx}{x^6\cdot x^2}=-\frac{1}{2}\int\frac{t^2dt}{(t^2+1)^2\frac{1}{(t^2+1)^{3/2}}}=-\frac{1}{2}\int\frac{t^2dt}{\sqrt{t^2+1}}=$$

$$=\begin{bmatrix}u=t, & du=dt,\\ dv=\frac{t\,dt}{\sqrt{t^2+1}}, & v=\frac{1}{2}\int\frac{d(t^2+1)}{\sqrt{t^2+1}}=\sqrt{t^2+1}\end{bmatrix}=$$

$$=-\frac{1}{2}\left(t\sqrt{t^2+1}-\int\sqrt{t^2+1}dt\right)=-\frac{1}{2}\left(t\sqrt{t^2+1}-\int\frac{t^2+1}{\sqrt{t^2+1}}\,dt\right)=$$

$$=-\frac{1}{2}\left(t\sqrt{t^2+1}-\int\frac{t^2dt}{\sqrt{t^2+1}}-\int\frac{dt}{\sqrt{t^2+1}}\right),$$

$$-\frac{1}{2}\int\frac{t^2dt}{\sqrt{t^2+1}}=-\frac{1}{2}t\sqrt{t^2+1}+\frac{1}{2}\int\frac{t^2dt}{\sqrt{1+t^2}}+\frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right|+C,$$

$$I=-\frac{1}{4}t\sqrt{t^2+1}+\frac{1}{4}\ln\left|t+\sqrt{t^2+1}\right|+C=\frac{1}{4}\ln\left|\frac{\sqrt{1-x^4}}{x^2}+\sqrt{\frac{1-x^4}{x^4}+1}\right|-$$

$$-\frac{\sqrt{1-x^4}}{4x^2}\sqrt{\frac{1-x^4}{x^4}+1}+C=\underline{\underline{\frac{1}{4}\ln\frac{\sqrt{1-x^4}+1}{x^2}-\frac{1}{4}\frac{\sqrt{1-x^4}}{x^4}+C}}.$$

408 (2083). $\displaystyle\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}=\left[\begin{matrix}\sqrt[3]{1+\sqrt[4]{x}}=t,\ x=(t^3-1)^4,\\ dx=12(t^3-1)^3t^2dt\end{matrix}\right]=\int\frac{12(t^3-1)^3t^3dt}{(t^3-1)^2}=$

$$=12\int(t^6-t^3)\,dt=12\left[\frac{t^7}{7}-\frac{t^4}{4}\right]+C=12\left[\frac{\sqrt[3]{\left(1+\sqrt[4]{x}\right)^7}}{7}-\frac{\sqrt[3]{\left(1+\sqrt[4]{x}\right)^4}}{4}\right]+$$

$$+C=\frac{12}{28}\sqrt[3]{1+\sqrt[4]{x}}\left(4\left(1+\sqrt[4]{x}\right)^2-7\left(1+\sqrt[4]{x}\right)\right)+C=$$

$$=\underline{\underline{\frac{3}{7}\sqrt[3]{1+\sqrt[4]{x}}\left(4\sqrt{x}+\sqrt[4]{x}-3\right)+C}}.$$

409 (2084). $\displaystyle\int\frac{\sqrt[3]{1+\sqrt{x}}}{x}dx=\left[\begin{matrix}\sqrt[3]{1+\sqrt{x}}=t,\ x=(t^3-1)^2\\ dx=6(t^3-1)t^2dt\end{matrix}\right]=6\int\frac{t^3dt}{t^3-1}=$

$$=6\left[\int dt+\int\frac{dt}{t^3-1}\right]=I.$$

$$\frac{1}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1},$$

$$1=A(t^2+t+1)+(t-1)(Bt+C).$$

$$\left.\begin{matrix}t=0: & A-C=1,\\ t=1: & 3A=1,\\ t=-1: & A+2B-2C=1\end{matrix}\right\}\Rightarrow A=\frac{1}{3},\ B=-\frac{1}{3},\ C=-\frac{2}{3}.$$

$$I=6t+2\int\frac{dt}{t-1}-2\int\frac{t+2}{t^2+t+1}dt=6t+2\ln|t-1|-2\int\frac{(2t+1)\frac{1}{2}+\frac{3}{2}}{t^2+t+1}dt=$$

$$= 6t + 2\ln|t-1| - \ln|t^2+t+1| - \frac{3\cdot 2}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + C =$$

$$= 6t + 2\ln\frac{|t-1|}{\sqrt{t^2+t+1}} - 2\sqrt{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + C.$$

410 (2085). $\displaystyle\int \frac{dx}{x\sqrt[3]{1+x^5}} = \left[\begin{array}{c} t = \sqrt[3]{1+x^5},\ t^3 = 1+x^5, \\ 3t^2dt = 5x^4dx \end{array}\right] = \frac{1}{5}\int \frac{5x^4dx}{x^5\sqrt[3]{1+x^5}} =$

$$= \frac{1}{5}\int \frac{3t^2dt}{(t^3-1)t} = \frac{3}{5}\int \frac{t\ dt}{t^3-1} = I.$$

$$\frac{1}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1},$$

$$1 = A(t^2+t+1) + (t-1)(Bt+C).$$

$$\left.\begin{array}{ll} t=0: & A-C=0, \\ t=1: & 3A=1, \\ t=-1: & A+2B-2C=-1 \end{array}\right\} \Rightarrow A = C = \frac{1}{3},\ B = -\frac{1}{3}.$$

$$I = \frac{3}{5}\left(\frac{1}{3}\int\frac{dt}{t-1} - \frac{1}{3}\int\frac{t-1}{t^2+t+1}\,dt\right) = \frac{1}{5}\left(\ln|t-1| - \frac{1}{2}\ln|t^2+t+1| +\right.$$

$$\left.+\sqrt{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}\right) + C = \frac{1}{5}\ln\frac{|t-1|}{\sqrt{t^2+t+1}} + \frac{\sqrt{3}}{5}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + C.$$

411 (2086). $\displaystyle\int \frac{\sqrt[3]{1+x^3}}{x^2}\,dx = \left[\begin{array}{c} \frac{\sqrt[3]{1+x^3}}{x} = t,\ \frac{1+x^3}{x^3} = t^3, \\ dx = x^4\cdot t^2dt \end{array}\right] = -\int \frac{t\cdot x^4t^2dt}{x} =$

$$= -\int t^3\cdot x^3dt = -\int\frac{t^3dt}{t^3-1} = -\int\frac{(t^3-1)+1}{t^3-1}\,dt = -\left(\int dt + \int\frac{dt}{t^3-1}\right) = I.$$

$$\frac{1}{t^3-1}=\frac{1}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1},$$

$$1=A(t^2+t+1)+(t-1)(Bt+C).$$

$$\left.\begin{array}{ll} t=0: & A-C=1,\\ t=1: & 3A=1,\\ t=-1: & A+2B-2C=1 \end{array}\right\}\Rightarrow A=\frac{1}{3},\ B=-\frac{1}{3},\ C=-\frac{2}{3}.$$

$$I=-t-\frac{1}{3}\ln|t-1|+\int\frac{t+2}{t^2+t+1}\,dt=-t-\frac{1}{3}\ln|t-1|+\frac{1}{3}\int\frac{(2t+1)\frac{1}{2}+\frac{3}{2}}{t^2+t+1}\,dt=$$

$$=-t-\frac{1}{3}\ln|t-1|+\frac{1}{2\cdot 3}\ln\left|t^2+t+1\right|+\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+C=\underline{\underline{C-\frac{\sqrt[3]{1+x^3}}{x}+}}$$

$$\underline{\underline{+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{1+x^3}+x}{x\sqrt{3}}-\frac{1}{3}\ln\left|\frac{\sqrt[3]{1+x^3}-x}{\sqrt{\sqrt[3]{(1+x^3)^2}+x\sqrt[3]{1+x^3}+x^2}}\right|}}.$$

412 (2087). $$\int\frac{dx}{x^{11}\sqrt{1+x^4}}=\left[\begin{array}{l}\sqrt{\dfrac{1+x^4}{x^4}}=t,\ \dfrac{1+x^4}{x^4}=t^4,\\ x^4=\dfrac{1}{t^2-1},\ dx=-\dfrac{1}{2}x^5t\,dt\end{array}\right]=$$

$$=-\frac{1}{2}\int\frac{x^5t\,dt}{x^{11}\sqrt{1+x^4}}=-\frac{1}{2}\int\frac{t\,dt}{x^8\sqrt{\dfrac{1+x^4}{x^4}}}=-\frac{1}{2}\int\frac{t\,dt}{\dfrac{1}{(t^2-1)^2}\cdot t}=$$

$$=-\frac{1}{2}\int(t^2-1)^2\,dt=-\frac{1}{2}\int(t^4-2t^2+1)\,dt=-\frac{1}{2}\left(\frac{t^5}{5}-\frac{2t^3}{3}+t\right)+C=$$

$$\underline{\underline{=-\frac{1}{10}\sqrt{\left(\frac{1+x^4}{x^4}\right)^5}+\frac{1}{3}\sqrt{\left(\frac{1+x^4}{x^4}\right)^3}-\frac{1}{2}\sqrt{\frac{1+x^4}{x^4}}+C.}}$$

413 (2088). $\int \sqrt[3]{x(1-x^2)}\,dx = \left[\begin{array}{c} \sqrt[3]{\dfrac{1-x^2}{x^2}} = t,\ x^2 = \dfrac{1}{t^3+1}, \\ dx = -\dfrac{3}{2}x^3t^2dt \end{array}\right] =$

$$= -\frac{3}{2}\int \sqrt[3]{x^3\,\frac{1-x^2}{x^2}}\;x^3t^2dt = -\frac{3}{2}\int x^4t^3dt = -\frac{3}{2}\int \frac{t^3dt}{(t^3+1)^2} =$$

$$= \left[\begin{array}{ll} u = t, & du = dt, \\ dv = \dfrac{t^2dt}{(t^2+1)^2}, & v = -\dfrac{1}{3(t^3+1)} \end{array}\right] = -\frac{3}{2}\left(-\frac{t}{3(t^3+1)} + \frac{1}{3}\int \frac{dt}{t^3+1}\right) = I.$$

$$\frac{1}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1},$$

$$1 = A(t^2+t+1) + (t-1)(Bt+C).$$

$$\left.\begin{array}{ll} t=0: & A+C=1, \\ t=-1: & 3A=1, \\ t=1: & A+2B+2C=1 \end{array}\right\} \Rightarrow\ A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}.$$

$$I = \frac{t}{2(t^3+1)} - \frac{1}{2}\left(\int \frac{dt}{t+1} - \frac{1}{3}\int \frac{t-2}{t^2-t+1}\,dt\right) = \frac{t}{2(t^3+1)} - \frac{1}{2}\ln|t+1| +$$

$$+\frac{1}{6}\int \frac{(2t-1)\frac{1}{2}-\frac{3}{2}}{t^2-t+1}\,dt = \frac{t}{2(t^3+1)} - \frac{1}{2}\ln|t+1| + \frac{1}{12}\ln\left|t^2-t+1\right| -$$

$$-\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + C =$$

$$= \frac{t}{2(t^3+1)} - \frac{1}{6}\ln\frac{|t+1|}{\sqrt{t^2-t+1}} - \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + C.$$

414 (2089). $\int \sqrt[3]{1+\sqrt[4]{x}}\,dx = \begin{bmatrix} 1+\sqrt[4]{x}=t,\ x=(t-1)^4, \\ dx = 4(t-1)^3\,dt \end{bmatrix} = 4\int \sqrt[3]{t}\,(t-1)^3\,dt =$

$$= 4\int (t^{10/3} - 3t^{7/3} + 3t^{4/3} - t^{1/3})\,dt = 4\left(\frac{3}{13}t^{13/3} - \frac{9}{10}t^{10/3} + \frac{9}{7}t^{7/3} - \right.$$

$$\left. - \frac{3}{4}t^{4/3}\right) + C = \underline{\underline{12\left(\frac{\sqrt[3]{t^{13}}}{13} - \frac{3\sqrt[3]{t^{10}}}{10} + \frac{3}{7}\sqrt[3]{t^7} - \frac{1}{4}\sqrt[3]{t^4}\right) + C}}.$$

3.3. Тригонометрические функции

При решении примеров этого раздела используем тригонометрические подстановки видов:

$$1)\ \operatorname{tg}\frac{x}{2} = t \ \Rightarrow\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2};$$

$$2)\ \operatorname{tg} x = t \ \Rightarrow\ \sin x = \frac{t^2}{1+t^2},\ \cos x = \frac{1}{1+t^2},\ dx = \frac{dt}{1+t^2}.$$

415 (2090). $\int \sin^3 x \cdot \cos^2 x\,dx = \int \cos^2 x(1-\cos^2 x)\sin x\,dx =$

$$= -\int (\cos^2 x - \cos^4 x)\,d(\cos x) = \underline{\underline{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}}.$$

416 (2091). $\int \frac{\sin^3 x}{\cos^4 x}\,dx = -\int \frac{(1-\cos^2 x)\,d(\cos x)}{\cos^4 x} =$

$$= -\int (\cos^{-4} x - \cos^{-2} x)\,d\cos x = \underline{\underline{\frac{1}{\cos^3 x} - \frac{1}{\cos x} + C}}.$$

417 (2092). $\int \frac{dx}{\cos x \cdot \sin^3 x} = \int \frac{\cos^2 x + \sin^2 x}{\cos x \cdot \sin^3 x}\,dx = \int \frac{dx}{\sin x \cdot \cos x} +$

$$+ \int \frac{d(\sin x)}{\sin^3 x} = \int \frac{d(2x)}{\sin 2x} + \int \sin^{-3} x\,d(\sin x) = \underline{\underline{\ln|\operatorname{tg} x| - \frac{1}{2\sin^2 x} + C}}.$$

418 (2093). $$\int \frac{\sin^4 x}{\cos^2 x}\,dx = \int \frac{(1-\cos^2 x)^2\,dx}{\cos^2 x} = \int \frac{dx}{\cos^2 x} - 2\int dx +$$

$$+\int \cos^2 x\ dx = \operatorname{tg} x - 2x + \frac{1}{2}\int (1+\cos 2x)\,dx = \operatorname{tg} x - 2x + \frac{1}{2}x +$$

$$+\frac{1}{4}\sin 2x + C = \underline{\underline{\operatorname{tg} x - \frac{3}{2}x + \frac{1}{4}\sin 2x + C}}.$$

419 (2094). $$\int \frac{dx}{\sin^3 x\cdot\cos^3 x} = \int \frac{(\sin^2 x+\cos^2 x)^2}{\sin^3 x\cdot\cos^3 x}\,dx =$$

$$= \int \frac{\sin^4 x + 2\cos^2 x\cdot\sin^2 x + \cos^4 x}{\sin^3 x\cos^3 x}\,dx = -\int \frac{d(\cos x)}{\cos^3 x} + 2\int \frac{d(2x)}{\sin 2x} +$$

$$+\int \frac{d(\sin x)}{\sin^3 x} = \underline{\underline{\frac{1}{2\cos^2 x} + 2\ln\left|\operatorname{tg} x\right| - \frac{1}{2\sin^2 x} + C}}.$$

420 (2095). $$\int \frac{dx}{\sin^4 x\cdot\cos^4 x} = \int \frac{(\sin^2 x+\cos^2 x)^2}{\sin^4 x\cdot\cos^4 x}\,dx = \int \frac{dx}{\sin^4 x} +$$

$$+2\int \frac{dx}{\sin^2 x\cdot\cos^2 x} + \int \frac{dx}{\cos^4 x} = I,$$

$$\int \frac{dx}{\sin^4 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin^4 x}\,dx = \int \frac{dx}{\sin^2 x} - \int \operatorname{ctg}^2 x\,d\,(\operatorname{ctg} x) =$$

$$= -\operatorname{ctg} x - \frac{\operatorname{ctg}^3 x}{3},$$

$$\int \frac{dx}{\cos^4 x} = \int \frac{\sin^2 x+\cos^2 x}{\cos^4 x}\,dx = \int \operatorname{tg}^2 x\,d\,(\operatorname{tg} x) + \int \frac{dx}{\cos^2 x} =$$

$$= \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x,$$

$$2\int \frac{dx}{\cos^2 x\cdot\sin^2 x} = 2\int \frac{\sin^2 x+\cos^2 x}{\cos^2 x\cdot\sin^2 x}\,dx = 2\operatorname{tg} x - 2\operatorname{ctg} x,$$

$$\underline{\underline{I = 3\operatorname{tg} x - 3\operatorname{ctg} x + \frac{1}{3}\operatorname{tg}^3 x - \frac{1}{3}\operatorname{ctg}^3 x + C}}.$$

Можно использовать подстановку $\operatorname{tg} x = t$: $\sin x = \dfrac{t^2}{1+t^2}$, $\cos x = \dfrac{1}{1+t^2}$, $dx = \dfrac{dt}{1+t^2}$, также получим,

$$I = 3\operatorname{tg} x - 3\operatorname{ctg} x + \frac{1}{3}\operatorname{tg}^3 x - \frac{1}{3}\operatorname{ctg}^3 x + C.$$

421 (2096). $\displaystyle\int \frac{\sin x\,dx}{(1-\cos x)^2} = \int \frac{d(1-\cos x)}{(1-\cos x)^2} = \frac{1}{\cos x - 1} + C.$

422 (2097). $\displaystyle\int \frac{\cos x\,dx}{(1-\cos x)^2} = -\int \frac{(1-\cos x)-1}{(1-\cos x)^2}dx = -\int \frac{dx}{1-\cos x} +$

$$+\int \frac{dx}{(1-\cos x)^2} = -\frac{1}{2}\int \frac{dx}{\sin^2 \frac{x}{2}} + \frac{1}{4}\int \frac{dx}{\sin^4 \frac{x}{2}} = -\int \frac{d\left(\frac{x}{2}\right)}{\sin^2 \frac{x}{2}} +$$

$$+\frac{1}{4}\int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{\sin^4 \frac{x}{2}}dx = -\int \frac{d\left(\frac{x}{2}\right)}{\sin^2 \frac{x}{2}} + \frac{1}{2}\int \frac{d\left(\frac{x}{2}\right)}{\sin^2 \frac{x}{2}} +$$

$$+\frac{1}{2}\int \frac{\operatorname{ctg}^2 \frac{x}{2}}{\sin^2 \frac{x}{2}} d\left(\frac{x}{2}\right) = -\frac{1}{2}\int \frac{d\left(\frac{x}{2}\right)}{\sin^2 \frac{x}{2}} + \frac{1}{2}\int \operatorname{ctg}^2 \frac{x}{2} d\left(\operatorname{ctg}\frac{x}{2}\right) =$$

$$= \frac{1}{2}\operatorname{ctg}\frac{x}{2} - \frac{1}{6}\operatorname{ctg}^3 \frac{x}{2} + C.$$

423 (2098). $\displaystyle\int \cos^6 x\,dx = \int (\cos^2 x)^3 dx = \frac{1}{8}\int (1+\cos 2x)^3 dx =$

$$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\,dx = \frac{1}{8}\int \Bigg(1 + 3\cos 2x +$$

$$+\frac{3}{2}(1+\cos 4x)+(1-\sin^2 2x)\cos 2x\Bigg)dx=\frac{1}{8}\Bigg(\int\Bigg(\frac{5}{2}+3\cos 2x+$$

$$+\frac{3}{2}\cos 4x\Bigg)dx+\frac{1}{2}\int(1-\sin^2 2x)\,d(\sin 2x)\Bigg)=\frac{5}{16}x+\frac{3}{16}\sin 2x+$$

$$+\frac{3}{64}\sin 4x+\frac{1}{16}\sin 2x-\frac{1}{48}\sin^3 2x+C=\frac{5}{16}x+\frac{3}{16}\sin 2x+$$

$$+\frac{3}{32}\sin 2x\cdot\cos 2x+\frac{1}{16}\sin 2x-\frac{1}{48}\sin 2x\cdot 4\sin^2 x\cdot\cos^2 x=$$

$$=\underline{\underline{\frac{5}{16}x+\frac{1}{12}\sin 2x\left(\frac{15}{8}+\frac{5}{4}\cos^2 x+\cos^4 x\right)+C.}}$$

424 (2099). $\displaystyle\int \operatorname{ctg}^4 x\,dx=\int \operatorname{ctg}^2 x\frac{1-\sin^2 x}{\sin^2 x}\,dx=-\int \operatorname{ctg}^2 x\,d(\operatorname{ctg} x)-$

$$-\int\frac{1-\sin^2 x}{\sin^2 x}\,dx=\underline{\underline{-\frac{\operatorname{ctg}^3 x}{3}+\operatorname{ctg} x+x+C.}}$$

425 (2100). $\displaystyle\int \operatorname{tg}^5 x\,dx=\left[\operatorname{tg} x=t,\ dx=\frac{t\,dt}{1+t^2}\right]=\int\frac{t^5dt}{1+t^2}=\int(t^3-t)\,dt+$

$$+\int\frac{t\,dt}{1+t^2}=\frac{t^4}{4}-\frac{t^2}{2}+\frac{1}{2}\ln(1+t^2)+C=$$

$$=\underline{\underline{\frac{\operatorname{tg}^4 x}{4}-\frac{\operatorname{tg}^2 x}{2}+\frac{1}{2}\ln(1+\operatorname{tg}^2 x)+C.}}$$

426 (2101). $\displaystyle\int\frac{dx}{\operatorname{tg}^8 x}=\left[\operatorname{tg} x=t,\ dx=\frac{dt}{1+t^2}\right]=\int\frac{dt}{t^8(1+t^2)}=I.$

$$\frac{1}{t^8(1+t^2)}=\frac{A}{t^8}+\frac{B}{t^7}+\frac{C}{t^6}+\frac{D}{t^5}+\frac{K}{t^4}+\frac{N}{t^3}+\frac{P}{t^2}+\frac{Q}{t}+\frac{Mt+L}{1+t^2},$$

$$1 = A + At^2 + Bt + Bt^3 + Ct^2 + Ct^4 + Dt^3 + Dt^5 + Kt^4 + Kt^6 +$$
$$+ Nt^5 + Nt^3 + Pt^6 + Pt^8 + Qt^7 + Qt^9 + Mt^9 + Lt^8.$$

Приравнивая коэффициенты при одинаковых степенях t, получим $A = L = K = 1$, $P = C = -1$, $D = B = Q = N = M = 0$.

$$I = \int \frac{dt}{t^8} - \int \frac{dt}{t^6} + \int \frac{dt}{t^4} - \int \frac{dt}{t^2} + \int \frac{dt}{1+t^2} =$$

$$= -\frac{1}{7\operatorname{tg}^7 x} + \frac{1}{5\operatorname{tg}^5 x} - \frac{1}{3\operatorname{tg}^3 x} + \frac{1}{\operatorname{tg} x} + x + C$$

$$\left(\int \frac{dt}{1+t^2} = \operatorname{arctg} t = \operatorname{arctg}(\operatorname{tg} x) = x \right).$$

427 (2102). $$\int \frac{dx}{\sin^3 x} = \left[\operatorname{tg}\frac{x}{2} = t,\ \sin x = \frac{2t}{1+t^2},\ dx = \frac{2dt}{1+t^2} \right] =$$

$$= \frac{1}{4}\int \frac{(1+t^2)^2}{t^3}\,dx = \frac{1}{4}\int \frac{1+2t^2+t^4}{t^3}\,dt = \frac{1}{4}\int \left(t^{-3} + \frac{2}{t} + t \right) dt =$$

$$= \frac{1}{4}\left[-\frac{1}{2t^2} + 2\ln|t| + \frac{t^2}{2} \right] + C = -\frac{1}{8t^2} + \frac{1}{2}\ln|t| + \frac{t^2}{8} + C = \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right| -$$

$$-\frac{1}{8}\left(\frac{\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}} - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \right) = \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right| -$$

$$-\frac{1}{8}\left(\frac{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}{\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}} \right) = \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right| - \frac{\cos x}{2\sin^2 x} + C.$$

428 (2103). $$\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,dx = \int \frac{dx}{\cos 2x} - \frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx =$$

$$= \int \frac{dx}{\cos 2x} - \frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx = \frac{1}{2}\int \frac{dx}{\cos 2x} - \frac{1}{2}\int \cos 2x\,dx =$$

$$= \frac{1}{4}\int \frac{d(2x)}{\cos 2x} - \frac{1}{4}\int \cos 2x\,d(2x) = \frac{1}{4}\int \frac{d\left(\frac{\pi}{2}+2x\right)}{\sin\left(\frac{\pi}{2}+2x\right)} - \frac{1}{4}\sin 2x + C =$$

$$= \frac{1}{4}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+x\right)\right| - \frac{1}{2}\cos x\cdot\sin x + C = \frac{1}{4}\ln\frac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}x}{1-\operatorname{tg}\frac{\pi}{4}\cdot\operatorname{tg}x} -$$

$$- \frac{1}{2}\cos x\cdot\sin x + C = \underline{\underline{\frac{1}{4}\ln\left|\frac{1+\operatorname{tg}x}{1-\operatorname{tg}x}\right| - \frac{1}{2}\cos x\cdot\sin x + C}}.$$

429 (2104). $\displaystyle\int \frac{dx}{(\sin x+\cos x)^2} = \int \frac{dx}{1+\sin 2x} =$

$$= \left[\operatorname{tg}x = t,\ dx = \frac{dt}{1+t^2},\ \sin 2x = \frac{2\sin x\cdot\cos x}{\sin^2 x+\cos^2 x} = \frac{2\operatorname{tg}x}{1+\operatorname{tg}^2 x} = \frac{2t}{1+t^2}\right] =$$

$$= \int \frac{(1+t^2)\,dt}{(1+t^2)(1+t)^2} = \int \frac{d(1+t)}{(1+t)^2} = -\frac{1}{1+t} + C = \underline{\underline{-\frac{1}{1+\operatorname{tg}x} + C}}.$$

430 (2105). $\displaystyle\int \frac{dx}{\sin x+\cos x} = \int \frac{dx}{\sin x+\operatorname{tg}\frac{\pi}{4}\cdot\cos x} =$

$$= \frac{\sqrt{2}}{2}\int \frac{dx}{\sin x\cdot\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cdot\cos x} = \frac{\sqrt{2}}{2}\int \frac{dx}{\sin\left(x+\frac{\pi}{4}\right)} =$$

$$= \underline{\underline{\frac{\sqrt{2}}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{8}+\frac{x}{2}\right)\right| + C}}.$$

431 (2106). $$\int \frac{dx}{a\cos x + b\sin x} = \frac{1}{b}\int \frac{dx}{\frac{a}{b}\cos x + \sin x} = \left[\frac{a}{b} = \operatorname{tg}\varphi\right] =$$

$$= \frac{1}{b}\int \frac{\cos\varphi\, dx}{\sin\varphi\cdot\cos x + \sin x\cdot\cos\varphi} = \frac{\cos\varphi}{b}\int \frac{dx}{\sin(x+\varphi)} =$$

$$= \frac{\cos\varphi}{b}\ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\varphi}{2}\right)\right| + C = \left[\varphi = \operatorname{arctg}\frac{a}{b};\ \cos\varphi = \frac{1}{\sqrt{1+\operatorname{tg}^2\varphi}} =\right.$$

$$\left. = \frac{b}{\sqrt{a^2+b^2}}\right] = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\frac{x + \operatorname{arctg}\frac{a}{b}}{2}\right| + C.$$

432 (2107). $$\int \frac{dx}{\operatorname{tg}x\cdot\cos 2x} = \int \frac{dx}{\operatorname{tg}(\cos^2 x - \sin^2 x)} =$$

$$= \begin{bmatrix} \operatorname{tg}x = t,\ dt = \dfrac{dt}{1+t^2}, \\ \cos^2 x = \dfrac{1}{1+t^2},\ \sin^2 x = \dfrac{t^2}{1+t^2} \end{bmatrix} = \int \frac{dt}{(1+t^2)t\left(\dfrac{1}{1+t^2} - \dfrac{t^2}{1+t^2}\right)} =$$

$$= \int \frac{dt}{t(1-t)(1+t)} = I.$$

$$\frac{1}{t(1-t)(1+t)} = \frac{A}{t} + \frac{B}{1-t} + \frac{C}{1+t},$$

$$1 = A(1-t^2) + Bt(1+t) + Ct(1-t).$$

$$\left.\begin{array}{ll} t=0: & A=1, \\ t=1: & 2B=1, \\ t=-1: & -2C=1 \end{array}\right\} \Rightarrow A=1,\ B=\frac{1}{2},\ C=-\frac{1}{2}.$$

$$I=\int\frac{dt}{t}-\frac{1}{2}\int\frac{dt}{t-1}-\frac{1}{2}\int\frac{dt}{t+1}=\ln\frac{|C\,t|}{\sqrt{t^2-1}}=\ln\frac{|\mathrm{ctg}\,x\cdot\cos x|}{\sqrt{\cos 2x}}=$$

$$=\underline{\underline{\ln\frac{|C\sin x|}{\sqrt{\cos 2x}}}}.$$

433 (2108). $\displaystyle\int\frac{\cos^2 x\,dx}{\sin x\cdot\cos 3x}=\left[\cos 3x=\cos^3 x-3\sin^2 x\cdot\cos x\right]=$

$$=\int\frac{\cos^2 x\,dx}{\cos^3 x\cdot\sin x-3\sin^3 x\cdot\cos x}=\int\frac{\cos x dx}{\sin x(\cos^2 x-3\sin^2 x)}=$$

$$=\left[\begin{array}{c}\mathrm{tg}\,x=t,\ dx=\dfrac{dt}{1+t^2},\\ \sin x=\dfrac{t}{\sqrt{1+t^2}},\ \cos x=\dfrac{1}{\sqrt{1+t^2}}\end{array}\right]=\int\frac{dt}{t(1-3t^2)}=I.$$

$$\frac{1}{t\left(1-\sqrt{3}t\right)\left(1+\sqrt{3}t\right)}=\frac{A}{t}+\frac{B}{1-\sqrt{3}t}+\frac{C}{1+\sqrt{3}t},$$

$$1=A(1-3t^2)+Bt\left(1+\sqrt{3}t\right)+Ct\left(1-\sqrt{3}t\right).$$

$$\left.\begin{array}{ll}t=0: & A=1,\\ t=\dfrac{1}{\sqrt{3}}: & 1=\dfrac{2}{\sqrt{3}}B,\\ t=-\dfrac{1}{\sqrt{3}}: & 1=-\dfrac{2}{\sqrt{3}}C\end{array}\right\}\Rightarrow A=1,\ B=\frac{\sqrt{3}}{2},\ C=-\frac{\sqrt{3}}{2}.$$

$$I=\int\frac{dt}{t}+\frac{\sqrt{3}}{2}\int\frac{dt}{1-\sqrt{3}t}-\frac{\sqrt{3}}{2}\int\frac{dt}{1+\sqrt{3}t}=\ln|t|-\ln\sqrt{1-3t^2}+\ln C=$$

$$=\ln\frac{|C\,t|}{\sqrt{1-3t^2}}=\ln\frac{|C\,\mathrm{tg}\,x|}{\sqrt{1-3\,\mathrm{tg}^2 x}}=\underline{\underline{\ln\frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}}}.$$

434 (2109). $\displaystyle\int \frac{dt}{1+\operatorname{tg} x} = \left[\operatorname{tg} x = t,\ dt = \frac{dt}{1+t^2}\right] = \int \frac{dt}{(1+t^2)(1+t)} = I.$

$$\frac{1}{(1+t^2)(1+t)} = \frac{At+B}{1+t^2} + \frac{C}{1+t},$$

$$1 = (At+B)(1+t) + C(1+t^2).$$

$$\left.\begin{array}{ll} t=0: & 1 = B+C, \\ t=1: & 1 = 2A+2B+2C, \\ t=-1: & 1 = 2C \end{array}\right\} \Rightarrow A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2}.$$

$$I = -\frac{1}{2}\int \frac{t-1}{1+t^2}\,dt + \frac{1}{2}\int \frac{dt}{1+t} = -\frac{1}{4}\int \frac{d(1+t^2)}{1+t^2} + \frac{1}{2}\int \frac{dt}{1+t^2} +$$

$$+\frac{1}{2}\int \frac{dt}{1+t} = -\frac{1}{4}\ln(1+t^2) + \frac{1}{2}\operatorname{arctg} t + \frac{1}{2}\ln(1+t) + C =$$

$$= \frac{1}{2}\left[x + \ln\frac{|1+t|}{\sqrt{1+t^2}}\right] + C = \frac{1}{2}\left(x + \ln\frac{|1+\operatorname{tg} x|}{\sqrt{1+\operatorname{tg}^2 x}}\right) + C =$$

$$= \frac{1}{2}\big(x + \ln|\sin x + \cos x|\big) + C.$$

435 (2110). $\displaystyle\int \frac{dx}{5-3\cos x} = \left[\operatorname{tg}\frac{x}{2} = t,\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2}\right] =$

$$= \int \frac{2dt}{(1+t^2)\left(5 - \dfrac{3-3t^2}{1+t^2}\right)} = \int \frac{dt}{1+4t^2} = \frac{1}{2}\int \frac{d(2t)}{1+(2t)^2} = \frac{1}{2}\operatorname{arctg} 2t + C =$$

$$= \frac{1}{2}\operatorname{arctg}\left(2\operatorname{tg}\frac{x}{2}\right) + C.$$

436 (2111). $\displaystyle\int \frac{dx}{5+4\sin x} = \left[\operatorname{tg}\frac{x}{2}=t,\ \sin x = \frac{2t}{1+t^2},\ dx = \frac{2dt}{1+t^2}\right] =$

$$= \int \frac{2dt}{(1+t^2)\left(5+\dfrac{8t}{1+t^2}\right)} = \frac{2}{5}\int \frac{dt}{\left(t+\dfrac{4}{5}\right)^2 + \left(\dfrac{3}{5}\right)^2} = \frac{2}{3}\operatorname{arctg}\frac{5t+4}{3} + C =$$

$$= \underline{\underline{\frac{2}{3}\operatorname{arctg}\frac{5\operatorname{tg}\dfrac{x}{2}+4}{3} + C}}.$$

437 (2112). $\displaystyle\int \frac{2-\sin x}{2+\cos x}\,dx = \left[\begin{array}{l} \operatorname{tg}\dfrac{x}{2}=t,\ \cos x = \dfrac{1-t^2}{1+t^2}, \\ \sin x = \dfrac{2t}{1+t^2},\ dt = \dfrac{2dt}{1+t^2} \end{array}\right] =$

$$= \int \frac{2-\dfrac{2t}{1+t^2}}{2+\dfrac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = 4\int \frac{t^2-t+1}{(t^2+3)(t^2+1)}\,dt = I.$$

$$\frac{t^2-t+1}{(t^2+3)(t^2+1)} = \frac{At+B}{t^2+3} + \frac{Ct+D}{t^2+1},$$

$$t^2-t+1 = At^3+Bt^2+At+B+Ct^3+Dt^2+3Ct+3D.$$

$$\left.\begin{array}{l|l} t^3 & A+C=0, \\ t^2 & B+D=1, \\ t^1 & A+3C=-1, \\ t^0 & 3D+B=1 \end{array}\right\} \Rightarrow\ A=\frac{1}{2},\ B=1,\ C=-\frac{1}{2},\ D=0.$$

$$I = 4\left(\frac{1}{2}\int \frac{t+2}{t^2+3}\,dt - \frac{1}{2}\int \frac{t\,dt}{t^2+1}\right) = 4\left(\frac{1}{4}\int \frac{2t\,dt}{t^2+3} + \int \frac{dt}{t^2+\left(\sqrt{3}\right)^2} - \right.$$

$$-\frac{1}{4}\int\frac{2t\,dt}{t^2+1}\Bigg)=\ln(t^2+3)+\frac{4}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}-\ln(t^2+1)+C=\ln\frac{t^2+3}{t^2+1}+$$

$$+\frac{4}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}+C=\ln\left(1+\frac{2}{1+\operatorname{tg}^2\dfrac{x}{2}}\right)+\frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right)+C=$$

$$=\ln\left(1+2\cos^2\frac{x}{2}\right)+\frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right)+C=$$

$$=\underline{\underline{\ln(2+\cos x)+\frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right)+C.}}$$

438 (2113). $\displaystyle\int\frac{\sin^2 x\,dx}{1-\operatorname{tg}x}=\left[\operatorname{tg}x=t,\ \sin^2 x=\frac{t^2}{1+t^2},\ dx=\frac{dt}{1+t^2}\right]=$

$$=\int\frac{t^2dt}{(1+t^2)^2(1-t)}=I.$$

$$\frac{t^2}{(1+t^2)^2(1-t)}=\frac{A}{1-t}+\frac{Bt+C}{(1+t^2)^2}+\frac{Dt+K}{1+t^2},$$

$$t^2=A+2At^2+At^4+Bt+C-Bt^2-Ct+Dt+K-Dt^2-Kt-$$
$$-Dt^4-Kt^3+Dt^3+Kt^2.$$

$$\left.\begin{array}{l|l} t^4 & A-D=0,\\ t^3 & D-K=0,\\ t^2 & 2A-B-D+K=1,\\ t^1 & B-C+D-K=0,\\ t^0 & A+C+K=0\end{array}\right\}\Rightarrow\ A=D=K=\frac{1}{4},\ B=C=-\frac{1}{2}.$$

$$I=\frac{1}{4}\int\frac{dt}{1-t}-\frac{1}{2}\int\frac{(t+1)\,dt}{(1+t^2)^2}+\frac{1}{4}\int\frac{t+1}{1+t^2}\,dt=-\frac{1}{4}\ln|1-t|-$$

$$-\frac{1}{4}\int\frac{d(1+t^2)}{(1+t^2)^2}-\frac{1}{2}\int\frac{dt}{(1+t^2)^2}+\frac{1}{8}\int\frac{d(1+t^2)}{1+t^2}+\frac{1}{4}\int\frac{dt}{1+t^2}=$$

$$=\left[\begin{array}{c}\displaystyle\int\frac{dt}{(1+t^2)^2}=\int\frac{(1+t^2)^2-t^2}{(1+t^2)^2}\,dt=\int\frac{dt}{1+t^2}-\int\frac{t^2dt}{(1+t^2)^2},\\ u=t,\qquad\qquad du=dt,\\ \displaystyle dv=\frac{t\,dt}{(1+t^2)^2},\quad v=\frac{1}{2}\int\frac{d(1+t^2)}{(1+t^2)^2}=-\frac{1}{2(1+t^2)},\\ \displaystyle\int\frac{dt}{(1+t^2)^2}=\operatorname{arctg}t-\left(-\frac{t}{2(1+t^2)}+\frac{1}{2}\int\frac{dt}{1+t^2}\right)=\frac{t}{2(1+t^2)}+\frac{1}{2}\operatorname{arctg}t\end{array}\right]=$$

$$=-\frac{1}{4}\ln|1-t|+\frac{1}{4(1+t^2)}-\frac{t}{4(1+t^2)}-\frac{1}{4}\operatorname{arctg}t+\frac{1}{8}\ln(1+t^2)+\frac{1}{4}\operatorname{arctg}t+$$

$$+C=-\frac{1}{4}\ln|1-t|+\frac{1-t}{4(1+t^2)}+\frac{1}{8}\ln(1+t^2)+C=-\frac{1}{4}\ln|1-\operatorname{tg}x|+$$

$$+\frac{1-\operatorname{tg}x}{4(1+\operatorname{tg}^2x)}+\frac{1}{8}\ln(1+\operatorname{tg}^2x)+C=\frac{\cos x(\cos x-\sin x)}{4}-$$

$$-\frac{1}{4}\ln\frac{|1-\operatorname{tg}x|}{\sqrt{1+\operatorname{tg}^2x}}+C=\underline{\underline{\frac{\cos x(\cos x-\sin x)}{4}-\frac{1}{4}\ln|\cos x-\sin x|+C.}}$$

439 (2114). $\displaystyle\int\frac{dx}{4+\operatorname{tg}x+4\operatorname{ctg}x}=\left[\operatorname{tg}x=t,\ dx=\frac{dt}{1+t^2},\ \operatorname{ctg}x=\frac{1}{t}\right]=$

$$=\int\frac{t\,dt}{(1+t^2)(t^2+4t+4)}=I.$$

$$\frac{t}{(1+t^2)(t^2+4t+4)}=\frac{At+B}{1+t^2}+\frac{C}{(t+2)^2}+\frac{D}{t+2},$$

$$t=At^3+Bt^2+4At^2+4Bt+4At+4B+C+Ct^2+Dt+Dt^3+$$
$$+2D+2Dt^2.$$

$$\left.\begin{array}{l|l} t^3 & A+D=0, \\ t^2 & B+4A+C+2D=0, \\ t^1 & 4B+4A+D=1, \\ t^0 & 4B+C+2D=0 \end{array}\right\} \Rightarrow \begin{cases} B+2A+C, \\ 4B+3A=1, \\ 4B+C-2A=0 \end{cases} \Rightarrow$$

$$\begin{cases} 5B+2C=0, \\ 3B-4B=0, \\ 4B+3A=1 \end{cases} \Rightarrow\ A=\frac{3}{25},\ B=\frac{4}{25},\ C=-\frac{2}{5},\ D=-\frac{3}{25}.$$

$$I=\frac{1}{25}\int\frac{3t+4}{1+t^2}\,dt-\frac{2}{5}\int\frac{dt}{(t+2)^2}-\frac{3}{25}\int\frac{dt}{t+2}=\frac{3}{50}\ln(1+t^2)+$$

$$+\frac{4}{25}\operatorname{arctg} t+\frac{2}{5(2+t)}-\frac{3}{25}\ln|t+2|+C=$$

$$=-\frac{3}{25}\ln|\cos x|+\frac{4}{25}x+\frac{2}{5(\operatorname{tg}x+2)}-\frac{3}{25}\ln|\operatorname{tg}x+2|+C.$$

440 (2115). $\displaystyle\int\frac{dx}{(\sin x+2\sec)^2}=\int\frac{dx}{\left(\sin x+\dfrac{2}{\cos x}\right)^2}=\int\frac{\cos^2 x\,dx}{(\sin x\cdot\cos x+2)^2}=$

$$=\int\frac{(2\cos^2 x-1)\frac{1}{2}+\frac{1}{2}}{(\sin x\cdot\cos x+2)^2}\,dx=\frac{1}{2}\int\frac{(2\cos^2 x-1)dx}{(\sin x\cdot\cos x+2)^2}+$$

$$+\frac{1}{2}\int\frac{dx}{(\sin x\cdot\cos x+2)^2}=\frac{1}{2}\int\frac{\cos^2 x-\sin^2 x}{(\sin x\cdot\cos x+2)^2}\,dx+$$

$$+\frac{1}{2}\int\frac{dx}{(\sin x\cdot\cos x+2)^2}=\frac{1}{2}\int\frac{d(\sin x\cdot\cos x+2)}{(\sin x\cdot\cos x+2)^2}+$$

$$+\frac{1}{2}\int\frac{dx}{(\sin x\cdot\cos x+2)^2}=-\frac{1}{2(\sin x\cdot\cos x+2)}+2\int\frac{dx}{(\sin 2x+4)^2}=$$

$$=-\frac{1}{\sin 2x+4}+2\int\frac{dx}{(\sin 2x+4)^2}=I.$$

Пусть $\displaystyle\int \frac{dx}{(\sin 2x+4)^2}=\frac{A\cos 2x}{\sin 2x+4}+B\int\frac{dx}{\sin 2x+4}.$

Дифференцируем обе части:

$$\frac{1}{(\sin 2x+4)^2}=A\frac{-2\sin 2x(\sin 2x+4)-2\cos^2 2x}{(\sin 2x+4)^2}+\frac{B}{\sin 2x+4},$$

$$1=-2A\sin^2 2x-8A\sin 2x-2A\cos^2 2x+B\sin 2x+4B,$$

$$1=-2A-8A\sin 2x+B\sin 2x+4B,$$

$$\left.\begin{aligned}-8A+B&=0\\-2A+4B&=1\end{aligned}\right\}\Rightarrow\ A=\frac{1}{30},\ B=\frac{4}{15}.$$

Тогда $\displaystyle 2\int\frac{dx}{(\sin 2x+4)^2}=\frac{\cos 2x}{15(\sin 2x+4)}+\frac{8}{15}\int\frac{dx}{\sin 2x+4}=$

$$=\left[\begin{aligned}&\operatorname{tg} t=t,\ \ x=\operatorname{arctg} t,\\ &dx=\frac{dt}{1+t^2},\ \sin 2x=\frac{2t}{1+t^2}\end{aligned}\right]=\frac{\cos 2x}{15(\sin 2x+4)}+$$

$$+\frac{8}{15}\int\frac{dt}{(1+t^2)\left(\frac{2t}{1+t^2}+4\right)}=\frac{\cos 2x}{15(\sin 2x+4)}+\frac{8}{15}\int\frac{dt}{2t+4+4t^2}=$$

$$=\frac{\cos 2x}{15(\sin 2x+4)}+\frac{2}{15}\int\frac{d\left(t+\frac{1}{4}\right)}{\left(t+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}=\frac{\cos 2x}{15(\sin 2x+4)}+$$

$$+\frac{8}{15\sqrt{15}}\operatorname{arctg}\frac{4t+1}{\sqrt{15}}=\frac{\cos 2x}{15(\sin 2x+4)}+\frac{8}{15\sqrt{15}}\operatorname{arctg}\frac{4\operatorname{tg} x+1}{\sqrt{15}}.$$

$$I=\frac{\cos 2x-15}{15(\sin 2x+4)}+\frac{8}{15\sqrt{15}}\operatorname{arctg}\frac{4\operatorname{tg} x+1}{\sqrt{15}}.$$

Другой способ:

$$I=\int\frac{4\cos^2 x\,dx}{(\sin 2x+4)^2}=\int\frac{2+2\cos 2x}{(\sin 2x+4)^2}\,dx=\int\frac{2\cos 2x\,dx}{(\sin 2x+4)^2}+$$

$$+2\int\frac{dx}{(\sin 2x+4)^2}=-\frac{1}{\sin 2x+4}+2\int\frac{dx}{(\sin 2x+4)^2},$$

$$2\int\frac{dx}{(\sin 2x+4)^2}=\left[\begin{array}{c}\sin 2x+4=t,\\ dx=\dfrac{1}{2}\cdot\dfrac{dt}{\sqrt{1-(t-4)^2}}\end{array}\right]=\int\frac{dt}{t^2\sqrt{1-(t-4)^2}}=$$

$$=-\int\frac{d\left(\dfrac{1}{t}\right)}{\sqrt{-15+8t-t^2}}=-\int\frac{d\left(\dfrac{1}{t}\right)}{t\sqrt{\dfrac{1}{15}-\left(\dfrac{\sqrt{15}}{t}-\dfrac{4}{\sqrt{15}}\right)^2}}=$$

$$=\left[\begin{array}{c}\sqrt{15}\left(\dfrac{\sqrt{15}}{t}-\dfrac{4}{\sqrt{15}}\right)=\sin z,\\ \dfrac{1}{t}=\dfrac{\sin z+4}{15},\ \cos z\,dz=15d\left(\dfrac{1}{t}\right)\end{array}\right]=-\int\frac{\dfrac{1}{15}\cos z\dfrac{\sin z+4}{15}\,dz}{\dfrac{1}{\sqrt{15}}\cos z}=$$

$$=-\frac{1}{15\sqrt{15}}\int(\sin z+4)\,dz=\frac{1}{15\sqrt{15}}\cos z-\frac{4z}{15\sqrt{15}}+C=$$

$$=\left[\sin z=\frac{15}{t}-4,\ z=\arcsin\left(\frac{15}{t}-4\right)=\arcsin\left(\frac{15}{\sin 2x+4}-4\right)=\right.$$

$$=\arcsin\left(\frac{-4\sin 2x-1}{\sin 2x+4}\right)=-\arcsin\frac{4\sin 2x+1}{\sin 2x+4},\ \cos^2 z=1-\sin^2 z=$$

$$=1-\left(\frac{15-4t}{t}\right)^2=1-\left(\frac{15-4\sin 2x-16}{\sin 2x+4}\right)^2=1-$$

$$-\frac{\sin^2 2x+8\sin 2x+16-16\sin^2 2x-8\sin 2x-1}{(\sin 2x+4)^2}=$$

$$=\frac{15(1-\sin^2 2x)}{(\sin 2x+4)^2}=\frac{15\cos^2 2x}{(\sin 2x+4)^2},\ \cos z=\left.\frac{\sqrt{15}\cos 2x}{\sin 2x+4}\right]=$$

$$=\frac{1}{15\sqrt{15}}\frac{\sqrt{15}\cos 2x}{\sin 2x+4}+\frac{4}{15\sqrt{15}}\arcsin\frac{4\sin 2x+1}{\sin 2x+4}.$$

$$I=\frac{1}{15}\frac{\cos 2x}{\sin 2x+4}-\frac{1}{\sin 2x+4}+\frac{4}{15\sqrt{15}}\arcsin\frac{4\sin 2x+1}{\sin 2x+4}+C=$$

$$=\underline{\underline{\frac{\cos 2x-15}{15(4+\sin 2x)}+\frac{4}{15\sqrt{15}}\arcsin\frac{4\sin 2x+1}{4+\sin 2x}+C}}.$$

441 (2116). $\displaystyle\int\frac{dx}{5-4\sin x+3\cos x}=\left[\begin{array}{l} t=\operatorname{tg}\frac{x}{2},\ \sin x=\frac{2t}{1+t^2}, \\ \cos x=\frac{1-t^2}{1+t^2},\ dt=\frac{2dt}{1+t^2}\end{array}\right]=$

$$=\int\frac{2dt}{(1+t^2)\left(5-\frac{8t}{1+t^2}+\frac{3-3t^2}{1+t^2}\right)}=\int\frac{dt}{(t-2)^2}=\frac{1}{2-t}+C=\underline{\underline{\frac{1}{2-\operatorname{tg}\frac{x}{2}}+C}}.$$

442 (2117). $\displaystyle\int\frac{dx}{4-3\cos^2 x+5\sin^2 x}=\left[\begin{array}{l} \operatorname{tg}x=t,\ \cos^2 x=\frac{1}{1+t^2}, \\ \sin^2 x=\frac{t^2}{1+t^2},\ dx=\frac{dt}{1+t^2}\end{array}\right]=$

$$=\int\frac{dt}{(1+t^2)\left(4-\frac{3}{1+t^2}+\frac{5t^2}{1+t^2}\right)}=\int\frac{dt}{1+9t^2}=\frac{1}{3}\operatorname{arctg}3t+C=$$

$$=\underline{\underline{\frac{1}{3}\operatorname{arctg}(3\operatorname{tg}x)+C}}.$$

443 (2118). $\displaystyle\int \frac{dx}{1+\sin^2 x} = \left[\begin{array}{c} \operatorname{tg} x = t, \quad \sin^2 x = \dfrac{t^2}{1+t^2}, \\ dx = \dfrac{dt}{1+t^2} \end{array}\right] =$

$$= \int \frac{dt}{(1+t^2)\left(1+\dfrac{t^2}{1+t^2}\right)} = \int \frac{dt}{1+2t^2} = \frac{1}{\sqrt{2}} \int \frac{d\left(\sqrt{2}t\right)}{1+\left(\sqrt{2}t\right)^2} =$$

$$= \underline{\underline{\frac{1}{\sqrt{2}} \operatorname{arctg}\left(\sqrt{2}\operatorname{tg} x\right) + C}}.$$

444 (2119). $\displaystyle\int \frac{dx}{1-\sin^4 x} = \int \frac{dx}{(1-\sin^2 x)(1+\sin^2 x)} =$

$$= \int \frac{dt}{(1+t^2)\left(1-\dfrac{t^2}{1+t^2}\right)\left(1+\dfrac{t^2}{1+t^2}\right)} = \int \frac{(1+t^2)\,dt}{1+2t^2} = \int \frac{\frac{1}{2}(1+2t^2)\,dt}{1+2t^2} +$$

$$+ \frac{1}{2}\int \frac{dt}{1+2t^2} = \underline{\underline{\frac{1}{2}\operatorname{tg} x + \frac{1}{2\sqrt{2}} \operatorname{arctg}\left(\sqrt{2}\operatorname{tg} x\right) + C}}.$$

445 (2120). $\displaystyle\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \int \frac{dt}{\left(\dfrac{a^2t^2}{1+t^2} + \dfrac{b^2}{1+t^2}\right)(1+t^2)} =$

$$= \int \frac{dt}{b^2 + a^2t^2} = \frac{1}{a}\int \frac{d\,(at)}{b^2 + (at)^2} = \frac{1}{ab}\operatorname{arctg}\frac{at}{b} + C =$$

$$= \underline{\underline{\frac{1}{ab} \operatorname{arctg} \frac{a \operatorname{tg} x}{b} + C}}.$$

446 (2121). $\displaystyle\int \frac{dx}{\sin^2 x + \operatorname{tg}^2 x} = \left[\operatorname{tg} x = t,\ \ \sin^2 x = \frac{t^2}{1+t^2},\ \ dx = \frac{dt}{1+t^2} \right] =$

$$= \int \frac{dt}{t^2(2+t^2)} = I.$$

$$\frac{1}{t^2(2+t^2)} = \frac{A}{t^2} + \frac{B}{t} + \frac{Ct+D}{2+t^2},$$

$$1 = 2A + At^2 + Bt + Bt^3 + Ct^3 + Dt^2.$$

$$\left.\begin{array}{l|l} t^3 & B+C=0, \\ t^2 & A+D=0, \\ t^1 & B=0, \\ t^0 & 2A=1 \end{array}\right\} \Rightarrow\ A=\frac{1}{2},\ B=0,\ C=0,\ D=-\frac{1}{2}.$$

$$I = \frac{1}{2}\int \frac{dt}{t^2} - \frac{1}{2}\int \frac{dt}{2+t^2} = -\frac{1}{2t} - \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t}{\sqrt{2}} + C =$$

$$= \underline{\underline{C - \frac{1}{2}\left(\operatorname{ctg} x + \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{\operatorname{tg} x}{\sqrt{2}} \right).}}$$

447 (2122). $\displaystyle\int \frac{\cos x\, dx}{\sin^3 x - \cos^3 x} = \int \frac{\cos x\, dx}{(\sin x - \cos x)(1 + \sin x \cdot \cos x)} = [:\cos x] =$

$$= \int \frac{dx}{(\operatorname{tg} x - 1)(1 + \sin x \cdot \cos x)} = \begin{bmatrix} \operatorname{tg} x = t,\ \ \sin^2 x = \dfrac{t^2}{1+t^{2,}}, \\ \cos^2 x = \dfrac{1}{1+t^2},\ \ dx = \dfrac{dt}{1+t^2} \end{bmatrix} =$$

$$= \int \frac{dt}{(t-1)(1+t^2)\left(1 + \dfrac{t}{1+t^2}\right)} = \int \frac{dt}{(t-1)(t^2+t+1)} = I.$$

$$\frac{1}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt}{t^2+t+1},$$

$$1=At^2+At+A+Bt^2+Ct-Bt-C.$$

$$\left.\begin{array}{c|l} t^2 & A+B=0, \\ t^1 & A+C-B=0, \\ t^0 & A-C=1 \end{array}\right\} \Rightarrow\ A=\frac{1}{3},\ B=-\frac{1}{3},\ B=-\frac{1}{3},\ C=-\frac{2}{3}.$$

$$I=\frac{1}{3}\int\frac{dt}{t-1}-\frac{1}{3}\int\frac{t+2}{t^2+t+1}\,dt=\frac{1}{3}\ln|t-1|-\frac{1}{3}\int\frac{(2t+1)\frac{1}{2}+\frac{3}{2}}{t^2+t+1}\,dt=$$

$$=\frac{1}{3}\ln|t-1|-\frac{1}{6}\ln\left|t^2+t+1\right|-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+C=$$

$$=\ln\left|\frac{\sqrt[3]{\operatorname{tg}x-1}}{\sqrt[6]{\operatorname{tg}^2x+\operatorname{tg}x+1}}\right|-\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\operatorname{tg}x+1}{\sqrt{3}}+C.$$

448 (2123). $\int\sqrt{1+\sin x}\,dx=I.$

$$1+\sin x=1+\cos\left(\frac{\pi}{2}-x\right)=2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right),$$

$$I=\int\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)dx=-2\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)=-2\sqrt{2}\left(\sin\frac{\pi}{4}\cdot\cos\frac{x}{2}-\right.$$

$$\left.-\cos\frac{\pi}{4}\cdot\sin\frac{x}{2}\right)=\left(-2\cos\frac{x}{2}-\sin\frac{x}{2}\right)=$$

$$=\begin{cases} 2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right), & \text{если}\quad \sin\frac{x}{2}-\cos\frac{x}{2}\ge 0, \\ -2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right), & \text{если}\quad \sin\frac{x}{2}-\cos\frac{x}{2}\le 0. \end{cases}$$

449 (2124). $\displaystyle\int \frac{\sqrt{\operatorname{tg} x}}{\sin x \cdot \cos x}\,dx = [\operatorname{tg} x = t] = \int \frac{\sqrt{t}\,dt}{(1+t^2)\dfrac{t}{1+t^2}} = \int \frac{dt}{\sqrt{2}} = 2\sqrt{t} +$

$$+C = \underline{\underline{2\sqrt{\operatorname{tg} x} + C}}.$$

450 (2125). $\displaystyle\int \frac{\sqrt{\sin^3 3x}}{\sin^5 x}\,dx = \left[\operatorname{tg} x = t,\ \cos^2 x = \frac{1}{1+t^2},\ \sin^2 x = \frac{t^2}{1+t^2}\right] =$

$$= 2\sqrt{2}\int \frac{\sqrt{\sin x \cdot \cos^3 x}}{\sin^4 x}\,dx = 2\sqrt{2}\int \frac{\sqrt{\dfrac{t}{1+t^2}\cdot\dfrac{1}{1+t^2}}\,dt}{\dfrac{t^4}{(1+t^2)}(1+t^2)} = 2\sqrt{2}\int t^{-7/2}dt =$$

$$= -\frac{4\sqrt{2}}{5\sqrt{\operatorname{tg}^5 x}} = \underline{\underline{-\frac{4\sqrt{2}}{5}\sqrt{\operatorname{ctg}^5 x} + C}}.$$

451 (2126). $\displaystyle\int \frac{dx}{\sqrt[4]{\sin^3 x \cdot \cos^5 x}} = \begin{bmatrix} \operatorname{tg} x = t,\ \sin^2 x = \dfrac{t^2}{1+t^2}, \\ \cos^2 x = \dfrac{1}{1+t^2},\ dx = \dfrac{dt}{1+t^2} \end{bmatrix} =$

$$= \int \frac{dt}{(1+t^2)\sqrt[4]{\dfrac{t^3}{(1+t^2)^4}}} = \int t^{-3/4}dt = 4\sqrt[4]{t} + C = \underline{\underline{4\sqrt[4]{\operatorname{tg} x} + C}}.$$

452 (2127). $\displaystyle\int \frac{dx}{\sqrt{1-\sin^4 x}} = \left[\operatorname{tg} x = t,\ \sin^2 x = \frac{t^2}{1+t^2},\ dx = \frac{dt}{1+t^2}\right] =$

$$=\int\frac{dt}{(1+t^2)\sqrt{1-\dfrac{t^4}{(1+t^2)^2}}}=\int\frac{dt}{\sqrt{1+2t^2}}=\frac{1}{2}\int\frac{d\left(\sqrt{2}t\right)}{\sqrt{1+\left(\sqrt{2}t\right)^2}}=$$

$$=\frac{1}{\sqrt{2}}\ln\left|\sqrt{2}t+\sqrt{1+2t^2}\right|=\frac{1}{\sqrt{2}}\ln\left|\sqrt{2}\,\mathrm{tg}\,x+\sqrt{1+2\,\mathrm{tg}^2\,x}\right|+C.$$

453 (2128). $\int\sqrt{1+\mathrm{cosec}\,x}\ dx=\int\frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}\,dx=\int\frac{\sqrt{1-\sin^2 x}}{\sqrt{\sin x}\sqrt{1-\sin x}}=$

$$=2\int\frac{\cos x\,dx}{2\sqrt{\sin x}\sqrt{1-\sin x}}=2\int\frac{d\left(\sqrt{\sin x}\right)}{1-\left(\sqrt{\sin x}\right)^2}=2\arcsin\sqrt{\sin x}+C.$$

454 (2129). $\int\frac{(\cos 2x-3)\,dx}{\cos^4 x\sqrt{4-\mathrm{ctg}^2\,x}}=\int\frac{(\cos^2 x-\sin^2 x-3)\,dx}{\cos^4 x\sqrt{4-\mathrm{ctg}^2\,x}}=$

$$=\begin{bmatrix}\mathrm{tg}\,x=t,\quad x=\mathrm{arctg}\,t,\ dx=\dfrac{dt}{1+t^2},\\ \cos^2 x=\dfrac{1}{1+t^2},\quad \sin^2 x=\dfrac{t^2}{1+t^2}\end{bmatrix}=-2\int\frac{(1+2t^2)\,t\,dt}{\sqrt{4t^2-1}}=$$

$$=\begin{bmatrix}t=\dfrac{1}{2\cos z},\\ dt=\dfrac{\sin z\,dz}{2\cos^2 z}\end{bmatrix}=-2\int\frac{\left(1+\dfrac{1}{2\cos^2 z}\right)\dfrac{1}{2\cos z}\cdot\dfrac{\sin z}{2\cos^2 z}}{\sqrt{\dfrac{1}{\cos^2 z}-1}}\,dz=$$

$$=-\frac{1}{4}\int\frac{2\cos^2 z+1}{\cos^4 z}\,dz=-\frac{1}{4}\int\frac{3\cos^2 z+\sin^2 z}{\cos^4 z}\,dz=$$

$$=-\frac{1}{4}\left(3\int\frac{dz}{\cos^2 z}+\int\mathrm{tg}^2\,z\,d\,(\mathrm{tg}\,z)\right)=-\frac{1}{4}\left[3\,\mathrm{tg}\,z+\frac{\mathrm{tg}^3\,z}{3}\right]+C=I,$$

$$\operatorname{tg} z = \frac{\sin z}{\cos z} = \frac{\sqrt{1-\frac{1}{4t^2}}}{\frac{1}{2}t} = \sqrt{4t^2-1},$$

$$I = -\frac{1}{4}\left(3\sqrt{4t^2-1} + \frac{\sqrt{(4t^2-1)^3}}{3}\right) + C = C - \frac{1}{4}\sqrt{4t^2-1}\left(3 + \frac{4t^2-1}{3}\right) =$$

$$= C - \frac{1}{4}\sqrt{4-\operatorname{ctg}^2 x}\cdot \operatorname{tg} x\left(\frac{8}{3} + \frac{4}{3}\operatorname{tg}^2 x\right) =$$

$$= \underline{\underline{C - \frac{1}{3}\operatorname{tg} x\sqrt{4-\operatorname{ctg}^2 x}\,(2+\operatorname{tg}^2 x).}}$$

455 (2130). $\displaystyle\int \frac{dx}{\sin\frac{x}{2}\sqrt{\cos^3\frac{x}{2}}} = \left[\begin{array}{c} \cos\frac{x}{2} = t,\ \sin\frac{x}{2} = \sqrt{1-t^2}, \\ x = 2\arccos t,\ dx = -\frac{2dt}{\sqrt{1-t^2}} \end{array}\right] =$

$$= -2\int \frac{dt}{\left(\sqrt{1-t^2}\right)^2 t\sqrt{t}} = \left[t = z^2,\ dt = 2z\,dz\right] = -4\int \frac{dz}{(1-z^4)z^2} = I.$$

$$\frac{1}{(1-z)(1+z)(1+z^2)z^2} = \frac{A}{1-z} + \frac{B}{1+z} + \frac{Cz+D}{1+z^2} + \frac{L}{z^2} + \frac{N}{z},$$

$$1 = Az^5 + Az^4 + Az^3 + Az^2 - Bz^5 + Bz^4 - Bz^3 + Bz^2 + Cz^3 +$$
$$+ Dz^2 - Cz^5 - Dz^4 + L - Lz^4 + Nz - Nz^5.$$

$$\left.\begin{array}{l|l} z^5 & A-B-C-N=0, \\ z^4 & A+B-D-L=0, \\ z^3 & A-B+C=0, \\ z^2 & A+B+D=0, \\ z^1 & N=0, \\ z^0 & L=1 \end{array}\right\} \Rightarrow \quad \begin{array}{l} N=0,\ L=1,\ A=\frac{1}{4}, \\ B=\frac{1}{4},\ C=0,\ D=-\frac{1}{2}. \end{array}$$

$$I=-4\left(\frac{1}{4}\int\frac{dz}{1-z}+\frac{1}{4}\int\frac{dz}{1+z}-\frac{1}{2}\int\frac{dz}{1+z^2}+\int\frac{dz}{z^2}\right)=\ln|1-z|-\ln|1+z|+$$

$$+2\operatorname{arctg} z+\frac{4}{z}+C=\underline{\underline{\frac{4}{\sqrt{\cos\frac{x}{2}}}+2\operatorname{arctg}\sqrt{\cos\frac{x}{2}}-\ln\frac{1+\sqrt{\cos\frac{x}{2}}}{1-\sqrt{\cos\frac{x}{2}}}+C}}.$$

456 (2131). $\displaystyle\int\sqrt{\operatorname{tg}x}\,dx=\left[\begin{array}{c}\operatorname{tg}x=t,\ x=\operatorname{arctg}t,\\ dx=\dfrac{dt}{1+t^2}\end{array}\right]=\int\frac{\sqrt{t}\,dt}{1+t^2}=$

$$=\left[\begin{array}{c}\sqrt{t}=z,\ z^2=t,\\ 2z\,dz=dt\end{array}\right]=2\int\frac{z^2dz}{1+z^4}=\left[\begin{array}{l}1+z^4=(z^2+1)^2-2z^2=\\ =\left(z^2+z\sqrt{2}+1\right)\left(z^2-z\sqrt{2}+1\right)\end{array}\right]=I.$$

$$\frac{z^2}{\left(z^2+z\sqrt{2}+1\right)\left(z^2-z\sqrt{2}+1\right)}=\frac{Az+B}{z^2+z\sqrt{2}+1}+\frac{Cz+D}{z^2-z\sqrt{2}+1},$$

$$z^2=Az^3+Bz^2-A\sqrt{2}z^2-B\sqrt{2}z+Az+B+Cz^3+Dz^2+$$

$$+C\sqrt{2}z^2+D\sqrt{2}z+Cz+D.$$

$$\left.\begin{array}{l|l}z^3 & A+C=0,\\ z^2 & B-A\sqrt{2}+D+C\sqrt{2}=1,\\ z^1 & -B\sqrt{2}+A+D\sqrt{2}+C=0,\\ z^0 & B+D=0\end{array}\right\}\Rightarrow\quad\begin{array}{l}A=-C=-\dfrac{1}{2\sqrt{2}},\\ D=B=0.\end{array}$$

$$I=-\frac{1}{\sqrt{2}}\int\frac{z\,dz}{z^2+z\sqrt{2}+1}+\frac{1}{\sqrt{2}}\int\frac{z\,dz}{z-z\sqrt{2}+1}=$$

$$=\frac{1}{\sqrt{2}}\int\frac{\left(2z+\sqrt{2}\right)\frac{1}{2}-\frac{\sqrt{2}}{2}}{z^2+z\sqrt{2}+1}\,dz+\frac{1}{\sqrt{2}}\int\frac{\left(2z-\sqrt{2}\right)+\frac{\sqrt{2}}{2}}{z^2-z\sqrt{2}+1}\,dz=$$

$$=-\frac{1}{2\sqrt{2}}\ln\left|z^2+z\sqrt{2}+1\right|+\frac{1}{2}\int\frac{d\left(z+\frac{\sqrt{2}}{2}\right)}{\left(z+\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}+$$

$$+\frac{1}{2\sqrt{2}}\ln\left|z^2-z\sqrt{2}+1\right|+\frac{1}{2}\int\frac{d\left(z-\frac{\sqrt{2}}{2}\right)}{\left(z-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{z^2-z\sqrt{2}+1}{z^2+z\sqrt{2}+1}\right|+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{z+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{z-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}+C=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{z^2-2\sqrt{2}+1}{z^2+2\sqrt{2}+1}\right|+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\frac{2z+\sqrt{2}}{\sqrt{2}}+\frac{2z-\sqrt{2}}{\sqrt{2}}}{1-\frac{4z^2-2}{2}}+C=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{z^2-z\sqrt{2}+1}{z^2+z\sqrt{2}+1}\right|+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2}z}{1-z^2}+C=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\operatorname{tg}x-\sqrt{2\operatorname{tg}x}+1}{\operatorname{tg}x+\sqrt{2\operatorname{tg}x}+1}\right|+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2\operatorname{tg}x}}{1-\operatorname{tg}x}+C=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{(\sin x+\cos x)-\sqrt{2\cos x\cdot\sin x}}{(\sin x+\cos x)+\sqrt{2\cos x\cdot\sin x}}\right|+$$

$$+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2\sin x\cdot\cos x}}{\cos x-\sin x}+C.$$

Воспользуемся формулой

$$\operatorname{arctg} a = \arcsin\frac{a}{\sqrt{1+a^2}} = \arcsin\frac{\dfrac{\sqrt{2\sin x\cdot\cos x}}{\cos x-\sin x}}{\sqrt{1+\dfrac{2\sin x\cdot\cos x}{\cos^2 x-2\sin x\cdot\cos x+\sin^2 x}}} =$$

$$= \arcsin\sqrt{\sin 2x},$$

тогда

$$I = \frac{1}{2\sqrt{2}}\ln\left|\frac{\left((\sin x+\cos x)-\sqrt{\sin 2x}\right)^2}{\sin^2 x+\cos^2 x+2\cos x\cdot\sin x-2\cos x\cdot\sin x}\right| +$$

$$+\frac{1}{\sqrt{2}}\arcsin\sqrt{\sin 2x}+C = \frac{1}{\sqrt{2}}\ln\left|\sin x+\cos x-\sqrt{\sin 2x}\right| +$$

$$+\frac{1}{\sqrt{2}}\arcsin\sqrt{\sin 2x} =$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\sin x+\cos x-\sqrt{\sin 2x}\right| + \arcsin(\sin x-\cos x)\right)+C.$$

Примечание.

$$\left(\arcsin\sqrt{\sin 2x}\right)' = \frac{2\cos 2x}{2\sqrt{\sin 2x}\sqrt{1-\sin 2x}} = \frac{\cos^2 x-\sin^2 x}{\sqrt{\sin 2x}(\cos x-\sin x)} =$$

$$= \frac{\cos x+\sin x}{\sqrt{\sin 2x}},$$

$$(\arcsin(\sin x-\cos x))' = \frac{\cos x+\sin x}{\sqrt{1-\sin^2 x+2\sin x\cdot\cos x-\cos^2 x}} =$$

$$= \frac{\cos x+\sin x}{\sqrt{\sin 2x}}.$$

3.4. Гиперболические функции

$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2}, \ \operatorname{sh} x = \frac{e^x - e^{-x}}{2},$$

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1, \ \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x}, \ \operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x},$$

$$\operatorname{sh} 2x = 2\operatorname{sh} x \operatorname{ch} x, \ \operatorname{sh}^2 x = \operatorname{ch} 2t - 1, \ \operatorname{ch}^2 t = \operatorname{ch} 2t + 1.$$

457 (2132). $\int \operatorname{ch} x\, dx = \int \frac{e^x + e^{-x}}{2} dx = \frac{1}{2}(e^x - e^{-x}) + C = \underline{\underline{\operatorname{sh} x + C}}.$

458 (2133). $\int \operatorname{ch} x\, dx = \int \frac{e^x - e^{-x}}{2} dx = \frac{1}{2}(e^x + e^{-x}) + C = \underline{\underline{\operatorname{ch} x + C}}.$

459 (2134). $\int \frac{dx}{\operatorname{ch}^2 x} = \underline{\underline{\operatorname{th} x + C}}, \ (\operatorname{th} x)' = \left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)' = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x}.$

460 (2135). $\int \frac{e^x}{\operatorname{ch} x + \operatorname{sh} x} = \int \frac{e^x dx}{\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}} = \int dx = \underline{\underline{x + C}}.$

461 (2136). $\int (\operatorname{ch}^2 ax + \operatorname{sh}^2 ax)\, dx = \frac{1}{2}\int (\operatorname{ch} 2ax + 1)\, dx +$

$+\frac{1}{2}\int (\operatorname{ch} 2ax - 1)\, dx = \frac{1}{4a}\operatorname{sh} 2ax + \frac{1}{2}x + \frac{1}{4a}\operatorname{ch} 2ax - \frac{1}{2}x =$

$= \underline{\underline{\frac{1}{2a}\operatorname{ch} 2ax + C}}.$

462 (2137). $\int \operatorname{sh}^2 x\, dx = \frac{1}{2}\int (\operatorname{ch} 2x - 1)\, dx = \frac{1}{4}\operatorname{sh} 2x - \frac{1}{2}x =$

$= \underline{\underline{\frac{\operatorname{sh} x \cdot \operatorname{ch} x - x}{2} + C}}.$

463 (2138). $\int \operatorname{th}^2 x\,dx = \int \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}\,dx = \int \frac{\operatorname{ch}^2 x - 1}{\operatorname{ch}^2 x}\,dx = \int dx - \int \frac{dx}{\operatorname{ch}^2 x} =$

$= \underline{\underline{x - \operatorname{th} x + C}}.$

464 (2139). $\int \operatorname{cth}^2 x\,dx = \int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^2 x}\,dx = \int \frac{1 + \operatorname{sh}^2 x}{\operatorname{sh}^2 x}\,dx = \int \frac{dx}{\operatorname{sh}^2 x} + \int dx =$

$= \underline{\underline{x - \operatorname{cth} x + C}}.$

465 (2140). $\int \operatorname{sh}^3 x\,dx = \int \operatorname{sh}^2 x \cdot \operatorname{sh} x\,dx = \int (\operatorname{ch}^2 x - 1)\,d(\operatorname{ch} x) =$

$= \underline{\underline{-\operatorname{ch} x + \frac{\operatorname{ch}^3 x}{3} + C}}.$

466 (2141). $\int \operatorname{ch}^3 x\,dx = \int \operatorname{ch}^2 x \cdot \operatorname{ch} x\,dx = \int (1 + \operatorname{sh}^2 x)\,d(\operatorname{sh} x) =$

$= \underline{\underline{\operatorname{sh} x + \frac{\operatorname{sh}^3 x}{3} + C}}.$

467 (2142). $\int \operatorname{th}^4 x\,dx = \int \operatorname{th}^2 x \cdot \operatorname{th}^2 x\,dx = \int \operatorname{th}^2 x \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}\,dx =$

$= \int \operatorname{th}^2 x \frac{\operatorname{ch}^2 x - 1}{\operatorname{ch}^2 x}\,dx = \int \operatorname{th}^2 x\,dx - \int \frac{\operatorname{th}^2 x}{\operatorname{ch}^2 x}\,dx = \int \frac{\operatorname{ch}^2 x - 1}{\operatorname{ch}^2 x}\,dx -$

$- \int \operatorname{th}^2 x\,d(\operatorname{th} x) = \underline{\underline{x - \operatorname{th} x - \frac{\operatorname{th}^3 x}{3} + C}}.$

468 (2143). $\int \operatorname{sh}^2 x \cdot \operatorname{ch}^3 x\,dx = \int \operatorname{sh}^4 x (1 + \operatorname{sh}^2 x)\,d(\operatorname{sh} x) = \underline{\underline{\frac{\operatorname{sh}^3 x}{3} + \frac{\operatorname{sh}^5 x}{5} + C}}.$

469 (2144). $\int \operatorname{cth}^5 x\,dx = \int \frac{\operatorname{ch}^4 x \cdot \operatorname{ch} x}{\operatorname{sh}^5 x}\,dx = \int \frac{(1 + \operatorname{sh}^2 x)^2\,d(\operatorname{sh} x)}{\operatorname{sh}^5 x} =$

$$=\int \frac{1+2\operatorname{sh}^2 x+\operatorname{sh}^4 x}{\operatorname{sh}^5 x}\,d(\operatorname{sh} x)=\int \frac{d(\operatorname{sh} x)}{\operatorname{sh}^5 x}+2\int \frac{d(\operatorname{sh} x)}{\operatorname{sh}^3 x}+\int \frac{d(\operatorname{sh} x)}{\operatorname{sh} x}=$$

$$=\underline{\underline{-\frac{1}{4\operatorname{sh}^4 x}-\frac{1}{\operatorname{sh}^2 x}+\ln|\operatorname{sh} x|+C}}.$$

470 (2145). $\displaystyle\int \frac{dx}{\operatorname{sh} x\cdot\operatorname{ch} x}=\int \frac{dx}{\operatorname{th} x\cdot\operatorname{ch}^2 x}=\int \frac{d(\operatorname{th} x)}{\operatorname{th} x}=\underline{\underline{\ln|\operatorname{th} x|+C}}.$

471 (2146). $\displaystyle\int \frac{dx}{\operatorname{sh} x}=\int \frac{dx}{2\operatorname{sh}\frac{x}{2}\cdot\operatorname{ch}\frac{x}{2}}=\frac{1}{2}\int \frac{dx}{\operatorname{th}\frac{x}{2}\cdot\operatorname{ch}^2\frac{x}{2}}=\int \frac{d\left(\operatorname{th}\frac{x}{2}\right)}{\operatorname{th}\frac{x}{2}}=$

$$=\underline{\underline{\ln\left|\operatorname{th}\frac{x}{2}\right|+C}}.$$

472 (2147). $\displaystyle\int \frac{dx}{(1+\operatorname{ch} x)^2}=\int \frac{dx}{4\operatorname{ch}^4\frac{x}{2}}=\frac{1}{4}\int \frac{\operatorname{ch}^2\frac{x}{2}-\operatorname{sh}^2\frac{x}{2}}{\operatorname{ch}^4\frac{x}{2}}\,dx=\frac{1}{4}\int \frac{dx}{\operatorname{ch}^2\frac{x}{2}}-$

$$-\frac{1}{4}\int \operatorname{th}^2 x\,d(\operatorname{th} x)=\underline{\underline{\frac{1}{2}\operatorname{th}\frac{x}{2}-\frac{1}{6}\operatorname{th}^3\frac{x}{2}+C}}.$$

473 (2148). $\displaystyle\int \sqrt{\operatorname{th} x}\,dx=\left[\begin{array}{c}\operatorname{th} x=z,\quad \dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\dfrac{e^{2x}-1}{e^{2x}+1}=z,\\ x=\dfrac{1}{2}\ln\left|\dfrac{1+z}{1-z}\right|,\quad dx=\dfrac{dz}{1-z^2},\quad e^{2x}=\dfrac{1+z}{1-z}\end{array}\right]=$

$$=\int \frac{\sqrt{z}\,dz}{1-z^2}=\left[z=t^2,\ dz=2t\,dt\right]=2\int \frac{t^2dt}{1-t^4}=I.$$

$$\frac{t^2}{1-t^4}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{Ct+D}{1+t^2},$$

$$\begin{cases} A-B-C=0, \\ A+B-D=1, \\ A-B+C=0, \\ A+B+D=0 \end{cases} \Rightarrow\ A=B=\frac{1}{4},\ C=0,\ D=-\frac{1}{4}.$$

$$I=2\left(\frac{1}{4}\int\frac{dt}{1-t}+\frac{1}{4}\int\frac{dt}{1+t}-\frac{1}{2}\int\frac{dt}{1+t^2}\right)=\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|-\operatorname{arctg}t+C=$$

$$=\frac{1}{2}\ln\frac{\left|1+\sqrt{\operatorname{th}x}\right|}{\left|1-\sqrt{\operatorname{th}x}\right|}-\operatorname{arctg}\sqrt{\operatorname{th}x}+C.$$

474 (2149). $$\int\frac{x\,dx}{\operatorname{ch}^2x}=\begin{bmatrix} u=x, & du=dx, \\ dv=\dfrac{dx}{\operatorname{ch}^2x}, & v=\operatorname{th}x \end{bmatrix}=x\operatorname{th}x-\int\operatorname{th}x\,dx=$$

$$=x\operatorname{th}x-\ln|\operatorname{ch}x|+C.$$

475 (2150). $$\int\frac{e^{2x}dx}{\operatorname{sh}^4x}=\begin{bmatrix} e^{2x}=t,\ \ dx=\dfrac{dt}{2t}, \\ \operatorname{sh}x=\dfrac{e^x-e^{-x}}{2},\ \operatorname{sh}^2x=\dfrac{e^{2x}-2+e^{-2x}}{4}=\dfrac{(t-1)^2}{4t} \end{bmatrix}=$$

$$=8\int\frac{t^2dt}{(t-1)^4}=I.$$

$$\frac{t^2}{(t-1)^4}=\frac{A}{(t-1)^4}+\frac{B}{(t-1)^3}+\frac{C}{(t-1)^2}+\frac{D}{t-1},$$

$$t^2=A+B(t-1)+C(t-1)^2+D(t-1)^3.$$

$$\begin{cases} t=0: & 0=A-B+C-D, \\ t=1: & 1=A, \\ t=-1: & 1=A-2B+4C-8D, \\ t=2: & 4=A+B+C+D \end{cases} \Rightarrow\ A=C=1,\ B=2,\ D=0.$$

$$I=8\left(\int\frac{dt}{(t-1)^4}+2\int\frac{dt}{(t-1)^3}+\int\frac{dt}{(t-1)^2}\right)=$$

$$=-8\left(\frac{1}{3(t-1)^3}+\frac{1}{(t-1)^2}+\frac{1}{(t-1)}\right)+C_0=-\frac{8}{3}\cdot\frac{3t^2-3t+1}{(t-1)^3}+C_0=$$

$$=-\frac{8}{3}\cdot\frac{3e^{4x}-3e^{2x}+1}{(e^{2x}-1)^3}+C_0=-\frac{8}{3}\cdot\frac{3e^{4x}-3e^{2x}+1}{\left(\frac{e^x-e^{-x}}{2}\right)^3 8e^{3x}}+C_0=$$

$$=-\frac{(3e^x-3e^{-x}+e^{-3x}-e^{3x})+e^{3x}}{3\left(\frac{e^x-e^{-x}}{2}\right)^3}+C_0=C_0+\frac{(e^x-e^{-x})^3}{3\left(\frac{e^x-e^{-x}}{2}\right)^3}-$$

$$-\frac{e^{3x}}{3\operatorname{sh}^3 x}=\underbrace{C_0+\frac{8}{3}}_{C}-\frac{e^{3x}}{3\operatorname{sh}^3 x}=\underline{\underline{C-\frac{e^{3x}}{3\operatorname{sh}^3 x}}}.$$

3.5. Рациональные функции от x и $\sqrt{ax^2+bx+c}$

Подстановки Эйлера:

1. Если $a>0$ то $\sqrt{ax^2+bx+c}=\pm\sqrt{a}x+t$;

2. Если $c>0$ то $\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$;

3. Если α и β – действительные корни трехчлена ax^2+bx+c, то $\sqrt{ax^2+bx+c}=(x-\alpha)t$.

476 (2151). $\displaystyle\int\frac{dx}{x\sqrt{x^2+x+1}}=\left[x=\frac{1}{z},\ dx=-\frac{dz}{z^2}\right]=-\int\frac{z^2dz}{z^2\sqrt{1+z+z^2}}=$

$$= -\int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} = -\ln\left|z+\frac{1}{2}+\sqrt{1+z+z^2}\right| + \ln C_1 = \ln C_1 -$$

$$-\ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x}\right| = \ln\frac{|Cx|}{2+x+\sqrt{x^2+x+1}} \quad (C=2C_1).$$

477 (2152). $\int \frac{dx}{x\sqrt{x^2+4x-4}} = I.$

Используем подстановку Эйлера:

$$\sqrt{x^2+4x-4} = -x+t,\;\; x^2+4x-4 = x^2-2xt+t^2,\;\; x(4+2t) = t^2+4,$$

$$x = \frac{t^2+4}{4+2t},\;\; dx = \frac{t^2+4t-4}{2(t+2)^2}\,dt.$$

$$I = \int \frac{(t^2+4t-4)(t+2)^2\,4\,dt}{2(t+2)^2(t^2+4)(t^2+4t-4)} = 2\int\frac{dt}{t^2+4} = \operatorname{arctg}\frac{t}{2}+C =$$

$$= \operatorname{arctg}\frac{\sqrt{x^2+4x-4}+x}{2}+C.$$

Другой способ:

$$x = \frac{1}{z},\;\; dx = -\frac{dz}{z^2},\;\; x^2+4x-4 = \frac{1}{z^2}+\frac{4}{z}-4 = \frac{1+4z-4z^2}{z^2} =$$

$$= \frac{\left(\sqrt{2}\right)^2-(2z-1)^2}{z^2}.$$

$$I = -\int\frac{z^2dz}{z^2\sqrt{(2)^2-(2z-1)^2}} = -\frac{1}{2}\int\frac{d\,(2z+1)}{\sqrt{\left(\sqrt{2}\right)^2-(2z-1)^2}} =$$

$$= \frac{1}{2}\arccos\frac{2z-1}{\sqrt{2}}+C = \frac{1}{2}\arccos\frac{2-x}{\sqrt{2}x}+C.$$

478 (2153). $\displaystyle\int\frac{dx}{x\sqrt{x^2+2x-1}}=\left[\begin{array}{c} x=-\dfrac{1}{z},\ \ dx=\dfrac{dz}{z^2}, \\ x^2+2x-1=\dfrac{\left(\sqrt{2}\right)^2-(z+1)^2}{z^2}\end{array}\right]=$

$$=\int\frac{dz}{\sqrt{\left(\sqrt{2}\right)^2-(z+1)^2}}=\arcsin\frac{z+1}{\sqrt{2}}+C=\underline{\underline{\arcsin\frac{x-1}{x\sqrt{2}}+C}}.$$

479 (2154). $\displaystyle\int\frac{dx}{x\sqrt{2+x-x^2}}=I.$

$$2+x-x^2=-(x+1)(x-1).$$

Применим третью подстановку Эйлера:

$$\sqrt{-(x+1)(x-2)}=(x+1)t,\ -(x+1)(x-2)=(x+1)^2t^2,$$

$$-x+2=(x+1)t^2,\ x=\frac{2-t^2}{t^2+1},\ dx=\frac{-6t}{(t^2+1)^2}\,dt.$$

$$I=-6\int\frac{t(t^2+1)^2\,dt}{(t^2+1)(2-t^2)}=-2\int\frac{dt}{2-t^2}=-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|+C_0=C_0-$$

$$-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+\dfrac{\sqrt{2+x-x^2}}{x+1}}{\sqrt{2}-\dfrac{\sqrt{2+x-x^2}}{x+1}}\right|=C_0-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}(x+1)+\sqrt{2+x-x^2}}{\sqrt{2}(x+1)-\sqrt{2+x-x^2}}\right|.$$

Умножим числитель и знаменатель дроби на числитель, тогда

$$I=C_0-\frac{1}{\sqrt{2}}\ln\left|\frac{2(x^2+2x+1)+2\sqrt{2}(x+1)\sqrt{2+x-x^2}+2+x-x^2}{3x(x+1)}\right|=$$

$$=C_0-\frac{1}{\sqrt{2}}\ln\left|\frac{2\sqrt{2}(x+1)\sqrt{2+x-x^2}}{3x(x+1)}+\frac{x^2+5x+4}{3x(x+1)}\right|=$$

$$=C_0-\frac{1}{\sqrt{2}}\ln\left|\frac{2\sqrt{2}\sqrt{2+x-x^2}+4}{3x}+\frac{1}{3}\right|=$$

$$=C_0-\frac{1}{\sqrt{2}}\ln\left|\frac{2\sqrt{2}}{3}\left(\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x}+\frac{1}{2\sqrt{2}}\right)\right|=$$

$$=\underline{\underline{C-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x}+\frac{1}{2\sqrt{2}}\right|}}\quad\left(C=C_0+\ln\frac{2\sqrt{2}}{3}\right).$$

480 (2155). $\int\frac{\sqrt{2x+x^2}}{x^2}=I.$

$$\sqrt{2x+x^2}=-x+t,\ x=\frac{t^2}{2(1+t)},\ dx=\frac{t(2+t)\,dt}{2(1+t)^2}.$$

$$I=\int\frac{(2+t)^2\,dt}{t^2(t+1)},$$

$$\frac{4+4t+t^2}{t^2(t+1)}=\frac{A}{t^2}+\frac{B}{t}+\frac{C}{t+1},$$

$$4+4t+t^2=A+Bt+Bt^2+At+Ct^2,$$

$$\begin{cases}B+C=1,\\ B+A=4,\\ A=4\end{cases}\Rightarrow\ A=4,\ B=0,\ C=1.$$

$$I=4\int\frac{dt}{t^2}+\int\frac{dt}{t+1}=-\frac{4}{t}+\ln|t+1|+C=$$

$$=\underline{\underline{-\frac{4}{\sqrt{2x+x^2}+x}+\ln\left|x+1+\sqrt{2x+x^2}\right|+C}}.$$

481 (2156). $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = I.$

$$\sqrt{x^2+x+1} = -x+t,\ x^2+x+1 = x^2+2xt+t^2,$$

$$x(1+2t) = t^2-1,\ x = \frac{t^2-1}{1+2t},$$

$$dx = \frac{2t(1+2t)-2(t^2-1)}{(1+2t)^2}\,dt = \frac{2(t^2+t+1)}{(1+2t)^2}\,dt,$$

$$x-1 = \frac{t^2-1}{1+2t}-1 = \frac{t^2-2t-2}{1+2t}.$$

$$I = \int \frac{2(t^2+t+1)(1+2t)^2\,dt}{(1+2t)^2(t^2-2t-2)(t^2+t+1)} = 2\int \frac{dt}{(t-1)^2-3} =$$

$$= -2\int \frac{dt}{\left(\sqrt{3}\right)^2-(t-1)^2} = -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+t-1}{\sqrt{3}-t+1}\right| + C =$$

$$= C - \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{x^2+x+1}+x-1}{\sqrt{3}-\sqrt{x^2+x+1}-x+1}\right|.$$

482 (2157). $\displaystyle\int \frac{dx}{(2x-3)\sqrt{4x-x^2}} = I.$

$$2x-3 = \frac{1}{t},\ 2dx = -\frac{dt}{t^2},\ t = -\frac{1}{2x-3},\ 4x-x^2 = \frac{15t^2+2t-1}{4t^2}.$$

$$I = -\int \frac{dt}{2t^2\cdot\frac{1}{t}\cdot\frac{\sqrt{15t^2+2t-1}}{2t}} = -\frac{1}{\sqrt{15}}\int \frac{dt}{\sqrt{t^2+\frac{2}{15}t-\frac{1}{15}}} =$$

$$= -\frac{1}{\sqrt{15}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{15}\right)^2-\left(\frac{4}{15}\right)^2}} = -\frac{1}{\sqrt{15}}\ln\left|t+\frac{1}{15}+\sqrt{t^2+\frac{2}{15}t-\frac{1}{15}}\right| +$$

$$+C_0=-\frac{1}{\sqrt{15}}\ln\left|\frac{1}{2x-3}+\frac{1}{15}+\frac{\sqrt{16x-4x^2}}{\sqrt{15}(2x-3)}\right|+C_0=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{1}{2x-3}+\frac{1}{15}+\frac{2\sqrt{60x-15x^2}}{15(2x-3)}\right|+C_0=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{2x+12+2\sqrt{60x-15x^2}}{15(2x-3)}\right|+C_0=$$

$$=\underline{\underline{C-\frac{1}{\sqrt{15}}\ln\left|\frac{x+6+\sqrt{60x-15x^2}}{2x-3}\right|}}\quad\left(C=C_0-\frac{1}{\sqrt{15}}\ln\frac{2}{15}\right).$$

483 (2158). $\displaystyle\int\sqrt{x^2-2x-1}\,dx=\begin{bmatrix}x-1=z,\\ dx=dz\end{bmatrix}=\int\sqrt{(x-1)^2-\left(\sqrt{2}\right)^2}\,dx=$

$$=\int\sqrt{z^2-2}=\begin{bmatrix}z=\sqrt{2}\operatorname{ch}t,\\ dz=\sqrt{2}\operatorname{sh}t\,dt\end{bmatrix}=2\int\sqrt{\operatorname{ch}^2t-1}\cdot\operatorname{sh}t\,dt=2\int\operatorname{sh}^2t\,dt=$$

$$=2\cdot\frac{1}{2}\int(\operatorname{ch}2t-1)\,dt=\frac{1}{2}\operatorname{sh}2t-t+C_0=\operatorname{sh}t\cdot\operatorname{ch}t-t+C_0=$$

$$=\sqrt{\operatorname{ch}^2t-1}\cdot\operatorname{ch}t-t+C_0=\sqrt{\frac{z^2}{2}-1}\cdot\frac{z}{\sqrt{2}}-\operatorname{arch}\frac{z}{\sqrt{2}}+C_0=$$

$$=\frac{1}{2}\sqrt{z^2-2}\cdot z-\ln\left(z+\sqrt{z^2-2}\right)-\ln\frac{1}{\sqrt{2}}+C_0=$$

$$=\underline{\underline{\frac{1}{2}(x-1)\sqrt{x^2-2x-1}-\ln\left(x-1+\sqrt{x^2-2x+1}\right)+C}}$$

$$\left(C+C_0-\ln\frac{1}{\sqrt{2}}\right).$$

Примечание.

$$\operatorname{ch} t=\frac{z}{\sqrt{2}},\ \Rightarrow\ \frac{e^t+e^{-t}}{2}=\frac{z}{\sqrt{2}},\ e^{2t}-2\frac{z}{\sqrt{2}}e^t+1=0,$$

$$t=\ln\left(z+\sqrt{z^2-2}\right)+\ln\frac{1}{\sqrt{2}}=\operatorname{arch}\frac{z}{\sqrt{2}}.$$

484 (2159). $\int\sqrt{3x^2-3x+1}\,dx=I.$

$$3x^2-3x+1=3\left(x^2-x+\frac{1}{4}\right)-\frac{3}{4}+1=3\left(x-\frac{1}{2}\right)^2+\frac{1}{4},\ x-\frac{1}{2}=t,$$

$$dx=dt,\ \sqrt{3x^2-3x+1}=\sqrt{3\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}=\sqrt{3t^2+\frac{1}{4}}.$$

$$I=\int\sqrt{3t^2+\frac{1}{4}}\,dt=\int\frac{3t^2+\frac{1}{4}}{\sqrt{3t^2+\frac{1}{4}}}\,dt=3\int\frac{t^2dt}{\sqrt{3t^2+\frac{1}{4}}}+\frac{1}{4}\int\frac{dt}{\sqrt{3t^2+\frac{1}{4}}}=$$

$$=\left[\begin{array}{ll} t=u, & dt=du, \\ dv=\dfrac{t\,dt}{\sqrt{3t^2+\frac{1}{4}}}, & v=\dfrac{1}{6}\displaystyle\int\frac{d\left(3t^2+\frac{1}{4}\right)}{\sqrt{3t^2+\frac{1}{4}}}=\frac{1}{3}\sqrt{3t^2+\frac{1}{4}}\end{array}\right]=$$

$$=3\left(\frac{t}{3}\sqrt{3t^2+\frac{1}{4}}-\frac{1}{3}\int\sqrt{3t^2+\frac{1}{4}}dt\right)+\frac{1}{4\sqrt{3}}\ln\left|\sqrt{3}t+\sqrt{3t^2+\frac{1}{4}}\right|.$$

Итак, $\int\sqrt{3t^2+\frac{1}{4}}\,dt=t\sqrt{3t^2+\frac{1}{4}}-\int\sqrt{3t^2+\frac{1}{4}}\,dt+$

$$+\frac{1}{4\sqrt{3}}\ln\left|\sqrt{3}t+\sqrt{3t^2+\frac{1}{4}}\right| \Rightarrow \int\sqrt{3t^2+\frac{1}{4}}\,dt=\frac{t}{2}\sqrt{3t^2+\frac{1}{4}}+$$

$$+\frac{1}{8\sqrt{3}}\ln\left|\sqrt{3}t+\sqrt{3t^2+\frac{1}{4}}\right|+C.$$

$$I=\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{3x^2-3x+1}+\frac{1}{8\sqrt{3}}\ln\left|\sqrt{3x^2-3x+1}+\frac{\sqrt{3}}{2}(2x-1)\right|+C.$$

485 (2160). $\int\sqrt{1-4x-x^2}\,dx=\int\sqrt{5-(x+2)^2}\,dx=\begin{bmatrix}x+2=z,\\ dx=dz\end{bmatrix}=$

$$=\int\sqrt{\left(\sqrt{5}\right)^2-z^2}\,dz=\begin{bmatrix}z=\sqrt{5}\sin t,\\ dz=\sqrt{5}\cos t\,dt\end{bmatrix}=5\int\cos^2 t\,dt=$$

$$=\frac{5}{2}\int(1+\cos 2t)\,dt=\frac{5}{2}t+\frac{5}{4}\sin 2t+C=\frac{5}{2}\arcsin\frac{z}{5}+$$

$$+\frac{5}{2}\sin\left(\arcsin\frac{z}{\sqrt{5}}\right)\cdot\cos\left(\arcsin\frac{z}{\sqrt{5}}\right)+C=\frac{5}{2}\arcsin\frac{x+2}{\sqrt{5}}+$$

$$+\frac{5}{2}\cdot\frac{x+2}{\sqrt{5}}\sqrt{1-\frac{(x+2)^2}{5}}+C=\frac{5}{2}\arcsin\frac{x+2}{\sqrt{5}}+\frac{x+2}{2}\sqrt{1-4x-x^2}+C=$$

$$=\frac{1}{2}\left((x+2)\sqrt{1-4x-x^2}+5\arcsin\frac{x+2}{\sqrt{5}}\right)+C.$$

486 (2161). $\int\frac{dx}{x-\sqrt{x^2-x+1}}=I.$

$$\sqrt{x^2-x+1}=x+t,\ x=\frac{-t^2+1}{2t+1},\ dx=\frac{-2t^2-2t-2}{(2t+1)^2}\,dt.$$

$$I=\int\frac{-2t^2-2t-2}{(2t+1)^2t}\,dt,$$

$$\frac{2t^2+2t+2}{(2t+1)^2 t}=\frac{A}{(2t+1)^2}+\frac{B}{(2t+1)}+\frac{C}{t},$$

$$2t^2+2t+2=At+B(2t+1)+C(2t+1)^2.$$

$$\begin{cases} t=0: & 2=C, \\ t=-\dfrac{1}{2}: & \dfrac{3}{2}=-\dfrac{A}{3}, \\ t=1: & 6=A+3B+9C \end{cases} \Rightarrow A=-3,\ B=-3,\ C=2.$$

$$I=-3\int\frac{dt}{(2t+1)^2}-3\int\frac{dt}{2t+1}+2\int\frac{dt}{t}=\frac{3}{2(2t+1)}-\frac{3}{2}\ln|2t+1|+$$

$$+2\ln|t|+C=\frac{3}{2\left(2\sqrt{x^2-x+1}-2x+1\right)}-\frac{3}{2}\ln\left|2\sqrt{x^2-x+1}-2x+1\right|+$$

$$+2\ln\left|\sqrt{x^2-x+1}-1\right|+C.$$

487 (2162). $\displaystyle\int\frac{dx}{x^2\left(x+\sqrt{1+x^2}\right)}=\left[x=\operatorname{tg} z,\ 1+x^2=\frac{1}{\cos^2 z},\ dx=\frac{dz}{\cos^2 z}\right]=$

$$=\int\frac{\cos^2 z\,dz}{\sin^2 z\left(\operatorname{tg} z+\dfrac{1}{\cos z}\right)}=\int\frac{\cos z\,dz}{\sin^2 z(\sin z+1)}=\int\frac{d(\sin z)}{\sin^2 z(\sin z+1)}=$$

$$=[\sin z=t]=\int\frac{dt}{t^2(t+1)}=I,$$

$$\frac{1}{t^2(t+1)}=\frac{A}{t^2}+\frac{B}{t}+\frac{C}{t+1}\ \Rightarrow\ A=C=1,\ B=-1.$$

$$I=-\frac{1}{t}+\ln\frac{t+1}{t}+C=-\frac{\sqrt{1+x^2}}{x}+\ln\frac{x\sqrt{1+x^2}}{x}+C.$$

488 (2163). $\displaystyle\int \frac{dx}{1+\sqrt{x^2+2x+2}} = \begin{bmatrix} x+1=z, \\ dx=dz \end{bmatrix} = \int \frac{dz}{1+\sqrt{z^2+1}} =$

$$= \begin{bmatrix} z=\operatorname{sh} t,\ dz=\operatorname{ch} t\,dt, \\ 1+z^2=1+\operatorname{sh}^2 t=\operatorname{ch}^2 t \end{bmatrix} = \int \frac{\operatorname{ch} t\,dt}{1+\operatorname{ch} t} = \int \frac{(1+\operatorname{ch} t)-1}{1+\operatorname{ch} t}\,dt =$$

$$= \int dt - \int \frac{dt}{1+\operatorname{ch} t} = I,$$

$$1+\operatorname{ch} t = \left(\operatorname{ch}^2\frac{t}{2}-\operatorname{sh}^2\frac{t}{2}\right) + \underbrace{\left(\operatorname{ch}^2\frac{t}{2}+\operatorname{sh}^2\frac{t}{2}\right)}_{\operatorname{ch} t} = 2\operatorname{ch}^2\frac{t}{2}.$$

$$I = t-\frac{1}{2}\int \frac{dt}{\operatorname{ch}^2\frac{t}{2}} = t-\operatorname{th}\frac{t}{2}+C,$$

$$\operatorname{sh} t = z,\ \frac{e^t-e^{-t}}{2}=z,\ e^{2t}-2ze^t-1=0,\ e^t=z+\sqrt{z^2+1},$$

$$t=\ln\left|z+\sqrt{z^2+1}\right| = \ln\left|x+1+\sqrt{x^2+2x+2}\right|,$$

$$\operatorname{th}\frac{t}{2} = \frac{\operatorname{ch} t-1}{\operatorname{sh} t} = \frac{\sqrt{1+\operatorname{sh}^2 t}-1}{\operatorname{sh} t} = \frac{\sqrt{x^2+2x+2}-1}{x+1}.$$

$$I = \ln\left|x+1+\sqrt{x^2+2x+2}\right| + \frac{1-\sqrt{x^2+2x+2}}{x+1}.$$

489 (2164). $\displaystyle\int \frac{x^2dx}{\sqrt{1-2x-x^2}} = I.$

$$1-2x-x^2 = 2-(x+1)^2 = 2-z^2,\ x+1=z,\ dx=dz.$$

$$I = \int \frac{(z-1)^2dz}{\sqrt{2-z^2}} = \int \frac{z^2-2z+1}{\sqrt{2-z^2}}dz = \int \frac{z^2dz}{\sqrt{2-z^2}} + \int \frac{d(2-z^2)}{\sqrt{2-z^2}} +$$

$$+\int\frac{dz}{\sqrt{2-z^2}}=\left[\begin{array}{ll} z=u, & dz=du, \\ v=-\frac{1}{2}\int\frac{(2-z^2)}{\sqrt{2-z^2}}=-\sqrt{2-z^2}, & dv=\frac{z\,dz}{\sqrt{2-z^2}}\end{array}\right]=$$

$$=-z\sqrt{2-z^2}+\int\sqrt{2z^2}\,dz+2\sqrt{2-z^2}+\arcsin\frac{z}{\sqrt{2}}+C=$$

$$=\left[\begin{array}{l} z=\sqrt{2}\sin t, \\ dz=\sqrt{2}\cos t\,dt\end{array}\right]=-z\sqrt{2-z^2}+2\int\cos^2 t\,dt+2\sqrt{1-z^2}+$$

$$+\arcsin\frac{z}{\sqrt{2}}+C=-z\sqrt{1-z^2}+\int(1+\cos 2t)\,dt+2\sqrt{1-z^2}+$$

$$+\arcsin\frac{z}{\sqrt{2}}+C=-2z\sqrt{2-z^2}+t+\frac{1}{2}\sin 2t+2\sqrt{2-z^2}+$$

$$+\arcsin\frac{z}{\sqrt{2}}+C=\sqrt{2-z^2}\,(2-z)+2\arcsin\frac{z}{\sqrt{2}}+\frac{z}{\sqrt{2}}\sqrt{1-\frac{z^2}{2}}+C=$$

$$=\sqrt{2-z^2}\,(2-z)+2\arcsin\frac{z}{\sqrt{2}}+\frac{z}{\sqrt{2}}\sqrt{2-z^2}+C=$$

$$=\frac{1}{2}\sqrt{1-2x-x^2}\,(3-x)+2\arcsin\frac{x+1}{\sqrt{2}}+C.$$

490 (2165). $$\int\frac{(2x^2-3x)\,dx}{\sqrt{x^2-2x+5}}\,dx=\left[\begin{array}{c} 2x^3-3x=2(x^2-2x+5)+ \\ +\frac{1}{2}(2x-2)-9\end{array}\right]=$$

$$=2\int\frac{x^2-2x+5}{\sqrt{x^2-2x+5}}\,dx+\frac{1}{2}\int\frac{2x-2}{\sqrt{x^2-2x+5}}\,dx-9\int\frac{dx}{\sqrt{x^2-2x+5}}=$$

$$=2\int\sqrt{x^2-2x+5}\,dx+\frac{1}{2}\int\frac{d\,(x^2+2x+5)}{\sqrt{x^2-2x+5}}-9\int\frac{dx}{\sqrt{x^2-2x+5}}=I.$$

1) $2\int\sqrt{x^2-2x+5}\,dx=2\int\sqrt{(x-1)^2+2^2}\,dx=\begin{bmatrix}x-1=z,\\ dx=dz\end{bmatrix}=$

$=2\int\sqrt{z^2+2^2}\,dz=z\sqrt{z^2+4}+\ln\left|z+\sqrt{z^2+4}\right|=(x+1)\sqrt{x^2-2x+5}+$

$+\ln\left|x-1+\sqrt{x^2-2x+5}\right|$

(первый интеграл берется, например, подстановкой z = 2sh t).

2) $\frac{1}{2}\int\frac{d\,(x^2-2x+5)}{\sqrt{x^2-2x+5}}=\sqrt{x^2-2x+5}.$

3) $-9\int\frac{dx}{\sqrt{x^2-2x+5}}=-9\int\frac{d\,(x-1)}{\sqrt{(x-1)+2^2}}=-9\ln\left|x-1+\sqrt{x^2-2x+5}\right|.$

$I=(x-1)(x^2-2x+5)+4\ln\left|x-1+\sqrt{x^2-2x+5}\right|+\sqrt{x^2-2x+5}-$

$-9\ln\left|x-1+\sqrt{x^2-2x+5}\right|+C=$

$=-x\sqrt{x^2-2x+5}-5\ln\left|x-1+\sqrt{x^2-2x+5}\right|+C.$

491 (2166). $\int\frac{3x^2-5x}{\sqrt{3-2x-x^2}}\,dx=\begin{bmatrix}3x^2-5x=-3(3-2x-x^2)+\\ +\frac{11}{2}(-2-2x)+20\end{bmatrix}=$

$=-3\int\frac{3-2x-x^2}{\sqrt{3-2x-x^2}}+\frac{11}{2}\int\frac{(-2-2x)\,dx}{\sqrt{3-2x-x^2}}+20\int\frac{dx}{\sqrt{3-2x-x^2}}=$

$=-3\int\sqrt{2^2-(x+1)^2}\,dx+\frac{11}{2}\int\frac{d\,(3-2x-x^2)}{\sqrt{3-2x-x^2}}+20\int\frac{dx}{\sqrt{2^2-(x+1)^2}}=$

$$=-\frac{3x}{2}\sqrt{3-2x-x^2}-6\arcsin\frac{x+1}{2}+\frac{11}{2}\cdot 2\sqrt{3-2x-x^2}+20\arcsin\frac{x+1}{2}+$$

$$+C=C-\frac{1}{2}(3x-22)\sqrt{3-2x-x^2}+14\arcsin\frac{x+1}{2}.$$

492 (2167). $\displaystyle\int\frac{3x^3dx}{\sqrt{x^2+4x+5}}=I.$

Применим формулу $\displaystyle\int\frac{P_n(x)}{Y}dx=Q_{n-1}(x)Y+\lambda\int\frac{dx}{Y}$; где

$Y=\sqrt{ax^2+bx+C}$; $P_n(x)$ – многочлен степени n; $Q_{n-1}(x)$ – многочлен степени $(n-1)$; λ – некоторое число. Тогда

$$\int\frac{3x^3dx}{\sqrt{x^2+4x+5}}=(Ax^2+Bx+C)\sqrt{x^2+4x+5}+\lambda\int\frac{dx}{\sqrt{x^2+4x+5}}.$$

Дифференцируя это тождество и приведя к общему знаменателю, получим

$$3x^3=(2Ax+B)(x^2+4x+5)+(Ax^2+Bx+C)(x+2)+\lambda.$$

$$\left.\begin{array}{l|l}x^3 & 3=3A,\\ x^2 & 0=10A+2B,\\ x^1 & 0=10A+6B+C,\\ x^0 & 0=5B+2C+\lambda\end{array}\right\}\Rightarrow A=1,\ B=-5,\ C=20,\ \lambda=-15.$$

$$I=(x^2+5x+20)\sqrt{x^2+4x+5}-15\int\frac{dx}{\sqrt{(x+2)^2+1}}=$$

$$=(x^2+5x+20)\sqrt{x^2+4x+5}-15\ln\left|x+2+\sqrt{x^2+4x+5}\right|+C.$$

493 (2168). $\displaystyle\int \frac{x^3 - x + 1}{\sqrt{x^2 + 2x + 2}}\,dx = (Ax^2 + Bx + C)\sqrt{x^2 + 2x + 2} +$

$$+\lambda\int \frac{dx}{\sqrt{x^2 + 2x + 2}} = I.$$

Дифференцируя это тождество и приводя к общему знаменателю, получим

$$x^3 - x + 1 = (Ax + B)(x^2 + 2x + 2) + (Ax^2 + Bx + C)(x + 1) + \lambda\ .$$

$$\left.\begin{array}{l|l} x^3 & 1 = 3A, \\ x^2 & 0 = 5A + 2B, \\ x^1 & -1 = 4A + 3B + C, \\ x^0 & 1 = 2B + C + \lambda \end{array}\right\} \Rightarrow \quad \begin{array}{l} A = \dfrac{1}{3},\ \ C = \dfrac{1}{6}, \\ B = -\dfrac{5}{6},\ \lambda = \dfrac{5}{2}. \end{array}$$

$$I = \left(\frac{1}{3}x^2 - \frac{5}{6}x + \frac{1}{6}\right)\sqrt{x^2 + 2x + 2} + \frac{5}{2}\int \frac{dx}{(x+1)^2 + 1} =$$

$$= \underline{\underline{\left(\frac{1}{3}x^2 - \frac{5}{6}x + \frac{1}{6}\right)\sqrt{x^2 + 2x + 2} + \frac{5}{2}\ln\left|x + 1 + \sqrt{x^2 + 2x + 2}\right| + C}}.$$

494 (2169). $\displaystyle\int \frac{3x^3 - 8x + 5}{\sqrt{x^2 - 4x - 7}}\,dx = (Ax^2 + Bx + C)\sqrt{x^2 - 4x - 7} +$

$$+\lambda\int \frac{dx}{\sqrt{x^2 - 4x - 7}} = I.$$

$$3x^3 - 8x + 5 = (2Ax + B)(x^2 - 4x - 7) + (Ax^2 + Bx + C)(x - 2) + \lambda.$$

$$\left.\begin{array}{l|l} x^3 & 3 = 3A, \\ x^2 & 0 = -10A + 2B, \\ x^1 & -8 = -14A - 6B + C, \\ x^0 & 5 = -7B - 2C + \lambda \end{array}\right\} \Rightarrow\ A = 1,\ B = 5,\ C = 36,\ \lambda = 112.$$

$$I=(x^2+5x+36)\sqrt{x^2-4x-7}+112\int\frac{dx}{\sqrt{(x-2)^2-11}}=$$

$$=(x^2+5x+36)\sqrt{x^2-4x-7}+112\ln\left|x-2+\sqrt{x^2-4x-7}\right|+C.$$

495 (2170). $\displaystyle\int\frac{x^4dx}{\sqrt{x^2+4x+5}}=(Ax^3+Bx^2+Cx+D)\sqrt{x^2+4x+5}+$

$$+\lambda\int\frac{dx}{\sqrt{x^2+4x+5}}.$$

$$x^4=(3Ax^2+2Bx+C)(x^2+4x+5)+(Ax^3+Bx^2+Cx+D)(x+2)+\lambda.$$

$$\left.\begin{array}{l|l} x^4 & 1=4A, \\ x^3 & 0=14A+3B, \\ x^2 & 0=15A+10B+2C, \\ x^1 & 0=10B+6C+D, \\ x^0 & 0=5C+2D+\lambda \end{array}\right\}\Rightarrow \quad \begin{array}{l} A=\dfrac{1}{4},\ \ B=-\dfrac{7}{6},\ \ C=\dfrac{95}{24}, \\ D=-\dfrac{145}{12},\ \ \lambda=\dfrac{35}{8}. \end{array}$$

$$I=\left(\frac{1}{4}x^3-\frac{7}{6}x^2+\frac{95}{24}x-\frac{145}{12}\right)\sqrt{x^2+4x+5}+$$

$$+\frac{35}{8}\ln\left|x+2+\sqrt{x^2+4x+5}\right|+C.$$

496 (2171). $\displaystyle\int\frac{dx}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}=\int\frac{d(x+1)}{(x+1)^3\sqrt{(x+1)^2-4}}=$

$$=[x+1=t,\ dx=dt]=\int\frac{dt}{t^3\sqrt{t^2-4}}=\left[t=\frac{2}{\cos z},\ dt=\frac{2\sin z}{\cos^2 z}dz\right]=$$

$$=\int\frac{2\sin z\cos^3 z\,dz}{8\cos^2 z\dfrac{2\sin z}{\cos z}}=\frac{1}{8}\int\cos^2 z dz=\frac{1}{8}\int\frac{1+\cos^2 z}{2}dz=$$

$$=\frac{1}{16}\left(z+\frac{1}{2}\sin 2z\right)+C=\frac{1}{16}\left(\arccos\frac{2}{t}+\frac{2}{t}\sqrt{1-\frac{4}{t^2}}\right)+C=$$

$$=\underline{\underline{\frac{1}{16}\arccos\frac{2}{x+1}+\frac{\sqrt{x^2+2x-3}}{8(x+1)^2}+C}}.$$

497 (2172). $\displaystyle\int\frac{\sqrt{1+x^2}}{2+x^2}\,dx=I.$

$$\frac{\sqrt{x^2+1}}{x^2+2}=\frac{x^2+1}{(x^2+2)\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}-\frac{1}{(x^2+2)\sqrt{x^2+1}},$$

$$\int\frac{dx}{\sqrt{x^2+1}}=\ln\left|x+\sqrt{x^2+1}\right|,$$

$$\int\frac{dx}{(x^2+2)\sqrt{x^2+1}}=\left[t=\frac{x}{\sqrt{x^2+1}},\ \ x^2=\frac{t^2}{1-t^2},\ x\,dx=\frac{t\,dt}{(1-t^2)^2}\right]=$$

$$=\int\frac{t^2dt}{(1-t^2)^2x^2\,\dfrac{2-t^2}{1-t^2}}=\int\frac{t^2dt}{(1-t^2)(2-t^2)\dfrac{t^2}{1-t^2}}=\int\frac{dt}{2-t^2}=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}-t}{\sqrt{2}+t}\right|=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2x^2+2}-x}{\sqrt{2x^2+2}+x}\right|.$$

$$I=\underline{\underline{\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2x^2+2}-x}{\sqrt{2x^2+2}+x}\right|+\ln\left|x+\sqrt{x^2+1}\right|+C}}.$$

498 (2173). $\displaystyle\int\frac{(x-1)\,dx}{x^2\sqrt{2x^2-2x+1}}=\left[\begin{array}{c}x=\dfrac{1}{1+t},\ dx=-\dfrac{dt}{(1+t)^2},\ t=\dfrac{1-x}{x},\\ \sqrt{2x^2-2x+1}=\dfrac{\sqrt{1+t^2}}{1+t}\end{array}\right]=$

$$=-\int\frac{\left(\frac{1}{1+t}-1\right)(1+t)^2(1+t)\,dt}{(1+t)^2\sqrt{1+t^2}}=\int\frac{t\,dt}{\sqrt{1+t^2}}=\frac{1}{2}\int\frac{d(1+t^2)}{\sqrt{1+t^2}}=$$

$$=\sqrt{1+t^2}+C=\underline{\underline{\frac{\sqrt{2x^2-2x+1}}{x}}}+C.$$

499 (2174). $$\int\frac{(2x+3)\,dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}=\int\frac{(2x+2)\,dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}+$$

$$+\int\frac{dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}=I_1+I_2=I.$$

$$x^2+2x+3=t,\ (2x+x)\,dx=dt.$$

$$I_1=\int\frac{dt}{t\sqrt{t+1}}=\begin{bmatrix}t+1=u^2,\\ dt=2u\,du\end{bmatrix}=2\int\frac{u\,du}{(u^2-1)u}=\ln\left|\frac{u-1}{u+1}\right|=$$

$$=\ln\left|\frac{\sqrt{t+1}-1}{\sqrt{t+1}+1}\right|=\ln\left|\frac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}\right|,$$

$$I_2=\int\frac{dx}{((x+1)^2+2)\sqrt{(x+1)^2+3}}=\begin{bmatrix}x+1=t,\\ dx=dt\end{bmatrix}=$$

$$=\int\frac{dt}{(t^2+2)\sqrt{t^2+3}}=\begin{bmatrix}t=\sqrt{3}\,\mathrm{tg}\,z,\\ dt=\dfrac{\sqrt{3}\,dz}{\cos^2 z}\end{bmatrix}=$$

$$=\int\frac{\sqrt{3}\,dz}{\cos^2 z(3\,\mathrm{tg}^2 z+2)\sqrt{3\,\mathrm{tg}^2 z+3}}=\int\frac{\cos z\,dz}{(3\sin^2 z+2\cos^2 z)}=$$

$$=\int\frac{d(\sin z)}{\sin^2 z+2}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sin z}{\sqrt{2}}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\operatorname{tg} z}{\frac{\sqrt{1+\operatorname{tg}^2 z}}{\sqrt{2}}}=$$

$$=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+1}{\sqrt{2(x^2+2x+4)}}.$$

Так как $\operatorname{arctg} a+\operatorname{arctg}\frac{1}{a}=\frac{\pi}{2}$, то $\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+1}{\sqrt{2(x^2+2x+4)}}=$

$$=\frac{\pi}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2(x^2+2x+4)}}{x+1}.$$

$\underline{\underline{I=\ln\left|\frac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}\right|-\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2(x^2+2x+4)}}{x+1}+C}}$ (константа $\frac{\pi}{2\sqrt{2}}$ включена в C).

3.6. Разные функции

500 (2175). $\int\frac{x^3dx}{(x-1)^{12}}=I.$

По формуле Тейлора,

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-1)+\frac{f''(a)}{2!}(x-1)^2+\frac{f'''(a)}{3!}(x-1)^3+\ldots$$

$a=1$, тогда $x^3=1+3(x-1)+3(x-1)^2+(x-1)^3$.

$$I=\int\frac{1+3(x-1)+3(x-1)^2+(x-1)^3}{(x-1)^{12}}\,dx=\int\frac{dx}{(x-1)^{12}}+3\int\frac{dx}{(x-1)^{11}}+$$

$$+3\int\frac{dx}{(x-1)^{10}}+\int\frac{dx}{(x-1)^9}=$$

$$\underline{\underline{=C-\frac{1}{11(x-1)^{11}}-\frac{3}{10(x-1)^{10}}-\frac{1}{3(x-1)^9}-\frac{1}{8(x-1)^8}}}.$$

501 (2176). $\displaystyle\int\frac{x\,dx}{x-\sqrt{x^2-1}}=\int\frac{x\left(x+\sqrt{x^2-1}\right)dx}{x^2-\left(\sqrt{x^2-1}\right)^2}=\frac{x^3}{3}+$

$$+\frac{1}{2}\int(x^2-1)^{1/2}d(x^2-1)=\frac{x^3}{3}+\frac{(x^2-1)^{3/2}}{3}+C=$$

$$=\underline{\underline{\frac{1}{3}\left(x^3+\sqrt{(x^2-1)^3}\right)+C}}.$$

502 (2177). $\displaystyle\int x\sqrt[3]{a+x}\,dx=\left[a+x=t^3,\ dx=3t^2dt,\ t=\sqrt[3]{a+x}\right]=$

$$=3\int(t^3-a)\,t^3dt=3\left(\frac{t^7}{7}-\frac{at^4}{4}\right)+C=\underline{\underline{\frac{3(4x-3a)\sqrt[3]{(a+x)^4}}{28}+C}}.$$

503 (2178). $\displaystyle\int\frac{dx}{ae^{mx}+be^{-mx}}=\int\frac{dx}{e^{-mx}(b+ae^{2mx})}=\int\frac{e^{mx}dx}{b+ae^{2mx}}=$

$$=\frac{1}{m\sqrt{a}}\int\frac{d\left(\sqrt{a}e^{mx}\right)}{\left(\sqrt{b}\right)^2+\left(\sqrt{a}e^{mx}\right)^2}=\underline{\underline{\frac{1}{m\sqrt{ab}}\operatorname{arctg}\left(e^{mx}\sqrt{\frac{a}{b}}\right)+C}}.$$

504 (2179). $\displaystyle\int\frac{x\sqrt{1+x}}{\sqrt{1-x}}\,dx=\left[\begin{array}{c}x=\cos t,\ dx=-\sin t\,dt,\\ \sqrt{\dfrac{1+x}{1-x}}=\sqrt{\dfrac{1+\cos t}{1-\cos t}}=\sqrt{\dfrac{\cos^2\frac{t}{2}}{\sin^2\frac{t}{2}}}=\dfrac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\end{array}\right]=$

$$=-\int\cos t\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin t\,dt=-2\int\cos t\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin\frac{t}{2}\cos\frac{t}{2}\,dt=$$

$$=-\int\cos t(1+\cos t)\,dt=-\sin t-\frac{1}{2}\int(1+\cos 2t)\,dt=-\sin t-\frac{1}{2}t-$$

$$-\frac{1}{4}\sin 2t+C_0=-\sin t-\frac{1}{2}t-\frac{1}{2}\sin t\cdot\cos t+C_0=-\frac{1}{2}t-$$

$$-\frac{2+\cos t}{2}\sin t+C_0=-\frac{1}{2}\arccos x-\frac{2+x}{2}\sqrt{1-x^2}+C_0.$$

Учитывая, что $\arcsin x+\arccos x=\frac{\pi}{2}\Rightarrow -\frac{1}{2}\arccos x=$

$=\frac{1}{2}\arcsin x-\frac{\pi}{4}$. Тогда

$$\underline{\underline{I=\frac{1}{2}\arcsin x-\frac{2+x}{2}\sqrt{1-x^2}+C}}\quad\left(C=C_0-\frac{\pi}{4}\right).$$

505 (2180). $\int\frac{x^4dx}{(x^2-1)(x+2)}=I.$

$$\frac{x^4}{(x-1)(x+1)(x+2)}=x-2+\frac{5x^2-4}{(x-1)(x+1)(x+2)},$$

$$\frac{5x^2-4}{(x-1)(x+1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+2},$$

$$5x^2-4=A(x+1)(x+2)+B(x-1)(x+2)+C(x^2-1)\ \Rightarrow$$

$$A=\frac{1}{6},\ B=-\frac{1}{2},\ C=\frac{16}{3}.$$

$$I=\frac{x^2}{2}-2x+\frac{1}{6}\ln|x-1|-\frac{1}{2}\ln|x+1|+\frac{16}{3}\ln|x+2|+C=$$

$$\underline{\underline{=\frac{x^2}{2}-2x+\frac{1}{6}\ln\frac{|x-1||x+2|^{32}}{|x+1|^3}+C.}}$$

506 (2181). $\int\frac{dx}{1-x^4}=I.$

$$\frac{1}{(1-x)(1+x)(1+x^2)}=\frac{A}{1-x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2},$$

$$1=A(1+x)(1+x^2)+B(1-x)(1+x^2)+(Cx+D)(1-x^2).$$

$$\left.\begin{array}{ll} x=0: & 1=A+B+D, \\ x=1: & 1=4A, \\ x=-1: & 1=4B, \\ x=2: & 1=15A-5B-6C-3D \end{array}\right\} \Rightarrow\ A=B=\frac{1}{4},\ C=0,\ D=\frac{1}{2}.$$

$$I=-\frac{1}{4}\ln|1-x|+\frac{1}{4}\ln|1+x|+\frac{1}{2}\operatorname{arctg}x+C=\underline{\underline{\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{2}\operatorname{arctg}x+C}}.$$

507 (2182). $\displaystyle\int\frac{dx}{(x^4-)^2}=I.$

Применяя метод Остроградского, интеграл представим в виде

$$\int\frac{dx}{(x^4-1)^2}=\frac{Ax^3+Bx^2+Cx+D}{x^4-1}+\int\frac{Ex^3+Fx^2+Gx+H}{x^4-1}\,dx.$$

Дифференцируя и приведя к общему знаменателю, получаем тождество

$$1=(3Ax^2+2Bx+C)(x^4-1)-4x^3(Ax^3+Bx^2+Cx+D)+$$
$$+(Ex^3+Fx^2+Gx+H)(x^4-1).$$

$$\left.\begin{array}{l|l} x^7 & 0=E, \\ x^6 & 0=-A+F, \\ x^5 & 0=-2B+G, \\ x^4 & 0=-3C+H, \\ x^3 & 0=-4D+E, \\ x^2 & 0=3A-F, \\ x^1 & 0=-2B-G, \\ x^0 & 1=-C-H \end{array}\right\} \Rightarrow\ \begin{array}{l} A=B=D=F=E=G=0, \\ C=-\dfrac{1}{4},\ H=-\dfrac{3}{4}. \end{array}$$

Воспользовавшись результатами задачи 506 (2181), получаем

$$I=-\frac{x}{4(x^4-1)}-\frac{3}{4}\int\frac{dx}{x^4-1}=-\frac{x}{4(x^4-1)}+\frac{3}{16}\ln\left|\frac{x-1}{x+1}\right|+\frac{3}{8}\operatorname{arctg}x+C=$$

$$=-\frac{x}{4(x^4-1)}-\frac{3}{16}\ln\left|\frac{x-1}{x+1}\right|+\frac{3}{8}\operatorname{arctg}x+C.$$

508 (2183). $\displaystyle\int\frac{\ln(x+1)\,dx}{\sqrt{x+1}}=\begin{bmatrix}x+1=t^2,\\ dx=2t\,dt\end{bmatrix}=2\int\ln t^2dt=$

$$=\begin{bmatrix}u=\ln t^2, & du=\dfrac{2dt}{t},\\ dv=dt, & v=t\end{bmatrix}=2\left(t\ln t^2-2\int dt\right)=2t(\ln t^2-2)+C=$$

$$=2\sqrt{x+1}\,(\ln(x+1)-2)+C.$$

509 (2084). $\displaystyle\int(x^2+3x+5)\cos 2x\,dx=I.$

Интеграл берется по частям. Воспользовавшись формулами

$$\int x\cos px\,dx=\frac{1}{p^2}\cos px+\frac{x}{p}\sin px,$$

$$\int x^2\cos px\,dx=\frac{2x\cos px}{p^2}+\frac{p^2x^2-2}{p^3}\sin px \text{ и положив}$$

$p=2$, имеем:

$$I=\frac{x\cos 2x}{2}+\frac{2x^2-1}{4}\sin 2x+\frac{3}{4}\cos 2x+\frac{3x}{2}\sin 2x+\frac{5}{2}\sin 2x+C=$$

$$=\left(\frac{1}{2}x+\frac{3}{4}\right)\cos 2x+\left(\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{4}\right)\sin 2x+C.$$

510 (2085). $\displaystyle\int x^2\operatorname{sh}x\,dx=\begin{bmatrix}u=x^2, & du=2x\,dx,\\ dv=\operatorname{sh}x\,dx, & v=\operatorname{ch}x\end{bmatrix}=x^2\operatorname{ch}x-$

$$-2\int x\operatorname{ch}x\,dx=\begin{bmatrix} u=x, & du=dx,\\ dv=\operatorname{ch}x\,dx, & v=\operatorname{sh}x\end{bmatrix}=x^2\operatorname{ch}x-2x\operatorname{sh}x+$$

$$+2\int\operatorname{sh}x\,dx=\underline{\underline{x^2\operatorname{ch}x-2x\operatorname{sh}x+2\operatorname{ch}x+C}}.$$

511 (2186). $\displaystyle\int\operatorname{arctg}\left(1+\sqrt{x}\right)dx=\left[1+\sqrt{x}=t,\ x=(t-1)^2,\ dx=2(t-1)\,dt\right]=$

$$=2\int(t-1)\operatorname{arctg}t\,dt=\begin{bmatrix} u=\operatorname{arctg}t, & du=\dfrac{dt}{1+t^2},\\ dv=(t-1)\,dt, & v=\dfrac{(t-1)^2}{2}\end{bmatrix}=$$

$$=2\left(\frac{(t-1)^2}{2}\operatorname{arctg}t-\frac{1}{2}\int\frac{(t-1)^2}{1+t^2}\,dt\right)=2\left(\frac{(t-1)^2}{2}\operatorname{arctg}t-\right.$$

$$\left.-\frac{1}{2}\int\frac{(1+t^2)-2t}{1+t^2}\,dt\right)=(t-1)^2\operatorname{arctg}t-\int dt+\int\frac{d\,(1+t^2)}{1+t^2}=$$

$$=\underline{\underline{x\operatorname{arctg}\left(1+\sqrt{x}\right)-\sqrt{x}+\ln\left|x+2\sqrt{x}+2\right|+C}}.$$

512 (2187). $\displaystyle\int\frac{\arcsin x\,dx}{x^2}=\begin{bmatrix} u=\arcsin x, & du=\dfrac{dx}{\sqrt{1-x^2}},\\ dv=\dfrac{dx}{x^2}, & v=-\dfrac{1}{x}\end{bmatrix}=-\frac{\arcsin x}{x}+$

$$+\int\frac{dx}{x\sqrt{1-x^2}}=\begin{bmatrix} x=\sin t,\\ dx=\cos t\,dt\end{bmatrix}=-\frac{\arcsin x}{x}+\int\frac{dt}{\sin t}=-\frac{\arcsin x}{x}+$$

$$+\ln\left|\operatorname{tg}\frac{x}{2}\right|+C=\ln\left|\frac{1-\cos x}{\sin x}\right|-\frac{\arcsin x}{x}+C=$$

$$=\underline{\underline{\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|-\frac{\arcsin x}{x}+C}}.$$

513 (2188). $\int e^{\sqrt[3]{x}}dx = \left[\sqrt[3]{x} = t,\ x = t^3,\ dx = 3t^2dt\right] = 3\int t^2e^t dx =$

$$= \begin{bmatrix} u = t, & du = 2t\,dt, \\ dv = e^t dt, & v = e^t \end{bmatrix} = 3\left(t^2e^t - 2\int te^t dt\right) =$$

$$= \begin{bmatrix} u = t, & du = dt, \\ dv = e^t dt, & v = e^t \end{bmatrix} = 3\left(t^2e^t - 2\left(te^t - \int e^t dt\right)\right) =$$

$$= 3(t^2e^t - 2te^t + 2e^t) + C = \underline{\underline{3e^{\sqrt[3]{x}}\left(\sqrt[3]{x^2} - 2\sqrt[3]{x} + 2\right)}} + C.$$

514 (2189). $\int xe^{\sqrt[3]{x}}dx = \left[\sqrt[3]{x} = t,\ \ x = t^3,\ dx = 3t^2dt\right] = 3\int t^5e^t dt = I.$

Беря по частям, аналогично задаче 513 (2188), получим

$$\underline{\underline{I = 3e^{\sqrt[3]{x}}\left(\sqrt[3]{x^5} - 5\sqrt[3]{x^4} + 20x - 60\sqrt[3]{x^2} + 120\sqrt[3]{x} - 120\right) + C}}.$$

515 (2190). $\int (x^3 - 2x^2 + 5)e^{3x}dx = I.$

Интегрирование по частям дает

$$\int x^3e^{ax}dx = \frac{a^3x^3 - 3a^2x^2 + 6ax - 6}{a^4}\,e^{ax},$$

$$\int x^2e^{ax}dx = \frac{a^2x^2 - 2ax + 2}{a^3}\,e^{ax},\ a = 3.$$

$$I = \frac{27x^3 - 27x^2 + 18x - 6}{81}e^{3x} - \frac{18x^2 - 12x + 4}{27}e^{3x} + \frac{5}{3}e^{3x} + C =$$

$$= \underline{\underline{e^{3x}\left(\frac{1}{3}x^3 - x^2 + \frac{2}{3}x + \frac{13}{9}\right) + C}}.$$

516 (2191). $\int \sin\sqrt{x}\,dx = \left[\sqrt{x} = t,\ x = t^2,\ dx = 2t\,dt\right] = 2\int t\sin t\,dt =$

$$=\begin{bmatrix} u=t, & du=dt, \\ dv=\sin t\,dt, & v=-\cos t \end{bmatrix}=2\left(-t\cos t+\int\cos t\,dt\right)=$$

$$=\underline{\underline{2\left(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x}\right)+C}}.$$

517 (2192). $\displaystyle\int\frac{dx}{x^3(x-1)^{1/2}}=\left[x-1=t^2,\ dx=2t\,dt,\ x^3=(t^2+1)^3\right]=$

$$=2\int\frac{dt}{(t^2+1)^3}=I.$$

Применив метод Остроградского, представим интеграл в виде

$$\int\frac{dt}{(t^2+1)^3}=\frac{At^3+Bt^2+Ct+D}{(t^2+1)^2}+\int\frac{Et+F}{t^2+1}dt.$$

Дифференцируя и приводя к общему знаменателю, получаем тождество

$$1=(3At^2+2Bt+C)(t^2+1)-4t(At^3+Bt^2+Ct+D)+$$
$$+(t^4+2t^2+1)(Et+F).$$

$$\left.\begin{array}{l|l} t^5 & E=0, \\ t^4 & -A+F=0, \\ t^3 & -2B+2E=0, \\ t^2 & 3A-3C+2F=0, \\ t^1 & 2B-4D+E=0, \\ t^0 & C+F=1 \end{array}\right\}\Rightarrow \begin{array}{l} B=E=D=0, \\ A=F=\dfrac{3}{8}, \\ C=\dfrac{5}{8}. \end{array}$$

$$I=\frac{\frac{3}{4}t^3+\frac{5}{4}t}{(t^2+1)^2}+\frac{3}{4}\int\frac{dt}{t^2+1}=\frac{\sqrt{x-1}(3(x-1)+5)}{4x^2}+\frac{3}{4}\operatorname{arctg}\sqrt{x-1}+C=$$

$$=\underline{\underline{\frac{\sqrt{x-1}(3x+2)}{4x^2}+\frac{3}{4}\operatorname{arctg}\sqrt{x-1}+C}}.$$

518 (2193). $$\int \frac{dx}{x-\sqrt{x^2-1}} = \int \frac{\left(x+\sqrt{x^2-1}\right)dx}{\left(x-\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)} = \int x\,dx +$$

$$+\int \sqrt{x^2-1}\,dx = \frac{x^2}{2} + I_1 = I.$$

$$I_1 = \int \sqrt{x^2-1}\;dx = \int \frac{\left(\sqrt{x^2-1}\right)^2}{\sqrt{x^2-1}}\,dx = \int \frac{x^2\,dx}{\sqrt{x^2-1}} - \int \frac{dx}{\sqrt{x^2-1}} =$$

$$= \left[\begin{array}{ll} u = x, & du = dx, \\ dv = \dfrac{x\,dx}{\sqrt{x^2-1}}, & v = \dfrac{1}{2}\displaystyle\int \frac{d\,(x^2-1)}{\sqrt{x^2-1}} = \sqrt{x^2-1} \end{array}\right] = x\sqrt{x^2-1} -$$

$$-\int \sqrt{x^2-1}\,dx - \int \frac{dx}{\sqrt{x^2-1}} \;\Rightarrow$$

$$2\int \sqrt{x^2-1}\;dx = x\sqrt{x^2-1} - \int \frac{dx}{\sqrt{x^2-1}},$$

$$\int \sqrt{x^2-1}\,dx = \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|.$$

$$\underline{\underline{I = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C.}}$$

519 (2194). $$\int \frac{\sqrt{(1+x^2)^5}}{x^6}\,dx = \left[x = \operatorname{tg} x,\;\; dx = \frac{dt}{\cos^2 t},\; 1+x^2 = \frac{1}{\cos^2 t}\right] =$$

$$= \int \frac{\cos^6 t\,dt}{\cos^2 t\cdot\sin^6 t\cdot\cos^5 t} = \int \frac{dt}{\cos t\cdot\sin^6 t} = \int \frac{(\sin^2 t+\cos^2 t)^2}{\cos t\cdot\sin^6 t}\,dt =$$

$$= \int \frac{dt}{\sin^2 t\cdot\cos t} + 2\int \frac{\cos t\,dt}{\sin^4 t} + \int \frac{\cos^3 t}{\sin^6 t}\,dt = I_1 + I_2 + I_3 = I.$$

$$I_1=\int\frac{\cos^2 t+\sin^2 t}{\sin^2 t\cdot\cos t}\,dt=\int\frac{\cos t}{\sin^2 t}\,dt+\int\frac{dt}{\sin\left(\frac{\pi}{2}+t\right)}=$$

$$=\int\frac{d(\sin t)}{\sin^2 t}+\int\frac{d\left(\frac{\pi}{2}+t\right)}{\sin\left(\frac{\pi}{2}+t\right)}=-\frac{1}{\sin t}+\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{t}{2}\right)\right|=$$

$$=-\frac{1}{\sin t}+\ln\left|\frac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}\frac{t}{2}}{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg}\frac{t}{2}}\right|=-\frac{1}{\sin t}+\ln\left|\frac{1+\operatorname{tg}\frac{t}{2}}{1-\operatorname{tg}\frac{t}{2}}\right|=$$

$$=-\frac{\sqrt{1+\operatorname{tg}^2 t}}{\operatorname{tg} t}+\ln\left|\frac{\cos\frac{t}{2}+\sin\frac{t}{2}}{\cos\frac{t}{2}-\sin\frac{t}{2}}\right|=-\frac{\sqrt{1+\operatorname{tg}^2 t}}{\operatorname{tg} t}+\ln\left|\frac{1+\sin t}{\cos t}\right|=$$

$$=-\frac{\sqrt{1+\operatorname{tg}^2 t}}{\operatorname{tg} t}+\ln\left|\frac{1+\frac{\operatorname{tg} t}{\sqrt{1+\operatorname{tg}^2 t}}}{\frac{1}{\sqrt{1+\operatorname{tg}^2 t}}}\right|=-\frac{\sqrt{1+x^2}}{x}+\ln\left|x+\sqrt{1+x^2}\right|.$$

$$I_2=2\int\frac{d(\sin t)}{\sin^4 t}=-\frac{2}{3\sin^3 t}=-\frac{2\sqrt{(1+\operatorname{tg}^2 t)^3}}{3\operatorname{tg}^3 t}=-\frac{2\sqrt{(1+x^2)^3}}{3x^3}.$$

$$I_3=\int\frac{\cos^3 t\,dt}{\sin^6 t}=\int\frac{(1-\sin^2 t)\,d(\sin t)}{\sin^6 t}=-\frac{1}{5\sin^5 t}-\frac{1}{3\sin^3 t}=$$

$$=-\frac{\sqrt{(1+x^2)^5}}{5x^5}+\frac{\sqrt{(1+x^2)^3}}{3x^3}.$$

$$\underline{\underline{I=\ln\left|x+\sqrt{1+x^2}\right|-\frac{\sqrt{1+x^2}}{x}-\frac{\sqrt{(1+x^2)^3}}{3x^3}-\frac{\sqrt{(1+x^2)^5}}{5x^5}+C.}}$$

520 (2195). $\int \frac{x^4 dx}{\sqrt{x^2+1}} = \left[x = \operatorname{tg} t,\ dx = \frac{dt}{\cos^2 t},\ x^2+1 = \frac{1}{\cos^2 t} \right] =$

$$= \int \frac{\sin^4 t}{\cos^5 t}\, dt = \frac{1}{4} \int \sin^3 t\, d\left(\frac{1}{\cos^4 t} \right) =$$

$$= \begin{bmatrix} u = \sin^3 t, & du = 3\sin^2 t \cdot \cos t\, dt, \\ dv = d\left(\frac{1}{\cos^4 t} \right), & v = \frac{1}{\cos^4 t} \end{bmatrix} =$$

$$= \frac{1}{4}\left(\frac{\sin^3 t}{\cos^4 t} - 3\int \frac{\sin^2 t\, dt}{\cos^3 t} \right) = \frac{1}{4}\left(\frac{\sin^3 t}{\cos^4 t} - 3\int \sin t\, d\left(\frac{1}{\cos^2 t} \right) \right) =$$

$$= \begin{bmatrix} u = \sin t, & du = \cos t\, dt, \\ dv = d\left(\frac{1}{\cos^2 t} \right), & v = \frac{1}{\cos^2 t} \end{bmatrix} = \frac{1}{4}\left(\frac{\sin^3 t}{\cos^4 t} - \frac{3}{2}\left(\frac{\sin t}{\cos^2 t} - \int \frac{\cos t\, dt}{\cos^2 t} \right) \right) =$$

$$= \frac{1}{4}\frac{\sin^3 t}{\cos^4 t} - \frac{3}{8}\frac{\sin t}{\cos^2 t} + \frac{3}{8}\int \frac{dt}{\cos t} = \frac{\sin t}{\cos^2 t}\left(\frac{1}{4}\frac{\sin^2 t}{\cos^2 t} - \frac{3}{8} \right) +$$

$$+ \frac{3}{8}\ln\left| x + \sqrt{1+x^2} \right| + C = \frac{\operatorname{tg} t\,(1+\operatorname{tg}^2 t)}{\sqrt{1+\operatorname{tg}^2 t}} + \frac{3}{8}\ln\left| x + \sqrt{1+x^2} \right| + C =$$

$$= x\sqrt{1+x^2}\left(\frac{1}{4}x^2 - \frac{3}{8} \right) + \frac{3}{8}\ln\left| x + \sqrt{1+x^2} \right| + C.$$

Примечание. $\int \frac{dx}{\cos t} = \ln\left| x + \sqrt{1+x^2} \right|$ (см. задачу 520 (2195)).

521 (2196). $\int \sqrt{\frac{1-\sqrt[3]{x}}{1+\sqrt[3]{x}}}\, \frac{dx}{x} = \begin{bmatrix} \sqrt[3]{x} = u,\ x = u^3, \\ dx = 3u^2 du \end{bmatrix} = 3\int \sqrt{\frac{1-u}{1+u}}\, \frac{du}{u} =$

$$= 3\int \sqrt{\frac{(1-u)^2}{1-u^2}}\, \frac{du}{u} = 3\int \frac{1-u}{\sqrt{1-u^2}}\frac{du}{u} = 3\left(\int \frac{du}{\sqrt{1-u^2}\cdot u} - \int \frac{du}{\sqrt{1-u^2}} \right) = I.$$

$$\int \frac{du}{\sqrt{1-u^2}\cdot u} = \begin{bmatrix} u=\sin t, \\ du = \cos t\,dt \end{bmatrix} = \int \frac{\cos t\,dt}{\cos t \sin t} = \ln\left|\operatorname{tg}\frac{t}{2}\right| =$$

$$= \ln\left|\frac{\sin t}{1+\cos t}\right| = \ln\left|\frac{\sin t}{1+\sqrt{1-\sin^2 t}}\right| = \ln\left|\frac{u}{1+\sqrt{1-u^2}}\right|.$$

$$I = 3\left(\ln|u| - \ln\left(1+\sqrt{1-u^2}\right) - \arcsin u\right) + C =$$

$$= \underline{\underline{3\left(\ln\left|\sqrt[3]{x}\right| - \ln\left(1+\sqrt{1-\sqrt[3]{x^2}}\right) - \arcsin\sqrt[3]{x}\right) + C.}}$$

522 (2197). $\displaystyle\int \frac{dx}{x^3\sqrt{(1+x)^3}} = \begin{bmatrix} x+1=t^2,\ dx = 2t\,dt, \\ x=t^2-1,\ \sqrt{(1+x)^3}=t^3 \end{bmatrix} = 2\int \frac{t\,dt}{t^3(t^2-1)^3} =$

$$= 2\int \frac{dt}{t^2(t^2-1)^3} = I.$$

$$\frac{1}{t^2(t^2-1)^3} = \frac{t^2-(t^2-1)}{t^2(t^2-1)^3} = \frac{1}{(t^2-1)^3} - \frac{1}{t^2(t^2-1)^2} = \frac{1}{(t^2-1)^3} -$$

$$-\frac{t^2-(t^2-1)}{t^2(t^2-1)^2} = \frac{1}{(t^2-1)^3} - \frac{1}{(t^2-1)^2} + \frac{1}{t^2(t^2-1)} = \frac{1}{(t^2-1)^3} -$$

$$-\frac{1}{(t^2-1)^2} + \frac{t^2-(t^2-1)}{t^2(t^2-1)} = \frac{1}{(t^2-1)^3} - \frac{1}{(t^2-1)^2} + \frac{1}{(t^2-1)} - \frac{1}{t^2}.$$

$$\frac{1}{2}I = \underbrace{\int \frac{dt}{(t^2-1)^3}}_{I_1} - \underbrace{\int \frac{dt}{(t^2-1)^2}}_{I_2} + \int \frac{dt}{t^2-1} - \int \frac{dt}{t^2}.$$

$$I_1 = \int \frac{dt}{(t^2-1)^3} = \int \frac{t^2-(t^2-1)}{(t^2-1)^3}\,dt = \int \frac{t^2dt}{(t^2-1)^2} - I_2,$$

$$\int \frac{t^2 dt}{(t^2-1)^2} = \left[\begin{array}{ll} u = t, & du = dt, \\ dv = \dfrac{t\,dt}{(t^2-1)^3}, & v = \dfrac{1}{2}\displaystyle\int \frac{d(t^2-1)}{(t^2-1)^3} = -\frac{1}{4(t^2-1)^2} \end{array} \right] =$$

$$= -\frac{t}{4(t^2-1)^2} + \frac{1}{4}\int \frac{dt}{(t^2-1)^2} = -\frac{t}{4(t^2-1)^2} + \frac{1}{4} I_2.$$

$$\frac{1}{2} I = -\frac{t}{4(t^2-1)^2} - \frac{7}{4}\int \frac{dt}{(t^2-1)^2} + \int \frac{dt}{t^2-1} - \int \frac{dt}{t^2}.$$

$$I_2 = \int \frac{dt}{(t^2-1)^2} = \int \frac{t^2-(t^2-1)}{(t^2-1)^2}\, dt = \int \frac{t^2 dt}{(t^2-1)^2} - \int \frac{dt}{t^2-1} =$$

$$= \left[\begin{array}{ll} u = t, & du = dt, \\ dv = \dfrac{t\,dt}{(t^2-1)^2}, & v = \dfrac{1}{2}\displaystyle\int \frac{d(t^2-1)}{(t^2-1)^2} = -\frac{1}{2(t^2-1)} \end{array} \right] =$$

$$= -\frac{t}{2(t^2-1)} + \frac{1}{2}\int \frac{dt}{t^2-1}.$$

$$\frac{1}{2} I = -\frac{t}{4(t^2-1)^2} - \frac{7t}{8(t^2-1)} + \frac{15}{8}\int \frac{dt}{t^2-1} - \int \frac{dt}{t^2} = \frac{t(7(t^2-1)-2)}{8(t^2-1)^2} +$$

$$+ \frac{15}{8}\int \frac{dt}{t^2-1} - \int \frac{dt}{t^2} = \frac{t(7t^2-9)}{8(t^2-1)^2} + \frac{15}{8\cdot 2}\ln\left|\frac{t-1}{t+1}\right| + \frac{1}{t} + C =$$

$$= \frac{\sqrt{x+1}\,(7x-2)}{8x^2} + \frac{15}{16}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + \frac{1}{\sqrt{x+1}} + C =$$

$$= \frac{(x+1)(7x-2)+8x^2}{8x^2\sqrt{x+1}} + \frac{15}{16}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C = \frac{15x^2+5x-2}{8x^2\sqrt{x+1}} +$$

$$+ \frac{15}{16}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

$$I=\frac{15x^2+5x-2}{4x^2\sqrt{x+1}}+\frac{15}{8}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C.$$

523 (2198). $\displaystyle\int\frac{\sqrt{2x+1}}{x^2}\,dx=\left[2x+1=t^2,\ dx=2t\,dt,\ x=\frac{t^2-1}{2}\right]=$

$$=4\int\frac{t^2dt}{(t^2-1)^2}=I.$$

$$\frac{t^2}{(t-1)^2(t+1)^2}=\frac{A}{(t-1)^2}+\frac{B}{t-1}+\frac{C}{(t+1)^2}+\frac{D}{t+1},$$

$$t^2=A(t+1)^2+B(t-1)(t+1)^2+C(t-1)^2+D(t+1)(t-1)^2.$$

$$\left.\begin{array}{ll} t=0: & 0=A-B+C+D,\\ t=1: & 1=4A,\\ t=-1: & 1=4C,\\ t=2: & 4=9A+9B+C+3D \end{array}\right\}\Rightarrow\quad \begin{array}{l} A=B=C=\dfrac{1}{4},\\ D=-\dfrac{1}{4}. \end{array}$$

$$I=4\left(\frac{1}{4}\int\frac{dt}{(t-1)^2}+\frac{1}{4}\int\frac{dt}{t-1}+\frac{1}{4}\int\frac{dt}{(t+1)^2}-\frac{1}{4}\int\frac{dt}{t+1}\right)=-\frac{1}{t-1}+$$

$$+\ln|t-1|-\frac{1}{t+1}-\ln|t+1|+C=C-\frac{2t}{t^2-1}+\ln\left|\frac{t-1}{t+1}\right|=$$

$$=C-\frac{\sqrt{2x+1}}{x}+\ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right|.$$

524 (2199). $\displaystyle\int\frac{x^4dx}{x^{15}-1}=\left[\begin{array}{c} z=x^5,\\ dz=5x^4dx\end{array}\right]=\frac{1}{5}\int\frac{dz}{z^3-1}=I.$

$$\frac{1}{z^3-1}=\frac{A}{z-1}+\frac{Bz+C}{z^2+z+1},$$

$$1 = Az^2 + Az + A + Bz^2 + Cz - Bz - C.$$

$$\left.\begin{array}{l|l} z^2 & A + B = 0, \\ z^1 & A - B + C = 0, \\ z^0 & A - C = 1 \end{array}\right\} \Rightarrow\ A = \frac{1}{3},\ B = -\frac{1}{3},\ C = -\frac{2}{3}.$$

$$I = \frac{1}{5}\left(\frac{1}{3}\ln|z-1| - \frac{1}{3}\int \frac{z+2}{z^2+z+1}\,dz\right) = \frac{1}{5}\left(\frac{1}{3}\ln|z-1| - \frac{1}{6}\ln\left|z^2+z+1\right| -\right.$$

$$\left. - \frac{1}{2}\int \frac{dz}{\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}}\right) + C = \frac{1}{15}\left(\frac{1}{2}\ln\frac{(z-1)^2}{z^2+z+1} - \sqrt{3}\,\mathrm{arctg}\frac{2z+1}{\sqrt{3}}\right) + C =$$

$$= \underline{\underline{\frac{1}{15}\left(\frac{1}{2}\ln\frac{(x^5-1)^2}{x^{10}+x^5+1} - \sqrt{3}\,\mathrm{arctg}\frac{2x^5+1}{\sqrt{3}}\right) + C.}}$$

525 (2200). $\displaystyle\int \frac{dx}{\sin 2x - 2\sin x} = \int \frac{dx}{2\sin x(\cos x - 1)} = -\frac{1}{4}\int \frac{dx}{\sin x \cdot \sin^2\frac{x}{2}} =$

$$= -\frac{1}{8}\int \frac{dx}{\sin^3\frac{x}{2}\cdot\cos\frac{x}{2}} = -\frac{1}{8}\int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{\sin^3\frac{x}{2}\cdot\cos\frac{x}{2}}\,dx = -\frac{1}{8}\left(\int \frac{dx}{\sin\frac{x}{2}\cdot\cos\frac{x}{2}} +\right.$$

$$\left. + \int \frac{\cos\frac{x}{2}\,dx}{\sin^3\frac{x}{2}}\right) = -\frac{1}{8}\int \frac{dx}{\mathrm{tg}\frac{x}{2}\cdot\cos^2\frac{x}{2}} - \frac{2}{8}\int \frac{d\left(\sin\frac{x}{2}\right)}{\sin^3\frac{x}{2}} = -\frac{1}{4}\int \frac{d\left(\mathrm{tg}\frac{x}{2}\right)}{\mathrm{tg}\frac{x}{2}} -$$

$$- \frac{1}{4}\int \frac{d\left(\sin\frac{x}{2}\right)}{\sin^3\frac{x}{2}} = \underline{\underline{C - \frac{1}{4}\ln\left|\mathrm{tg}\frac{x}{2}\right| + \frac{1}{8\sin^2\frac{x}{2}}.}}$$

526 (2201). $\int \frac{dx}{1+\cos^2 x} = \int \frac{dx}{2\cos^2 x + \sin^2 x} = \int \frac{dx}{(2+\operatorname{tg}^2 x)\cos^2 x} =$

$$= \int \frac{d(\operatorname{tg} x)}{\operatorname{tg}^2 x + \left(\sqrt{2}\right)^2} = \underline{\underline{\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{\operatorname{tg} x}{\sqrt{2}} + C}}.$$

527 (2202). $\int \frac{dx}{a^2 - b^2 \cos^2 x} = \int \frac{dx}{a^2 \cos^2 x + a^2 \sin^2 x - b^2 \cos^2 x} =$

$$= \int \frac{dx}{(a^2 - b^2) \cdot \cos^2 x + a^2 \sin^2 x} = \int \frac{dx}{((a^2 - b^2) + a^2 \operatorname{tg}^2 x) \cdot \cos^2 x} =$$

$$= \frac{1}{a^2} \int \frac{d(\operatorname{tg} x)}{\left(\frac{a^2 - b^2}{a^2}\right) + \operatorname{tg}^2 x} = \frac{1}{a^2 \frac{\sqrt{a^2 - b^2}}{a}} \operatorname{arctg} \frac{\operatorname{tg} x}{\frac{\sqrt{a^2 - b^2}}{a}} = I.$$

Если $a^2 > b^2$, то $\frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}$; если $\frac{b}{a} = \cos\alpha$, то

$\frac{\sqrt{a^2 - b^2}}{a} = \sin\alpha$ и тогда $\underline{\underline{I = \frac{1}{a^2 \sin\alpha} \operatorname{arctg} \frac{\operatorname{tg} x}{\sin\alpha} + C}}$.

Другой способ:

$$\int \frac{dx}{a^2 - b^2 \cos^2 x} = \frac{1}{b^2} \int \frac{dx}{\frac{a^2}{b^2} - \cos^2 x} = \frac{1}{b^2} \int \frac{dx}{\cos^2\alpha - \cos^2 x} =$$

$$= \left[\begin{array}{l} b^2 > a^2, \\ \cos\alpha = \frac{a}{b} \end{array}\right] = \frac{1}{b^2} \int \frac{dx}{(\cos\alpha - \cos x)(\cos\alpha + \cos x)} = I.$$

$$(\cos\alpha-\cos x)(\cos\alpha+\cos x)=-2\sin\frac{x-\alpha}{2}\cdot\sin\frac{x+\alpha}{2}\times$$

$$\times 2\cos\frac{\alpha+x}{2}\cdot\cos\frac{\alpha-x}{2}=-\sin(\alpha-x)\cdot\sin(x+\alpha),$$

$$-\frac{1}{\sin(x-\alpha)\cdot\sin(x+\alpha)}=-\left(\frac{\cos(\alpha-x)}{\sin(\alpha-x)}+\frac{\cos(\alpha+x)}{\sin(\alpha+x)}\right)\frac{1}{\sin 2\alpha}.$$

$$I=\frac{1}{b^2\sin 2\alpha}\left(-\int\frac{\cos(\alpha-x)}{\sin(\alpha-x)}\,dx-\int\frac{\cos(\alpha+x)}{\sin(\alpha+x)}\,dx\right)=$$

$$=\frac{1}{b^2\sin 2\alpha}\left(\int\frac{d(\sin(\alpha-x))}{\sin(\alpha-x)}-\int\frac{d(\sin(\alpha+x))}{\sin(\alpha+x)}\right)=$$

$$=\underline{\underline{\frac{1}{b^2\sin 2\alpha}\ln\left|\frac{\sin(\alpha-x)}{\sin(\alpha+x)}\right|+C}}.$$

528 (2203). $\displaystyle\int x\ln(1+x^3)\,dx=\left[\begin{array}{ll} u=\ln(1+x^3), & du=\dfrac{3x^2dx}{1+x^3}, \\ dv=x\,dx, & v=\dfrac{x^2}{2}\end{array}\right]=$

$$=\frac{x^2}{2}\ln(1+x^3)-\frac{3}{2}\int\frac{x^4dx}{1+x^3}=\frac{x^2}{2}\ln(1+x^3)-\frac{3}{2}\left(\int x\,dx-\int\frac{x\,dx}{1+x^3}\right)=$$

$$=\frac{x^2}{2}\ln(1+x^3)-\frac{3}{4}x^2+\frac{3}{2}\int\frac{x\,dx}{1+x^3}=I.$$

$$\frac{x}{(1+x)(1-x+x^2)}=\frac{A}{1+x}+\frac{Bx+C}{x^2-x+1},$$

$$x=Ax^2-Ax+A+Bx+C+Bx^2+Cx.$$

$$\left.\begin{array}{l|l} x^2 & A+B=0, \\ x^1 & -A+B+C=1, \\ x^0 & A+C=0 \end{array}\right\}\Rightarrow\quad \begin{array}{l} A=-\dfrac{1}{3}, \\ B=C=\dfrac{1}{3}. \end{array}$$

$$\frac{3}{2}\int\frac{x\,dx}{1+x^3}=-\frac{1}{2}\int\frac{dx}{1+x}+\frac{1}{2}\int\frac{x+1}{x^2-x+1}\,dx=-\frac{1}{2}\ln|1+x|+$$

$$+\frac{1}{2}\int\frac{(2x-1)\frac{1}{2}+\frac{3}{2}}{x^2-x+1}\,dx=\frac{1}{4}\int\frac{d\,(x^2-x+1)}{x^2-x+1}+$$

$$+\frac{3}{4}\int\frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{4}\ln\left|x^2-x+1\right|+\frac{\sqrt{3}}{2}\ln\left|\frac{2x-1}{\sqrt{3}}\right|.$$

$$\underline{\underline{I=\frac{x^2}{2}\ln(1+x^3)-\frac{3}{4}x^2+\frac{1}{4}\ln\left|x^2-x+1\right|+\frac{\sqrt{3}}{2}\ln\left|\frac{2x-1}{\sqrt{3}}\right|+C.}}$$

529 (2204). $\displaystyle\int\frac{(\ln x-1)\,dx}{\ln^2 x}=\int d\left(\frac{x}{\ln x}\right)=\underline{\underline{\frac{x}{\ln x}+C.}}$

530 (2205). $\displaystyle\int\frac{x\ln x}{\sqrt{(x^2-1)^3}}\,dx=$

$$=\left[\begin{array}{ll} u=\ln x, & du=\dfrac{dx}{x}, \\ dv=\dfrac{x\,dx}{\sqrt{(x^2-1)^3}}, & v=\dfrac{1}{2}\displaystyle\int\frac{d\,(x^2-1)}{\sqrt{(x^2-1)^3}}=-\frac{1}{\sqrt{x^2-1}} \end{array}\right]=-\frac{\ln x}{\sqrt{x^2-1}}+$$

$$+\int\frac{dx}{x\sqrt{x^2-1}}=\left[\begin{array}{c} x=\dfrac{1}{\cos t},\ dx=\dfrac{\sin t\,dt}{\cos^2 t}, \\ t=\arccos\dfrac{1}{x}=\operatorname{arctg}\dfrac{\sqrt{1-\dfrac{1}{x^2}}}{\dfrac{1}{x}}=\operatorname{arctg}\sqrt{x^2-1} \end{array}\right]=$$

$$= -\frac{\ln x}{\sqrt{x^2-1}} + \int \frac{\sin t\, dt}{\cos^2 t \cdot \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t}} = -\frac{\ln x}{\sqrt{x^2-1}} + t + C =$$

$$= -\frac{\ln x}{\sqrt{x^2-1}} + \operatorname{arctg}\sqrt{x^2-1} + C.$$

531 (2206). $\int x^2 e^x \cos x \; dx = I.$

$$u = x^2,\ du = 2x\,dx,\ \ dv = e^x \cos x,$$

$$v = \int e^x \cos x\, dx = \begin{bmatrix} u_1 = e^x, & du_1 = e^x dx, \\ dv_1 = \cos x\, dx, & v_1 = \sin x \end{bmatrix} = e^x \sin x -$$

$$-\int e^x \sin x\, dx = \begin{bmatrix} u_2 = e^x, & du_2 = e^x dx, \\ dv_2 = \sin x\, dx, & v_2 = -\cos x \end{bmatrix} = e^x \sin x +$$

$$+ e^x \cos x - \int e^x \cos x\; dx \;\Rightarrow\; v = \int e^x \cos x\; dx = \frac{1}{2}(\sin x + \cos x).$$

$$I = \frac{x^2}{2} e^x (\sin x + \cos x) - \int x e^x (\sin x + \cos x)\, dx = \frac{x^2}{2} e^x (\sin x + \cos x) -$$

$$- I_1 - I_2.$$

$$I_1 = \int x e^x \sin x\, dx = \begin{bmatrix} u_3 = x,\ du_3 = dx,\ dv_3 = e^x \sin x\, dx, \\ v_3 = \int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x) \\ \left(\text{интеграл берется по частям}\right) \end{bmatrix} =$$

$$= \frac{x}{2} e^x (\sin x - \cos x) - \frac{1}{2} \int e^x (\sin x - \cos x)\, dx,$$

$$I_2 = \int xe^x \cos x\,dx = \begin{bmatrix} u_4 = x,\ du_4 = dx,\ dv_4 = e^x \cos x\,dx, \\ v_4 = \frac{1}{2}e^x(\sin x + \cos x) \end{bmatrix} =$$

$$= \frac{x}{2}e^x(\sin x + \cos x) - \frac{1}{2}\int e^x(\sin x + \cos x)\,dx.$$

$$I = \frac{x^2}{2}e^x(\sin x + \cos x) - \frac{x}{2}e^x(\sin x - \cos x) + \frac{1}{2}\int e^x(\sin x - \cos x)\,dx -$$

$$-\frac{x}{2}e^x(\sin x + \cos x) + \frac{1}{2}\int e^x(\sin x + \cos x)\,dx = \frac{x^2}{2}e^x(\sin x + \cos x) -$$

$$-\frac{x}{2}e^x(\sin x - \cos x) - \frac{x}{2}e^x(\sin x + \cos x) + \int e^x \sin x\,dx =$$

$$= \frac{x^2}{2}e^x(\sin x + \cos x) - xe^x \sin x + \frac{1}{2}e^x(\sin x - \cos x) + C =$$

$$= \frac{1}{2}e^x((x^2 - 1)\cos x + (x-1)^2 \sin x) + C.$$

532 (2207). $\displaystyle\int xe^{x^2}(x^2+1)\,dx = \begin{bmatrix} u = x^2 + 1, & du = 2x\,dx, \\ dv = xe^{x^2}dx, & v = \frac{1}{2}e^{x^2} \end{bmatrix} =$

$$= \frac{1}{2}(x^2+1)e^{x^2} - xe^{x^2}dx = \frac{1}{2}(x^2+1)e^{x^2}dx = \frac{1}{2}(x^2+1)e^{x^2} -$$

$$-\frac{1}{2}e^{x^2} + C = \frac{x^2 e^{x^2}}{2} + C.$$

533 (2208). $\displaystyle\int \frac{dx}{\sqrt{\sin^3 x \cdot \cos^5 x}} = \int \frac{dx}{\sin x \cdot \cos^2 x\sqrt{\sin x \cdot \cos x}} =$

$$=\int\frac{d(\operatorname{tg}x)}{\sin x\sqrt{\sin x\cdot\cos x}}=\int\frac{(\cos^2 x+\sin^2 x)\,d(\operatorname{tg}x)}{\sin x\sqrt{\sin x\cdot\cos x}}=\int\frac{\sin x\,d(\operatorname{tg}x)}{\sqrt{\sin x\cdot\cos x}}+$$

$$+\int\frac{\cos^2 x\,d(\operatorname{tg}x)}{\sin x\sqrt{\sin x\cdot\cos x}}=\int\sqrt{\frac{\sin x}{\cos x}}\,d(\operatorname{tg}x)+\int\frac{d(\operatorname{tg}x)}{\left(\dfrac{\sin x}{\cos x}\right)^{3/2}}=$$

$$=\int(\operatorname{tg}x)^{1/2}d(\operatorname{tg}x)+\int\frac{d(\operatorname{tg}x)}{(\operatorname{tg}x)^{3/2}}=\frac{2}{3}(\operatorname{tg}x)^{3/2}-\frac{2}{(\operatorname{tg}x)^{1/2}}+C=$$

$$=\underline{\underline{\frac{2}{3}\cdot\frac{\operatorname{tg}^2 x-3}{\operatorname{tg}x\sqrt{\operatorname{tg}x}}+C.}}$$

534 (2209). $\displaystyle\int\frac{dx}{\sin^5 x\cdot\cos^5 x}=\int\frac{(\sin^2 x+\cos^2 x)^3\,dx}{\sin^5 x\cdot\cos^5 x}=$

$$=\int\frac{(\sin^6 x+3\sin^4 x\cdot\cos^2 x+3\sin^2 x\cdot\cos^4 x+\cos^6 x)\,dx}{\sin^5 x\cdot\cos^5 x}=$$

$$=\int\operatorname{tg}x\frac{1}{\cos^2 x}\,d(\operatorname{tg}x)+3\int\operatorname{ctg}x\frac{1}{\cos^2 x}\,d(\operatorname{tg}x)-$$

$$-3\int\operatorname{tg}x\frac{1}{\sin^2 x}\,d(\operatorname{ctg}x)+\int\operatorname{ctg}x\frac{1}{\sin^2 x}\,d(\operatorname{ctg}x)=$$

$$=\left[\cos^2 x=\frac{1}{1+\operatorname{tg}^2 x},\quad \sin^2 x=\frac{1}{1+\operatorname{ctg}^2 x}\right]=\int\operatorname{tg}x(1+\operatorname{tg}^2 x)\,d(\operatorname{tg}x)+$$

$$+3\int\frac{1+\operatorname{tg}^2 x}{\operatorname{tg}x}\,d(\operatorname{tg}x)-3\int\frac{1+\operatorname{ctg}^2 x}{\operatorname{ctg}x}\,d(\operatorname{ctg}x)-$$

$$-\int\operatorname{ctg}x(1+\operatorname{ctg}^2 x)\,d(\operatorname{ctg}x)=\frac{\operatorname{tg}^2 x}{2}+\frac{\operatorname{tg}^4 x}{4}+3\ln|\operatorname{tg}x|+\frac{3}{2}\operatorname{tg}^2 x-$$

$$-3\ln|\operatorname{ctg} x|-\frac{3}{2}\operatorname{ctg}^2 x-\frac{\operatorname{ctg}^2 x}{2}-\frac{\operatorname{ctg}^4 x}{4}=$$

$$=\frac{1}{4}(\operatorname{tg}^4 x-\operatorname{ctg}^4 x)+2(\operatorname{tg}^2 x-\operatorname{ctg}^2 x)+6\ln|\operatorname{tg} x|+C.$$

535 (2210). $\displaystyle\int\frac{\sin 2x\,dx}{\sin^4 x+\cos^4 x}=2\int\frac{\sin x\cdot\cos x\,dx}{\sin^4 x+\cos^4 x}=[:\cos^2 x]=$

$$=2\int\frac{\operatorname{tg} x\,dx}{\cos^2 x(1+\operatorname{tg}^4 x)}=\int\frac{d(\operatorname{tg}^2 x)}{1+(\operatorname{tg}^2 x)^2}=\operatorname{arctg}(\operatorname{tg}^2 x)+C.$$

536 (2211). $\displaystyle\int\frac{dx}{1+\sin x+\cos x}=\int\frac{dx}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=$

$$=\int\frac{dx}{2\cos^2\frac{x}{2}\left(1+\operatorname{tg}\frac{x}{2}\right)}=\int\frac{d\left(\operatorname{tg}\frac{x}{2}\right)}{1+\operatorname{tg}\frac{x}{2}}=\ln\left|1+\operatorname{tg}\frac{x}{2}\right|+C.$$

537 (2212). $\displaystyle\int\sqrt{\operatorname{tg}^2 x+2}\,dx=\int\cos^2 x\frac{1}{\cos^2 x}\sqrt{\operatorname{tg}^2 x+2}\,dx=$

$$=\int\frac{1}{1+\operatorname{tg}^2 x}\sqrt{\operatorname{tg}^2 x+2}\,d(\operatorname{tg} x)=[\operatorname{tg} x=t]=\int\frac{1}{1+t^2}\sqrt{t^2+2}\,dt=$$

$$=\left[t=\sqrt{2}\operatorname{tg} z,\ dt=\frac{\sqrt{2}}{\cos^2 z}\,dz\right]=\int\frac{\sqrt{2-2\operatorname{tg}^2 z}}{1+2\operatorname{tg}^2 z}\sqrt{2}\frac{1}{\cos^2 z}\,dz=$$

$$=2\int\frac{dz}{\cos z(1+\sin^2 z)}=2\int\frac{\cos z\,dz}{(1-\sin^2 z)(1+\sin^2 z)}=$$

$$=2\int\frac{d(\sin z)}{(1-\sin^2 z)(1+\sin^2 z)}=[\sin z=u]=2\int\frac{du}{(1-u^2)(1+u^2)}=$$

$$=\int\left(\frac{1}{1-u^2}+\frac{1}{1+u^2}\right)du=\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+\operatorname{arctg}u+C=$$

$$=\frac{1}{2}\ln\left|\frac{\operatorname{tg}x+\sqrt{2+\operatorname{tg}^2 x}}{\sqrt{2+\operatorname{tg}^2 x}-\operatorname{tg}x}\right|+\operatorname{arctg}\frac{\operatorname{tg}x}{\sqrt{2+\operatorname{tg}^2 x}}+C=$$

$$=\ln\left|\sqrt{\frac{\left(\sqrt{2+\operatorname{tg}^2 x}+\operatorname{tg}x\right)\left(\sqrt{\operatorname{tg}^2 x+2}+\operatorname{tg}x\right)}{\left(\sqrt{\operatorname{tg}^2 x+2}-\operatorname{tg}x\right)\left(\sqrt{\operatorname{tg}^2 x+2}+\operatorname{tg}x\right)}}\right|+\operatorname{arctg}\frac{\operatorname{tg}x}{\sqrt{\operatorname{tg}^2 x+2}}+C=$$

$$=\ln\left|\operatorname{tg}^2 x+2+\operatorname{tg}x\right|+\operatorname{arctg}\frac{\operatorname{tg}x}{\sqrt{\operatorname{tg}^2 x+2}}+C.$$

538 (2213). $$\int\frac{(x^2-1)\,dx}{x\sqrt{x^4+3x^2+1}}=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\sqrt{x^2+3+\frac{1}{x^2}}}=\int\frac{d\left(x+\frac{1}{x}\right)}{\sqrt{\left(x+\frac{1}{x}\right)^2+1}}=$$

$$=\ln\left|x+\frac{1}{x}+\sqrt{\left(x+\frac{1}{x}\right)^2+1}\right|+C=\ln\frac{x^2+1+\sqrt{x^4+3x^2+1}}{|x|}+C.$$

539 (2214). $$\int\frac{dx}{(2x-3)\sqrt{4x-x^2}}=\int\frac{dx}{(2x-3)\sqrt{4-(x-2)^2}}=$$

$$=\begin{bmatrix}x-2=2\cos t,\\ dx=-2\sin t\,dt,\\ 2x-3=4\cos t+1\end{bmatrix}=-\int\frac{dt}{4\cos t+1}=\begin{bmatrix}t=\operatorname{tg}\frac{z}{2},\ dt=\frac{2dz}{1+z^2},\\ \cos t=\frac{1-z^2}{1+z^2}\end{bmatrix}=$$

$$=-2\int\frac{dz}{5-3z^2}=-2\int\frac{dz}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}z\right)^2}=-\frac{2}{\sqrt{3}}\int\frac{d\left(\sqrt{3}z\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}z\right)^2}=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{5}+\sqrt{3}z}{\sqrt{5}-\sqrt{3}z}\right|+C=-\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{5}+\sqrt{3}\operatorname{tg}\dfrac{t}{2}}{\sqrt{5}-\sqrt{3}\operatorname{tg}\dfrac{t}{2}}\right|+C=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{5}+\sqrt{3}\,\dfrac{\sin t}{1+\cos t}}{\sqrt{5}-\sqrt{3}\,\dfrac{\sin t}{1+\cos t}}\right|+C=-\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{5}+\sqrt{5}\cos t+\sqrt{3}\sin t}{\sqrt{5}+\sqrt{5}\cos t-\sqrt{3}\sin t}\right|+$$

$$+C=-\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{5}+\sqrt{5}\,\dfrac{x-2}{2}+\sqrt{3}\sqrt{1-\dfrac{(x-2)^2}{4}}}{\sqrt{5}+\sqrt{5}\,\dfrac{x-2}{2}-\sqrt{3}\sqrt{1-\dfrac{(x-2)^2}{4}}}\right|+C=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{\sqrt{5}+\sqrt{-3x^2+12x}}{\sqrt{5}-\sqrt{-3x^2+12x}}\right|+C=-\frac{1}{\sqrt{15}}\ln\left|\frac{5x+\sqrt{60x-15x^2}}{5x-\sqrt{60x-15x^2}}\right|+C=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{25x^2+10x\sqrt{60x-15x^2}+60x-15x^2}{25x^2-60x+15x^2}\right|+C=$$

$$=-\frac{1}{\sqrt{15}}\ln\left|\frac{10x^2+60x+10x\sqrt{60x-15x^2}}{40x-60x^2}\right|+C=$$

$$=C-\frac{1}{\sqrt{15}}\ln\left|\frac{x+6+\sqrt{60x-15x^2}}{2x-3}\right|.$$

540 (2215). $\displaystyle\int\frac{xe^x\,dx}{(1+x)^2}=\int d\left(\frac{e^x}{1+x}\right)=\frac{e^x}{1+x}+C.$

541 (2216). $$\int \frac{xe^x dx}{\sqrt{1+e^x}} = \left[\begin{array}{l} u = x,\ du = dx,\ dv = \dfrac{e^x dx}{\sqrt{1+e^x}}, \\ v = \displaystyle\int \frac{d(1+e^x)}{\sqrt{1+e^x}} = 2\sqrt{1+e^x} \end{array} \right] = 2x\sqrt{1+e^x} -$$

$$-4\sqrt{1+e^x} - 2\int \frac{dx}{\sqrt{1+e^x}} = \left[t = \sqrt{1+e^x},\ x = \ln(t^2-1),\ dx = \frac{2t}{t^2-1} \right] =$$

$$= 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - \int \frac{2t\,dt}{t(t^2-1)} = 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} -$$

$$-\ln\left|\frac{t-1}{t+1}\right| + C = 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C.$$

542 (2217). $$\int \frac{\operatorname{arctg} x\ dx}{x^4} = \left[\begin{array}{ll} u = \operatorname{arctg} x, & du = \dfrac{dx}{1+x^2}, \\ dv = \dfrac{dx}{x^4}, & v = -\dfrac{1}{3x^3} \end{array} \right] = -\frac{\operatorname{arctg} x}{3x^2} +$$

$$+\frac{1}{3}\int \frac{dx}{x^3(1+x^2)} = I.$$

$$\frac{1}{x^3(1+x^2)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{Dx+E}{1+x^2},$$

$$1 = A(1+x^2) + Bx(1+x^2) + x^3(Dx+E) + Cx^2(1+x^2),$$

$$1 = A + Ax^2 + Bx + Bx^3 + Dx^4 + Ex^3 + Cx^2 + Cx^4.$$

$$\left. \begin{array}{l|l} x^4 & C + D = 0, \\ x^3 & B + E = 0, \\ x^2 & A + C = 0, \\ x^1 & B = 0, \\ x^0 & A = 1 \end{array} \right\} \Rightarrow A = 1,\ B = 0,\ C = -1,\ D = -1,\ E = 0.$$

$$I=-\frac{\operatorname{arctg} x}{3x^2}+\frac{1}{3}\int\frac{dx}{x^3}-\frac{1}{3}\int\frac{dx}{x}+\frac{1}{3}\int\frac{x\,dx}{1+x^2}=-\frac{\operatorname{arctg} x}{3x^2}-\frac{1}{6x^2}-$$

$$-\frac{1}{3}\ln|x|+\frac{1}{6}\ln\left|1+x^2\right|+C=\underline{\underline{\frac{1}{6}\ln\frac{1+x^2}{x^2}-\frac{\operatorname{arctg} x}{3x^3}-\frac{1}{6x^2}+C}}.$$

543 (2218). $\displaystyle\int\frac{x\operatorname{arctg} x\,dx}{(1+x^2)^2}=$

$$=\left[\begin{array}{ll} u=\operatorname{arctg} x, & du=\dfrac{dx}{1+x^2}, \\ dv=\dfrac{x\,dx}{(1+x^2)^2}, & v=\dfrac{1}{2}\displaystyle\int\frac{d(1+x^2)}{(1+x^2)^2}=-\frac{1}{2(1+x^2)}\end{array}\right]=-\frac{\operatorname{arctg} x}{2(1+x^2)}+$$

$$+\frac{1}{2}\int\frac{dx}{(1+x^2)^2}=I.$$

$$\int\frac{dx}{(1+x^2)^2}=\int\frac{(1+x^2)-x^2}{(1+x^2)}dx=\int\frac{dx}{1+x^2}-\int\frac{x^2dx}{(1+x^2)^2}=$$

$$=\left[\begin{array}{ll} u=x, & du=dx, \\ dv=\dfrac{x\,dx}{(1+x^2)^2}, & v=-\dfrac{1}{2(1+x^2)}\end{array}\right]=\operatorname{arctg} x+\frac{x}{2(1+x^2)}+$$

$$+\frac{1}{2}\int\frac{dx}{1+x^2}=\frac{1}{2}\operatorname{arctg} x+\frac{x}{2(1+x^2)}.$$

$$I=\underline{\underline{C-\frac{\operatorname{arctg} x}{2(1+x^2)}+\frac{\operatorname{arctg} x}{4}+\frac{x}{4(1+x^2)}}}.$$

544 (2219). $\displaystyle\int\frac{\operatorname{arctg} x}{(1+x)^3}dx=\left[\begin{array}{ll} u=\operatorname{arctg} x, & du=\dfrac{dx}{1+x^2} \\ dv=\dfrac{dx}{(1+x)^3}, & v=-\dfrac{1}{2(1+x)^2}\end{array}\right]=-\frac{\operatorname{arctg} x}{2(1+x)^2}+$

$$+\frac{1}{2}\int\frac{dx}{(1+x^2)(1+x)^2}=I.$$

$$\frac{1}{(1+x^2)(1+x)^2}=\frac{Ax+B}{1+x^2}+\frac{C}{(1+x)^2}+\frac{D}{1+x},$$

$$1=(Ax+B)(1+x)^2+C(1+x^2)+D(1+x)(1+x^2),$$

$$1=Ax+2Ax^2+Ax^3+B+2Bx+Bx^2+C=Cx^2+D+Dx+$$
$$+Dx^2+Dx^3.$$

$$\left.\begin{array}{l|l} x^3 & A+D=0,\\ x^2 & 2A+B+C+D=0,\\ x^1 & A+2B+D=0,\\ x^0 & B+C+D=1 \end{array}\right\}\Rightarrow A=-\frac{1}{2},\ B=0,\ C=D=\frac{1}{2}.$$

$$I=C-\frac{\operatorname{arctg}x}{2(1+x)^2}-\frac{1}{4}\int\frac{x\,dx}{1+x^2}+\frac{1}{4}\int\frac{dx}{(1+x)^2}+\frac{1}{4}\int\frac{dx}{1+x}=$$

$$=C-\frac{\operatorname{arctg}x}{2(1+x)^2}+\ln\frac{|1+x|}{\sqrt{1+x^2}}-\frac{1}{4(x+1)}.$$

545 (2220). $\displaystyle\int\frac{dx}{(1-2^x)^4}=\left[\begin{array}{c}1-2^x=t;\ x=\log_2(1-t)\\ dx=-\dfrac{dt}{\ln^2(1-t)}\end{array}\right]=-\frac{1}{\ln 2}\int\frac{dt}{t^4(1-t)}=$

$$=\frac{1}{\ln 2}\int\frac{dt}{t^4(t-1)}=I.$$

$$\frac{1}{t^4(t-1)}=\frac{A}{t^4}+\frac{B}{t^3}+\frac{C}{t^2}+\frac{D}{t}+\frac{E}{t-1},$$

$$1=A(t-1)+Bt(t-1)+Ct^2(t-1)+Dt^3(t-1)+Et^4,$$

$$1=At-A+Bt^2-Bt+Ct^3-Ct^2+Dt^4-Dt^3+Et^4.$$

$$\left.\begin{array}{l|l} t^4 & D+E=0,\\ t^3 & C-D=0,\\ t^2 & B-C=0,\\ t^1 & A-B=0,\\ t^0 & -A=1, \end{array}\right\}\Rightarrow A=B=C=D=-1,\ E=1.$$

$$I=\left[-\int\frac{dt}{t^4}-\int\frac{dt}{t^3}-\int\frac{dt}{t}+\int\frac{dt}{t-1}-\int\frac{dt}{t^2}=\frac{1}{3t^3}+\frac{1}{2t^2}+\frac{1}{t}-\ln|t|+\right.$$

$$\left.+\ln|t-1|\right]\frac{1}{\ln 2}+C=$$

$$=\frac{1}{\ln 2}\left[\frac{1}{(1-2^x)^3}+\frac{1}{2(1-2^x)^2}+\frac{1}{1-2^x}\right]-\log_2\left|1-2^x\right|+x+C.$$

546 (2221). $\displaystyle\int\frac{(e^{3x}+e^x)dx}{e^{4x}-e^{2x}+1}=\int\frac{e^{2x}(e^x+e^{-x})\,dx}{e^{4x}-e^{2x}+1}=\int\frac{d\,(e^x-e^{-x})\,dx}{e^{2x}-1+e^{-2x}}=$

$$=\int\frac{d\,(e^x-e^{-x})\,dx}{1+(e^x-e^{-x})^2}=\underline{\underline{\operatorname{arctg}(e^x-e^{-x})+C}}.$$

547 (2222). $\displaystyle I=\int\frac{dx}{\sqrt{1+e^x+e^{2x}}}=\int\frac{dx}{e^x\sqrt{1+e^{-x}+e^{-2x}}}=$

$$=\int\frac{e^{-x}dx}{\sqrt{\left(e^{-x}+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}=-\int\frac{d\left(e^{-x}+\frac{1}{2}\right)}{\sqrt{\left(e^{-x}+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}=$$

$$=-\ln\left(e^{-x}+\frac{1}{2}+\sqrt{e^{-2x}+e^{-x}+1}\right)+C=\ln\frac{1}{e^{-x}+\frac{1}{2}+\sqrt{e^{-2x}+e^{-x}+1}}+$$

$$+C=\ln\frac{2e^x}{2+e^x+2\sqrt{e^{2x}+e^x+1}}+C=\underline{\underline{\ln\frac{e^x}{2+e^x+2\sqrt{e^{2x}+e^x+1}}+C}}.$$

Другой способ:

$e^x = t,\ x = \ln t,\ dx = \dfrac{dt}{t}.$ Подстановка Эйлера $\sqrt{t^2 + t + 1} = t + z \Rightarrow$

$$t = \frac{z^2 - 1}{1 - 2z},\ dt = -2\frac{z^2 - z + 1}{(1 - 2z)^2}\,dz.$$

$$I = -2\int \frac{(z^2 - z + 1)(1 - 2z)}{(z^2 - 1)\left(\dfrac{z^2 - 1}{1 - 2z} + z\right)}\,dz = 2\int \frac{dz}{z^2 - 1} = \ln\left|\frac{1 - z}{1 + z}\right| + C =$$

$$= \ln\left|\frac{1 - \sqrt{t^2 + t + 1} + t}{1 + \sqrt{t^2 + t + 1} - t}\right| + C = \ln\left|\frac{1 + e^x - \sqrt{e^{2x} + e^x + 1}}{1 - e^x + \sqrt{e^{2x} + e^x + 1}}\right| + C.$$

548 (2223). $\displaystyle\int \frac{\operatorname{tg} x\,dx}{1 + \operatorname{tg} x + \operatorname{tg}^2 x} = \left[\begin{gathered}\operatorname{tg} x = t,\ x = \operatorname{arctg} t,\\ dx = \frac{dt}{1 + t^2}\end{gathered}\right] =$

$$= \int \frac{t\,dt}{(1 + t^2)(t^2 + t + 1)} = I.$$

$$\frac{t}{(t^2 + 1)(t^2 + t + 1)} = \frac{At + B}{t^2 + 1} + \frac{Ct + D}{t^2 + t + 1},$$

$$t = At^3 + At^2 + At + Bt^2 + Bt + B + Ct^3 + Dt^2 + Ct + D.$$

$$\left.\begin{array}{l|l} t^3 & A + C = 0,\\ t^2 & A + B + D = 0,\\ t^1 & A + B + C = 1,\\ t^0 & B + D = 0 \end{array}\right\} \Rightarrow A = 0,\ B = 1,\ C = 0,\ D = -1.$$

$$I=\int\frac{dt}{t^2+1}-\int\frac{dt}{t^2+t+1}=\int\frac{dt}{t^2+1}-\int\frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\operatorname{arctg} t-$$

$$-\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+C=\underline{\underline{x-\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2\operatorname{tg}x+1}{\sqrt{3}}+C}}.$$

549 (2224). $\int\sin^8 x\,dx=I.$

$$\sin^8 x=(\sin^2 x)^4=\left(\frac{1-\cos 2x}{2}\right)^2=\frac{1}{16}(1-4\cos 2x+6\cos^2 2x-$$

$$-4\cos^3 2x+\cos^4 2x)=\frac{1}{16}\left(1-4\cos 2x+6\frac{1+\cos 4x}{2}-\right.$$

$$\left.-4(1-\sin^2 2x)\cos 2x+\left(\frac{1+\cos 4x}{2}\right)^2\right)=\frac{1}{16}(1-4\cos 2x+$$

$$+3+3\cos 4x-4\cos 2x+4\sin^2 2x\cdot\cos 2x+\frac{1}{4}+\frac{1}{2}\cos 4x+$$

$$\left.+\frac{1+\cos 8x}{8}\right)=\frac{1}{16}\left(\frac{35}{8}-8\cos 2x+\frac{7}{2}\cos 4x+4\sin^2 2x\cdot\cos 2x+\right.$$

$$\left.+\frac{1}{8}\cos 8x\right).$$

$$\underline{\underline{I=\frac{35}{128}x-\frac{1}{4}\sin 2x+\frac{7}{128}\sin 4x+\frac{1}{24}\sin^3 2x+\frac{1}{1024}\sin 8x+C}}.$$

550 (2225). $\int\frac{(3+x^2)^2x^3dx}{(1+x^2)^3}=I.$

$$\frac{((1+x^2)+2)^2x^3}{(1+x^2)^3}=\frac{x^3}{1+x^2}+\frac{4x^3}{(1+x^2)^2}=\frac{4x^3}{(1+x^2)^3}=-\frac{x}{x^2+1}+$$

$$+x+\frac{4x^3}{(1+x^2)^2}+\frac{4x^3}{(1+x^2)^3}=-\frac{x}{x^2+1}+x+4\frac{2x^3+x^5}{(1+x^2)^3},$$

$$\frac{2x^3+x^5}{(1+x^2)^3}=\frac{Ax+B}{(1+x^2)^3}+\frac{Cx+D}{(1+x^2)^3}+\frac{Ex+K}{1+x^2},$$

$$2x^3+x^5=Ax+B+Cx+D+Cx^3+Dx^2+Ex+K+2Ex^3+$$

$$+2Kx^2+Ex^5+Kx^4.$$

$$\left.\begin{array}{l|l} x^5 & E=1, \\ x^4 & K=0, \\ x^3 & C+2F=2, \\ x^2 & D+2K=0, \\ x^1 & A+C+E=0, \\ x^0 & B+D+K=0 \end{array}\right\} \Rightarrow \quad \begin{array}{l} B=K=C=D=0, \\ A=-1,\ E=1. \end{array}$$

$$I=-\frac{1}{2}\int\frac{d\,(1+x^2)}{1+x^2}+\int x\,dx+4\left(-\int\frac{x\,dx}{(1+x^2)^3}+\int\frac{x\,dx}{1+x^2}\right)=$$

$$=-\frac{1}{2}\ln(1+x^2)+\frac{x^2}{2}+\frac{1}{(1+x^2)^2}+2\ln(1+x^2)+C=$$

$$=\underline{\underline{\frac{x^2}{2}+\frac{3}{2}\ln(1+x^2)+\frac{1}{(1+x^2)^2}}}+C.$$

551 (2226). $\displaystyle\int\frac{x^2-8x+7}{(x^2-3x-10)^2}=I.$

$$\frac{x^2-8x+7}{(x+2)^2(x-5)^2}=\frac{A}{(x+2)^2}+\frac{B}{x+2}+\frac{C}{(x-5)^2}+\frac{D}{x-5},$$

$$x^2-8x+7=A(x-5)^2+B(x+2)(x-5)^2+C(x+2)^2+$$

$$+D(x-5)(x+2)^2.$$

$$\left.\begin{array}{ll} x=4: & -9=A+6B+36C-36D, \\ x=5: & -8=49C, \\ x=-2: & 27=49A, \\ x=-1: & 16=36A+36B+C-6D \end{array}\right\} \Rightarrow\ A=\frac{27}{49},\ C=-\frac{8}{49},$$

$$\begin{cases} B-6D=-\dfrac{30}{49}, \\ 6B-D=\dfrac{30}{343} \end{cases} \Rightarrow\ B=-\frac{30}{343},\ D=\frac{30}{343}.$$

$$I=\frac{8}{49(x+2)}-\frac{27}{49(x-5)}+\frac{30}{343}\ln\left|\frac{x-5}{x+2}\right|+C.$$

552 (2227). $\displaystyle\int\frac{dx}{\sin^4x+\cos^4x}=I.$

$$\sin^4+\cos^4x=2\sin^2x\cdot\cos^2x+(\sin^4x-2\sin^2x\cdot\cos^2x+\cos^4x)=$$

$$=2\sin^2x\cdot\cos^2x+(\cos^2x-\sin^2x)^2=\frac{1}{2}(4\sin^2x\cdot\cos^2x+2\cos^2 2x)=$$

$$=\frac{1}{2}(\sin^2 2x+2\cos^2 2x)=\frac{1}{2}\sin^2 2x\left(1+2\frac{\cos^2 2x}{\sin^2 2x}\right)=$$

$$=\frac{1}{2}\sin^2 2x(1+2\,\mathrm{ctg}^2\,2x).$$

$$I=\int\frac{\frac{2}{\sin^2 2x}}{1+\left(\sqrt{2}\operatorname{ctg}2x\right)^2}=-\frac{1}{\sqrt{2}}\int\frac{d\left(\sqrt{2}\operatorname{ctg}2x\right)}{1+\left(\sqrt{2}\operatorname{ctg}2x\right)^2}=$$

$$=\underline{\underline{C-\frac{\sqrt{2}}{2}\operatorname{arctg}\left(\sqrt{2}\operatorname{ctg}2x\right).}}$$

553 (2228). $$\int\frac{(x+\sin x)\,dx}{1+\cos x}=\int\frac{\left(x+2\sin\frac{x}{2}\cos\frac{x}{2}\right)dx}{2\cos^2\frac{x}{2}}=$$

$$=\int\left(\frac{x}{2\cos^2\frac{x}{2}}+\operatorname{tg}\frac{x}{2}\right)dx=\int d\left(x\operatorname{tg}\frac{x}{2}\right)=\underline{\underline{x\operatorname{tg}\frac{x}{2}+C}}.$$

554 (2229). $$\int\frac{x^2-1}{x^2+1}\frac{dx}{\sqrt{1+x^4}}=\int\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}\frac{dx}{\sqrt{(x^2+1)^2-2x^2}}=$$

$$=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}=\left[\begin{gathered}x+\frac{1}{x}=t;\\ \left(1-\frac{1}{x^2}\right)dx=dt\end{gathered}\right]=\int\frac{dt}{t\sqrt{t^2-2}}=$$

$$=\left[t=\frac{\sqrt{2}}{\cos z};\ dt=\frac{\sqrt{2}\sin z\,dz}{\cos^2 z}\right]=\int\frac{\cos^2 z\cdot\sin z\,dz}{\sqrt{2}\cos^2 z\cdot\sqrt{2}\sin z}=\frac{1}{2}z+C=$$

$$=\frac{1}{2}\arccos\frac{\sqrt{2}}{z}+C=\underline{\underline{\frac{1}{2}\arccos\frac{\sqrt{2}x}{x^2+1}+C}}.$$

555 (2230). $\int e^{\sin x} \frac{x\cos^3 x - \sin x}{\cos^2 x}\, dx =$

$$= \int e^{\sin x} \frac{(x\cos^3 x + \cos^2 x) - (\cos^2 x + \sin x)}{\cos^2 x}\, dx = \int e^{\sin x} (x\cos x + 1)\, dx -$$

$$- \int e^{\sin x} \left(1 + \frac{\sin x}{\cos^2 x}\right) dx = \int d\,(xe^{\sin x}) - \int d\left(\frac{e^{\sin x}}{\cos x}\right) = xe^{\sin x} - \frac{e^{\sin x}}{\cos x} +$$

$$+ C = \underline{\underline{e^{\sin x}(x - \sec x) + C}}.$$

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