МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
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ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ
DIFFERENTIAL EQUATIONS

Навчальний посібник / Tutorial

Рекомендовано Методичною радою НУК

ВИДАВНИЦТВО
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Представлено методичні розробки для проведення практичних занять з розділу «Диференціальні рівняння». Розроблено п'ять практичних занять, для кожного з яких викладено основні теоретичні відомості, завдання для аудиторної, самостійної роботи студентів та їх розв'язки.
Посібник призначено для студентів та викладачів.

The tutorial contains sample tasks for practical lessons on the unit «Differential equations». Each of the developed practical lessons comprises general theoretical information, exercises for practical lesson, exercises for individual work, and their solutions.
The tutorial is designed for students and university teachers.

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ВСТУП

Навчальний посібник «Диференціальні рівняння» створено для проведення практичних занять студентам, які навчаються за скороченим терміном, з огляду на сучасні вимоги щодо істотного підвищення рівня фундаментальної математичної підготовки і посилення прикладної спрямованості курсу.

Розроблений посібник має за мету допомогти студентам у самостійному оволодінні теоретичними знаннями та необхідними практичними навичками.

Навчальний матеріал посібника подано у вигляді п’яти практичних занять, кожне з яких містить основні теоретичні відомості, завдання для аудиторної, самостійної роботи студентів, а також відповіді та вказівки до їх розв’язання.

Посібник може бути корисним як для студентів, яким зручно вивчати вищу математику англійською мовою, так і для викладачів при підготовці до практичних занять.

INTRODUCTION

Tutorial «Differential equations» is designed for practical lessons for students in the context of contemporary requirements for considerable increase of fundamental mathematical training and encouragement of practical implementation of the course.

The provided tutorial aims to help students in self-mastery of theoretical knowledge and necessary practical skills.

The learning material of the tutorial is presented in the form of five practical lessons, each including general theoretical information, exercises for practical lessons, and exercises for student's individual work as well as solutions and keys to the exercises.

This tutorial will be useful both for students who are comfortable studying higher mathematics in English, and for teachers to prepare for practical lessons.

Practical lesson 1.

Ordinary differential equations of first order: separable equations, homogeneous equations

I. General theoretical information

Definition 1. *Differential equation of first order* is an equation of the form $F(x, y, y') = 0$, that connects independent variable x, unknown function $y = y(x)$ and its derivative y'.

In case when equation $F(x, y, y') = 0$ can be solved with respect to y', namely $y' = f(x, y)$, then this equation is called the ordinary differential equation of first order solved with respect to derivative, or the equation *in the usual form*.

The equation $y' = f(x, y)$ can be written as follows:

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad -f(x, y)\,dx + dy = 0 .$$

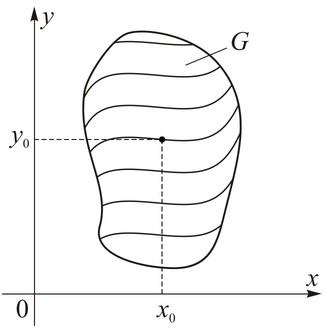
When we multiply the latter equation by function $Q(x, y) \neq 0$, then we get equation $P(x, y)\,dx + Q(x, y)\,dy = 0$, where $P(x, y)$ and $Q(x, y)$ are the known functions, and this equation is the ordinary differential equation of first order *in a differential form.*

Definition 2. *A solution of a differential equation* of first order on a certain interval $(a; b)$ is function $y = \phi(x)$, differentiated on this interval, and being substituted into the given equation, the function turns it into congruence of x on $(a;b)$, that is:

$$\forall x \in (a;b)\ :\ \phi'(x) \equiv f(x, \phi(x)).$$

Definition 3. Graphical solution to differential equation is called *an integral curve* of this equation.

Theorem 1 (Cauchy's Theorem) (existence and uniqueness theorem). If function $f(x,y)$ and its partial derivative $f_y'(x,y)$ are determined and continuous in an open domain G of area Oxy and point $(x_0;y_0) \in G$, then there is a singular solution $y = \phi(x)$ of equation $y' = f(x,y)$, that satisfies the condition $y = y_0$ when $x = x_0$, that is $\phi(x_0) = y_0$.



Drawing 1.1

Geometric meaning of Cauchy theorem: through every point $(x_0;y_0) \in G$ goes one integral curve; if x_0 is fixed and we change y_0 and we do not leave area G, then we get other integral curves (Drawing 1.1).

Definition 4. The condition, under which a solution $y = \phi(x)$ gets a predetermined value y_0 in a given point x_0, is called an initial condition of a solution and is written as follows: $y|_{x=x_0} = y_0$ or $y(x_0) = y_0$. The task of finding a solution of a differential equation $y' = f(x,y)$, that satisfies the initial condition is called *the Cauchy task*.

Definition 5. If the right side of differential equation $y' = f(x,y)$ in area G satisfies the conditions of Cauchy theorem, then function $y = \phi(x,C)$ is called *general solution* of a given differential equation under the following conditions:

1) function $\phi(x,C)$ is a solution of an equation under any value of constant C;

2) for arbitrary point $(x_0;y_0)\in G$ we can find such value $C=C_0$, that function $y=\phi(x,C_0)$ satisfies the initial condition: $\phi(x_0,C_0)=y_0$.

Definition 6. *Particular solution* of equation $y'=f(x,y)$ is function $y=\phi(x,C_0)$, made up out of general solution $y=\phi(x,C)$ with certain value of constant $C=C_0$.

Definition 7. If a general solution of a differential equation calculated in the form $\Phi(x,y,C)=0$, $\Phi(x,y,C)=0$, then such solution is called a general integral of a differential equation.

In this case equality $\Phi(x,y,C_0)=0$ is called *a particular integral* of the equation.

If it is possible to reduce a task of calculation of all the solutions of a differential equation to calculation of a limited number of integrals and derivatives of the known functions, then we say that a differential equation is *integrated in quadratures* or *reduced to quadratures*.

Let us consider separate types of differential equations that are integrated in quadratures.

Separable, equations

are equations of the form $y'=f(x)\cdot\phi(y)$ (1), where $f(x)$ and $\phi(y)$ are determined and continuous on a certain interval functions. If in equation (1) we substitute $y'=\dfrac{dy}{dx}$, divide both sides of the equation by $\phi(y)$, assuming that $\phi(y)\neq 0$, and multiply it by *dx*, then we have differential equation

$$\frac{dy}{\phi(y)}=f(x)dx$$

which is a separable equation that is solved in quadratures:

$$\int\frac{dy}{\phi(y)}=\int f(x)dx+C.$$

Homogeneous differential equations of first order

– are equations of the form $y' = f(x, y)$, if function $f(x, y)$ is a homogeneous function of zero-degree.

Function $f(x, y)$ is a homogeneous function of *n*-degree відносно variables *x* and *y*, if the following congruency is valid

$$f(tx, ty) = t^n \cdot f(x, y), \qquad t \neq 0.$$

Homogeneous equations can be reduced to separable equations by means of substitution $y = u \cdot x$, where $u = u(x)$ is an unknown function.

II. Exercises for practical lesson

Exercise 1. Calculate whether a given function is a solution of a given differential equation:

a) $y = C \cdot e^{-2x}, \quad y' + 2y = 0$;

b) $y = C_1 x + C_2 x^2, \quad x^2 y'' - 2xy' + 2y = 0$.

Solution: a) We calculate a derivative of a given function: $y' = -2C \cdot e^{-2x}$. We replace the resulting derivative into the equation and prove that it becomes a congruence: $-2C \cdot e^{-2x} + 2C \cdot e^{-2x} = 0$, $0 \equiv 0$. So, function $y = C \cdot e^{-2x}$ is a solution of the given equation.

b) We differentiate function $y(x)$ twice and calculate: $y' = C_1 + 2C_2 x$, $y'' = 2C_2$. Then we have:

$$x^2 \cdot 2C_2 - 2x(C_1 + 2C_2 x) + 2(C_1 x + C_2 x^2) = 0,$$

$$2C_2 x^2 - 2C_1 x - 4C_2 x^2 + 2C_1 x + 2C_2 x^2 = 0,$$

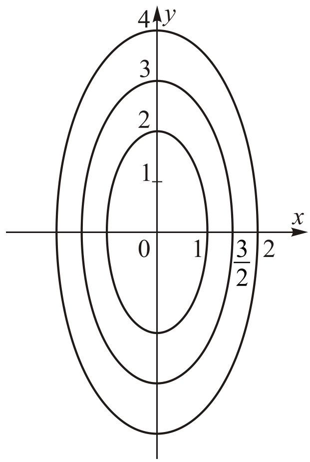
$0 \equiv 0$ is a congruence, thus the given function is a solution of the differential equation.

Exercise 2. Given general solution $4x^2 + y^2 = C^2$ of first-order differential equation, calculate and build the sketch of its integral curves that run trough points $B_1(-1;0)$, $B_2(0;-3)$, $B_3(2;0)$.

Solution:

We replace into the general solution of differential equation coordinates of points B_1, B_2, B_3, and get the value of parameter C, with respect to which out of the general solution we get the equation of an integral curve running through points B_1, B_2, B_3 correspondingly:

p. $B_1(-1;0)$: $C^2 = 4 \quad \Rightarrow \quad 4x^2 + y^2 = 4$;
p. $B_2(0;-3)$: $C^2 = 9 \quad \Rightarrow \quad 4x^2 + y^2 = 9$;
p. $B_3(2;0)$: $C^2 = 16 \quad \Rightarrow \quad 4x^2 + y^2 = 16$.



Drawing 1.2

The resulting equations define the integral curves that run through points B_1, B_2, B_3, and make up concentric ellipses (Drawing 1.2).

Exercise 3. Calculate general solutions of the following differential equations:

a) $(y + xy)dx + (x - xy)dy = 0$;

b) $1 + (1 + y') \cdot e^y = 0$;

c) $2^{x+y} + 3^{x-2y} \cdot y' = 0$;

d) $y - x \cdot y' = x + y \cdot y'$;

e) $y\,dy + (x - 2y)dx = 0$;

f) $y\,dx + \left(2\sqrt{xy} - x\right)dy = 0$.

Solution:

Equations a)-c) are separable equations, and equations d)-f) are homogeneous equations.

a) $(y + xy)dx + (x - xy)dy = 0$;

$$y(1+x)dx - x(y-1)dy = 0\ ; \quad \frac{y-1}{y}dy = \frac{1+x}{x}dx\ ;$$

$$\int\left(1-\frac{1}{y}\right)dy = \int\left(\frac{1}{x}+1\right)dx\ ;$$

$$y - \ln|y| = x + \ln|x| + C\ ; \quad y - x - \ln|xy| = C\ .$$

b) $1+(1+y')\cdot e^{y} = 0$;

$$1+y' = -\frac{1}{e^{y}}\ ; \qquad \frac{dy}{dx} = -\frac{1}{e^{y}} - 1\ ;$$

$$dy = -\frac{1+e^{y}}{e^{y}}dx\ ; \quad \int\frac{e^{y}}{1+e^{y}}dy = -\int dx\ ;$$

$$\ln\left(1+e^{y}\right) = -x + C_1\ ; \quad 1+e^{y} = e^{-x+C_1}\ ; \quad 1+e^{y} = e^{-x}\cdot C\ ,$$

де $C = e^{C_1}$; $\left(1+e^{y}\right)\cdot e^{x} = C$.

c) $2^{x+y} + 3^{x-2y}\cdot y' = 0$;

$$\frac{3^{x}}{3^{2y}}\cdot y' = -2^{x}\cdot 2^{y}\ ; \qquad \frac{1}{9^{y}\cdot 2^{y}}\cdot\frac{dy}{dx} = -\frac{2^{x}}{3^{x}}\ ;$$

$$\int 18^{-y}\,dy = -\int\left(\frac{2}{3}\right)^{x}dx\ ; \quad -\frac{18^{y}}{\ln 18} = -\frac{\left(\frac{2}{3}\right)^{x}}{\ln\frac{2}{3}} + C_1\ ;$$

$$\frac{18^{y}}{\ln 18} - \frac{\left(\frac{2}{3}\right)^{x}}{\ln\frac{2}{3}} = C\ ,$$

where $C = -C_1$.

d) $y - xy' = x + y \cdot y'$;

$$(y + x) \cdot y' = y - x; \qquad y' = \frac{y - x}{y + x}.$$

We label $y = ux$, then $y' = u'x + u$ hence:

$$u'x + u = \frac{ux - x}{ux + x}; \quad u'x = \frac{u - 1}{u + 1} - u; \quad u'x = \frac{u - 1 - u^2 - u}{u + 1};$$

$$\frac{du}{dx} \cdot x = -\frac{1 + u^2}{u + 1}; \quad \int \frac{u + 1}{1 + u^2} du - \int \frac{dx}{x}.$$

Calculate

$$I = \int \frac{u + 1}{1 + u^2} du = \frac{1}{2} \int \frac{2u}{1 + u^2} du + \int \frac{du}{1 + u^2} = \frac{1}{2} \ln\left(1 + u^2\right) + \operatorname{arctg} u + \ln C.$$

Therefore: $\frac{1}{2} \ln\left(1 + \frac{y^2}{x^2}\right) + \operatorname{arctg} \frac{y}{x} + \ln C = -\ln|x|$;

$$\ln \sqrt{\frac{x^2 + y^2}{x^2}} \cdot C|x| + \operatorname{arctg} \frac{y}{x} = 0; \quad \ln C \cdot \sqrt{x^2 + y^2} + \operatorname{arctg} \frac{y}{x} = 0.$$

e) $y\,dy + (x - 2y)\,dx = 0$;

$$y\,dy = (2y - x)\,dx; \qquad y' = \frac{2y - x}{y}.$$

We apply substitution $y = ux$, $y' = u'x + u$,

then $u'x + u = \frac{2ux - x}{ux}$;

$$u'x = \frac{2u - 1}{u} - u; \quad \frac{du}{dx} \cdot x = \frac{2u - 1 - u^2}{u}; \quad -\int \frac{u\,du}{u^2 - 2u + 1} = \int \frac{dx}{x};$$

$$I = -\int \frac{u - 1}{u^2 - 2u + 1} - \int \frac{du}{(u - 1)^2} = -\frac{1}{2} \int \frac{(2u - 1)\,du}{u^2 - 2u + 1} + \frac{1}{u - 1} =$$

$$=-\frac{1}{2}\ln(u-1)^2+\frac{1}{u-1}-\ln C=-\ln(u-1)+\frac{1}{u-1}-\ln C;$$

$$-\ln(u-1)+\frac{1}{u-1}-\ln C=\ln|x|;$$

$$\frac{1}{\frac{y}{x}-1}=\ln C\cdot|x|\cdot\left(\frac{y}{x}-1\right);\qquad \frac{x}{y-x}=\ln C(y-x);$$

$$x=(y-x)\cdot\ln C(y-x).$$

f) $y\,dx+\left(2\sqrt{xy}-x\right)dy=0$;

$$\left(x-2\sqrt{xy}\right)dy=y\,dx;\qquad \frac{dy}{dx}=\frac{y}{x-2\sqrt{xy}}.$$

The substitution is: $y=ux$, $y'=u'x+u$, hence

$$u'x+u=\frac{ux}{x-2x\sqrt{u}};\ u'x=\frac{u}{1-2\sqrt{u}}-u;$$

$$\frac{du}{dx}\cdot x=\frac{u-u+2u\sqrt{u}}{1-2\sqrt{u}};\quad \int\frac{1-2\sqrt{u}}{2u\sqrt{u}}du=\int\frac{dx}{x};$$

$$\frac{1}{2}\cdot\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}-\ln|u|=\ln|x|\cdot C;$$

$$-\frac{1}{\sqrt{u}}=\ln\left|\frac{y}{x}\right|+\ln C|x|;\quad -\sqrt{\frac{x}{y}}=\ln Cy;\quad \sqrt{y}\ln Cy+\sqrt{x}=0.$$

Exercise 4. Calculate particular solutions of differential equations, which satisfy initial conditions:

a) $y'\sin x=y\ln y$, $\quad y\left(\frac{\pi}{2}\right)=e$;

b) $2\left(1+e^x\right)\cdot y\cdot y'=e^x$, $\quad y(0)=0$;

c) $\dfrac{y-xy'}{x+yy'}=2$, $\quad y(1)=1$.

Solution:

a) $y'\sin x = y\ln y$;

$$\int\frac{dy}{y\ln y}=\int\frac{dx}{\sin x}; \quad \ln|\ln y|=\ln\left|\operatorname{tg}\frac{x}{2}\right|+\ln C; \quad \ln y = C\operatorname{tg}\frac{x}{2};$$

we apply initial conditions: $\ln e = C\operatorname{tg}\dfrac{\pi}{4}$, hence $C=1$; $\ln y=\operatorname{tg}\dfrac{x}{2}$ or $y=e^{\operatorname{tg}\frac{x}{2}}$.

b) $2\left(1+e^x\right)\cdot yy'=e^x$;

$$2y\cdot\frac{dy}{dx}=\frac{e^x}{1+e^x}; \quad 2\int y\,dy=\int\frac{e^x dx}{1+e^x}; \quad y^2=\ln\left(1+e^x\right)+C;$$

if $y(0)=0$, then $0=\ln 2+C \quad \Rightarrow \quad C=-\ln 2$;

$$y^2=\ln\left(1+e^x\right)-\ln 2; \quad y^2=\ln\frac{1+e^x}{2}; \quad \frac{1+e^x}{2}=e^{y^2}; \quad 1+e^x=2e^{y^2}.$$

c) $\dfrac{y-xy'}{x+yy'}=2$;

$$y-xy'=2x+2yy'; \quad (2y+x)y'=y-2x; \quad y'=\frac{y-2x}{2y+x};$$

$$y=ux, \quad y'=u'x+u; \quad u'x+u=\frac{ux-2x}{2ux+x}; \quad u'x+u=\frac{u-2}{2u+1};$$

$$u'x=\frac{u-2-2u^2-u}{2u+1}; \quad \frac{du}{dx}\cdot x=-\frac{2+2u^2}{2u+1}; \quad \int\frac{2u+1}{2u^2+2}du=-\int\frac{dx}{x};$$

$$\frac{1}{2}\int\frac{2u+1}{u^2+1}du=-\int\frac{dx}{x}; \quad \frac{1}{2}\int\frac{2u}{u^2+1}du+\frac{1}{2}\int\frac{du}{u^2+1}=-\int\frac{dx}{x};$$

$$\frac{1}{2}\ln\left(u^2+1\right)+\frac{1}{2}\operatorname{arctg}u=-\ln|x|+C_1;$$

$$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right)+\frac{1}{2}\operatorname{arctg}\frac{y}{x}+\ln|x|=C_1;$$

$$\ln\left(\frac{y^2}{x^2}+1\right)+\operatorname{arctg}\frac{y}{x}+2\ln|x|=C,$$

where $C=2C_1$;

$$\ln\left(\frac{y^2}{x^2}+1\right)\cdot x^2+\operatorname{arctg}\frac{y}{x}=C; \qquad \ln\left(x^2+y^2\right)+\operatorname{arctg}\frac{y}{x}=C.$$

Under the given initial conditions $y(1)=1$ we have:

$$\ln 2+\operatorname{arctg}1=C \quad \Rightarrow \quad C=\ln 2+\frac{\pi}{4}.$$

Therefore, $\ln\left(x^2+y^2\right)+\operatorname{arctg}\frac{y}{x}=\ln 2+\frac{\pi}{4}$.

III. Exercises for individual work

Exercise 1. Calculate whether a given function is a solution of a given differential equation:

a) $y=\sqrt{x}$, $2yy'=1$;

b) $S=-t-\frac{1}{2}\sin 2t$, $\frac{d^2S}{dt^2}+\operatorname{tg}t\cdot\frac{dS}{dt}=\sin 2t$.

Exercise 2. Calculate general solutions of the following differential equations:

a) $(x+1)^3\,dy-(y-2)^2\,dx=0$;

b) $yy'+x=1$;

c) $x^2(2yy'-1)=1$;

d) $y=x\left(y'-\sqrt[x]{e^y}\right)$;

e) $y^2 + x^2 \cdot y' = xyy'$;

f) $xy' = y \cdot \ln \dfrac{y}{x}$.

Exercise 3. Calculate particular solutions of differential equations that satisfy the initial conditions:

a) $y^2 + x^2 \cdot y' = 0$, $y(-1) = 1$;

b) $(1 + x^2) y^3 dx - (y^2 - 1) x^3 dy = 0$, $y(1) = -1$.

Practical lesson 2.

Linear differential equations. The Bernoulli equation. Total differential equations

I. General theoretical information

Equation of the form

$$y' + p(x)y = q(x), \tag{1}$$

where $p(x)$, $q(x)$ are determined and continuous on a certain interval functions, are called *linear differential equations of first order*.

The Bernoulli method of integration of equations (1) means finding a solution in the form $y = u \cdot v$, where $u = u(x)$ and $v = v(x)$ are the unknown functions.

Considering that $y' = u'v + uv'$, equation (1) will have the form:

$$u'v + uv' + p(x) \cdot uv = q(x). \tag{2}$$

We choose function $v(x)$ so that

$$v' + p(x) \cdot v = 0; \tag{3}$$

hence $v = e^{-\int p(x)\,dx}$ is one of solutions of equation (3).

Then $u' \cdot e^{-\int p(x)\,dx} = q(x)$, $u = \int q(x) \cdot e^{\int p(x)\,dx} dx + C$.

So, $y = u \cdot v = e^{-\int p(x)\,dx} \cdot \int q(x) \cdot e^{-\int p(x)\,dx} dx$ is a general solution of equation (1).

The Bernoulli equation

is equation of the form $y' + p(x) \cdot y = q(x) \cdot y^{\alpha}$, where $\alpha \in R$, $\alpha \neq 0$, $\alpha \neq 1$.

Substitution $z = y^{1-\alpha}$ makes the Bernoulli equation a linear differential equation.

Total differential equations

Equation of the form

$$P(x,y)dx + Q(x,y)dy = 0 \qquad (4)$$

is called total differential equation if its left side is a total differential of some function $u(x,y)$, which is

$$P(x,y)dx + Q(x,y)dy = du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\,. \qquad (5)$$

For equation (4) to be a total differential equation, a necessary and sufficient condition is: $\dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$.

We find general integral of equation (4) in the form $u(x,y) = C$, where $u(x,y) = \int P(x,y)dx = F(x,y) + \phi(y)$ (6), $F(x,y)$ is antiderivative of function $P(x,y)$ on *x*, $\phi(y)$ is an arbitrary differentiated function.

Differentiating equality (6) on *y* and considering that $\dfrac{\partial u}{\partial y} = Q(x,y)$, we get an equation to find an unknown function $\phi(y)$:

$$\frac{\partial u(x,y)}{\partial y}=\frac{\partial F(x,y)}{\partial y}+\frac{d\phi}{dy}=Q(x,y)\,.$$

II. Exercises for practical lesson

Exercise 1. Find general solutions of differential equations:

a) $y'-y\operatorname{ctg}x=\sin x$;

b) $(x\ \ +1)\cdot y'+4xy=3$;

c) $y\,dx-(3x+1+\ln y)\,dy=0$;

d) $x^2y^2\cdot y'+xy^3=1$;

e) $y'+y=x\sqrt{y}$;

f) $(e^y+ye^x+3)\,dx=(2-xe^y-e^x)\,dy$;

g) $(x+\ln|y|)\,dx+\left(1+\dfrac{x}{y}+\sin y\right)dy=0$.

Solution:

a) $y'-y\operatorname{ctg}x=\sin x$ is a linear equation, then $y=u\cdot v$, $y'=u'v+uv'$;

$$u'v+uv'-uv\cdot\operatorname{ctg}x=\sin x;\qquad u'v+u(v'-v\operatorname{ctg}x)=\sin x;$$

$$v'-v\operatorname{ctg}x=0;\qquad v'=v\operatorname{ctg}x;\qquad \int\frac{dv}{v}=\int\operatorname{ctg}x\,dx;$$

$$\ln|v|=\ln|\sin x|\quad\Rightarrow\quad v=\sin x;$$

$$u'\sin x=\sin x;\quad u'=1\quad\Rightarrow\quad u=x+C.$$

So, $y=uv=\underline{(x+C)\cdot\sin x}$.

b) $(x^2+1)\cdot y'+4xy=3$; $y'+\dfrac{4xy}{x^2+1}=\dfrac{3}{x^2+1}$ is a linear equation, then $y=uv$, $y'=u'v+uv'$;

$$u'v + uv' + \frac{4xuv}{x^2+1} = \frac{3}{x^2+1}; \qquad u'v + u\left(v' + \frac{4xv}{x^2+1}\right) = \frac{3}{x^2+1};$$

$$v' + \frac{4xv}{x^2+1} = 0; \qquad \int \frac{dv}{v} = -\int \frac{4x}{x^2+1}dx;$$

$$\ln|v| = -2\ln\left(x^2+1\right) \quad \Rightarrow \quad v = \frac{1}{\left(x^2+1\right)^2};$$

$$u' \cdot \frac{1}{\left(x^2+1\right)^2} = \frac{3}{x^2+1};$$

$$\int du = 3\int\left(x^2+1\right)dx \quad \Rightarrow \quad u = 3\left(\frac{x^3}{3} + x\right) + C = x^3 + 3x + C.$$

Therefore, $y = uv = \left(x^3 + 3x + C\right) \cdot \dfrac{1}{\left(x^2+1\right)^2}$ or $y\left(x^2+1\right)^2 = x^3 + 3x + C$.

c) $y\,dx - \left(3x + 1 + \ln y\right)dy = 0$; $\dfrac{dx}{dy} = \dfrac{3x + 1 + \ln y}{y}$ or $x' - \dfrac{3x}{y} = \dfrac{1 + \ln y}{y}$ is a linear equation, $x = x(y)$, then $x = uv$, $x' = u'v + uv'$;

$$u'v + uv' - \frac{3uv}{y} = \frac{1 + \ln y}{3}; \quad u'v + u\left(v' - \frac{3v}{y}\right) = \frac{1 + \ln y}{3};$$

$$v' - \frac{3v}{y} = 0; \quad \frac{dv}{v} = 3\int \frac{dy}{y};$$

$$\ln|v| = 3\ln|y| \quad \Rightarrow \quad v = y^3;$$

$$u' \cdot y^3 = \frac{1 + \ln y}{y}; \quad \int du = \int \frac{1 + \ln y}{y^4}dy;$$

$$u=\int\frac{1+\ln y}{y^4}dy=\begin{bmatrix}1+\ln y=t, & dt=\frac{1}{y}dy,\\ \frac{dy}{y^4}=ds, & s=\frac{y^{-3}}{-3}\end{bmatrix}=-\frac{1+\ln y}{3y^3}+\frac{1}{3}\int\frac{1}{y^4}dy=$$

$$=-\frac{1+\ln y}{3y^3}-\frac{1}{9y^3}+C=-\frac{1}{3y^3}-\frac{\ln y}{3y^3}-\frac{1}{9y^3}+C=-\frac{4}{9y^3}-\frac{\ln y}{3y^3}+C.$$

So, $x=uv=\left(-\frac{4}{9y^3}-\frac{\ln y}{3y^3}+C\right)\cdot y^3=-\frac{4}{9}-\frac{\ln y}{3}+Cy^3$ or $x=Cy^3-\frac{\ln y}{3}-\frac{4}{9}$.

d) $x^2y^2\cdot y'+xy^3=1$;

$y^2y'+\frac{y^3}{x}=\frac{1}{x^2}$ or $y'+\frac{y}{x}=x^{-2}y^{-2}$ is the Bernoulli equation ($\alpha=-2$), substitution is $z=y^{1-\alpha}=y^3$, then $z'=3y^2y'\Rightarrow y^2y'=\frac{z'}{3}$; $\frac{z'}{3}+\frac{z}{x}=\frac{1}{x^2}$ or $z'+\frac{3z}{x}=\frac{3}{x^2}$ is a linear equation, $z=z(x)$, then $z=uv$, $z'=u'v+uv'$;

$$u'v+uv'+\frac{3uv}{x}=\frac{3}{x^2};\qquad u'v+u\left(v'+\frac{3v}{x}\right)=\frac{3}{x^2};$$

$$v'+\frac{3v}{x}=0;\qquad \int\frac{dv}{v}=-3\int\frac{dx}{x};$$

$$\ln|v|=-3\ln|x|\quad\Rightarrow\quad v=\frac{1}{x^3};$$

$$u'\cdot\frac{1}{x^3}=\frac{3}{x^2};\qquad u'=3x;\qquad \int du=3\int dx;\qquad u=\frac{3x^2}{2}+C.$$

So, $z = uv = \left(\frac{3x^2}{2} + C\right) \cdot \frac{1}{x^3} = \frac{3}{2x} + \frac{C}{x^3}$.

Since $z = y^3$, then $y = \sqrt[3]{z} = \underline{\sqrt[3]{\frac{3}{2x} + \frac{C}{x^3}}}$.

e) $y' + y = x\sqrt{y}$ or $\frac{y'}{\sqrt{y}} + \frac{y}{\sqrt{y}} = x$ is the Bernoulli equation ($\alpha = \frac{1}{2}$), then substitution is $z = y^{1-\alpha} = y^{\frac{1}{2}} = \sqrt{y}$, $z' = \frac{1}{2\sqrt{y}} \cdot y' \Rightarrow$ $\frac{y'}{\sqrt{y}} = 2z'$; $2z' + z = x$ or $z' + \frac{z}{2} = \frac{x}{2}$ is a linear equation, that is why $z = uv$, $z' = u'v + uv'$;

$$u'v + uv' + \frac{uv}{2} = \frac{x}{2}; \qquad u'v + u\left(v' + \frac{v}{2}\right) = \frac{x}{2};$$

$$v' + \frac{v}{2} = 0; \qquad \frac{dv}{dx} = -\frac{v}{2}; \qquad \int \frac{dv}{v} = -\frac{1}{2}\int dx;$$

$$\ln|v| = -\frac{1}{2}x \quad \Rightarrow \quad v = e^{-\frac{x}{2}};$$

$$u' \cdot e^{-\frac{x}{2}} = \frac{x}{2}; \qquad \frac{du}{dx} = e^{\frac{x}{2}} \cdot \frac{x}{2}; \qquad \int du = \frac{1}{2}\int x \cdot e^{\frac{x}{2}}\, dx;$$

$$u = \frac{1}{2}\int x \cdot e^{\frac{x}{2}}\, dx = \begin{bmatrix} x = t, & dx = dt, \\ e^{\frac{x}{2}}\, dx = ds, & s = 2e^{\frac{x}{2}} \end{bmatrix} = \frac{1}{2}\left(2xe^{\frac{x}{2}} - 2\int e^{\frac{x}{2}}\, dx\right) =$$

$$= \frac{1}{2}\left(2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}}\right) + C = xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + C.$$

Therefore, $z = uv = \left(xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + C \right) \cdot e^{-\frac{x}{2}} = x - 2 + C \cdot e^{-\frac{x}{2}}$;
considering that $z = \sqrt{y}$, we get $y = z^2 = \underline{\left(x - 2 + C \cdot e^{-\frac{x}{2}} \right)^2}$.

f) $\left(e^y + ye^x + 3\right)dx = \left(2 - xe^y - e^x\right)dy$;

$\left(e^y + ye^x + 3\right)dx + \left(xe^y + e^x - 2\right)dy = 0$;

$\frac{\partial P}{\partial y} = e^y + e^x$, $\frac{\partial Q}{\partial x} = e^y + e^x$, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\Rightarrow$ total differential equation.

We find

$u(x, y) = \int \left(e^y + ye^x + 3\right)dx + \phi(y) = e^y x + ye^x + 3x + \phi(y)$;

$\frac{\partial u}{\partial y} = xe^y + e^x + \phi'(y)$.

Since $\frac{\partial u}{\partial y} = Q(x, y)$ we get: $xe^y + e^x + \phi'(y) = xe^y + e^x - 2$;
$\phi'(y) = -2 \Rightarrow \phi(y) = -2y + C_1$, then $u = xe^y + ye^x + 3x - 2y + C_1$.

Since $u(x, y) = C_2$, then $xe^y + ye^x + 3x - 2y + C_1 = C_2$ or $xe^y + ye^x + 3x - 2y = C$.

g) $\left(x + \ln|y|\right)dx + \left(1 + \frac{x}{y} + \sin y\right)dy = 0$ is a total differential equation, since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1 + \frac{1}{y}$.

We find $u(x, y) = \int \left(x + \ln|y|\right)dx + \phi(y) = \frac{x^2}{2} + x\ln|y| + \phi(y)$;

$\frac{\partial u}{\partial y} = \frac{x}{y} + \phi'(y)$; $\frac{\partial u}{\partial y} = Q(x, y) \Rightarrow \frac{x}{y} + \phi'(y) = 1 + \frac{x}{y} + \sin y$;

$\phi'(y)=1+\sin y$; $\phi(y)=\int(1+\sin y)dy = y-\cos y + C_1$, then

$u(x,y)=\dfrac{x^2}{2}+x\ln|y|+y-\cos y+C_1$.

Since $u(x,y)=C_2$, then $\dfrac{x^2}{2}+x\ln|y|+y-\cos y=C$.

Exercise 2. Find partial solutions of differential equations under initial conditions:

a) $xy'+y-e^x=0$, $\quad y(a)=b$;

b) $xy'-\dfrac{y}{x+1}=x$, $\quad y(1)=0$;

c) $y\,dx+2x\,dy=2y\sqrt{x}\sec^2 y\,dy$, $\quad y(0)=\pi$;

d) $\left(2x+y\cdot e^{xy}\right)dx+\left(1+x\cdot e^{xy}\right)dy=0$, $\quad y(0)=1$.

Solution:

a) $xy'+y-e^x=0$ or $y'+\dfrac{y}{x}=\dfrac{e^x}{x}$ is a linear equation, then $y=uv$, $y'=u'v+uv'$;

$$u'v+uv'+\frac{uv}{x}=\frac{e^x}{x};\qquad u'v+u\left(v'+\frac{v}{x}\right)=\frac{e^x}{x};$$

$$v'+\frac{v}{x}=0;\qquad \frac{dv}{dx}=-\frac{v}{x};\qquad \int\frac{dv}{v}=-\int\frac{dx}{x};$$

$$\ln|v|=-\ln|x| \quad\Rightarrow\quad v=\frac{1}{x};$$

$$u'\cdot\frac{1}{x}=\frac{e^x}{x};\qquad \int du=\int e^x\,dx;\qquad u=e^x+C.$$

Therefore, $y=uv=\left(e^x+C\right)\cdot\dfrac{1}{x}$ is a general solution of equation.

We apply initial condition $y(a)=b$:

$$b=\left(e^a+C\right)\cdot\frac{1}{a} \quad\Rightarrow\quad ab=e^a+C \quad\Rightarrow\quad C=ab-e^a;$$

$$y=\frac{1}{x}\cdot\left(e^x+ab-e^a\right).$$

b) $xy'-\frac{y}{x+1}=x$ or $y'-\frac{y}{x(x+1)}=1$ is a linear equation, then $y=uv$, $y'=u'v+vu'$;

$$u'v+uv'-\frac{uv}{x(x+1)}=1; \quad u'v+u\left(v'-\frac{v}{x(x+1)}\right)=1;$$

$$v'-\frac{v}{x(x+1)}=0; \quad \frac{dv}{dx}=\frac{v}{x(x+1)}; \quad \int\frac{dv}{v}=\int\frac{dx}{x(x+1)};$$

$\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1} \quad\Rightarrow A(x+1)+Bx\equiv 1$, we assume that $x=0$, then $A=1$; if $x=-1$, then $B=-1$, so

$$\int\frac{dv}{v}=\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx; \quad \ln|v|=\ln|x|-\ln|x+1| \quad\Rightarrow\quad v=\frac{x}{x+1};$$

$$u'\cdot\frac{x}{x+1}=1; \quad \frac{du}{dx}=\frac{x+1}{x}; \quad \int du=\int\left(1+\frac{1}{x}\right)dx; \quad u=x+\ln|x|+C.$$

Therefore, $y=uv=\left(x+\ln|x|+C\right)\cdot\frac{x}{x+1}$ is a general solution of equation.

Out of initial condition $y(1)=0$ we find C:

$$0=(1+C)\cdot\frac{1}{2} \Rightarrow C=-1,$$

then $y=\left(x+\ln|x|-1\right)\cdot\frac{x}{x+1}$ is a partial solution.

c) $y\,dx + 2x\,dy = 2y\sqrt{x}\sec^2 y\,dy$ or, divided by $y\,dy$, $\dfrac{dx}{dy} + \dfrac{2x}{y} = 2\sqrt{x}\sec^2 y$ is a Bernoulli equation, $x = x(y)$, $\alpha = \dfrac{1}{2}$,

$z = x^{1-\alpha} = x^{\frac{1}{2}}$; $z' = \dfrac{1}{2\sqrt{x}} \cdot x' \;\Rightarrow\; \dfrac{x'}{\sqrt{x}} = 2z'$.

Then $\dfrac{x'}{\sqrt{x}} + \dfrac{2\sqrt{x}}{y} = 2\sec^2 y$ after substitution is: $2z' + \dfrac{2z}{y} = 2\sec^2 y$ or $z' + \dfrac{z}{y} = \sec^2 y$ is a linear equation, then $z = uv$, $z' = u'v + uv'$;

$$u'v + uv' + \frac{uv}{y} = \sec^2 y; \qquad u'v + u\left(v' + \frac{v}{y}\right) = \sec^2 y;$$

$$v' + \frac{v}{y} = 0; \qquad \frac{dv}{dy} = -\frac{v}{y}; \qquad \int \frac{dv}{v} = -\int \frac{dy}{y} \;\Rightarrow\; v = \frac{1}{y};$$

$$u' \cdot \frac{1}{y} = \sec^2 y; \qquad \int du = \int y\sec^2 y\,dy;$$

$$\int y \cdot \sec^2 y\,dy = \begin{bmatrix} y = t, & dy = dt, \\ \sec^2 y\,dy = dv, & v = \operatorname{tg} y \end{bmatrix} =$$

$$= y\operatorname{tg} y - \int \operatorname{tg} y\,dy = y\operatorname{tg} y + \ln|\cos y| + C; \quad u = y\operatorname{tg} y + \ln|\cos y| + C.$$

Hence $z = uv = \left(y\operatorname{tg} y + \ln|\cos y| + C\right) \cdot \dfrac{1}{y}$. Since $x = z^2$, then $x = \left(y\operatorname{tg} y + \ln|\cos y| + C\right)^2 \cdot \dfrac{1}{y^2}$ is a general solution of the equation.

Under the initial condition $y(0) = \pi$ we calculate the value of C:

$$0=\left(\pi\operatorname{tg}\pi+\ln\left|\cos\pi\right|+C\right)^2\cdot\frac{1}{\pi^2}\quad\Rightarrow\quad C=0\,.$$

So, $x=\left(y\operatorname{tg}y+\ln\left|\cos y\right|\right)^2\cdot\dfrac{1}{y^2}$ or $xy^2=\left(y\operatorname{tg}y+\ln\left|\cos y\right|\right)^2$.

d) $\left(2x+y\cdot e^{xy}\right)dx+\left(1+x\cdot e^{xy}\right)dy=0$ is a total differential equation, since $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}=e^{xy}+xye^{xy}$.

We find function

$u(x,y)=\int\left(2x+ye^{xy}\right)dx+\phi(y)=x^2+e^{xy}+\phi(y)$;

$\dfrac{\partial u}{\partial y}=x\cdot e^{xy}+\phi'(y)$. Since $\dfrac{\partial u}{\partial y}=Q(x,y)$:

$$xe^{xy}+\phi'(y)=1+xe^{xy}\Rightarrow\phi'(y)=1\ \Rightarrow\ \phi(y)=y+C_1\,.$$

Therefore, $u(x,y)=x^2+e^{xy}+y+C_1$; but $u(x,y)=C_2$, hence $x^2+e^{xy}+y+C_1=C_2$ or $x^2+e^{xy}+y=C$ is a general solution of the equation.

Let us now calculate a particular solution under the initial condition $y(0)=1$:

$$e^0+1=C\quad\Rightarrow\quad C=2\text{, so } x^2+e^{xy}+y=2\,.$$

III. Exercises for individual work

Exercise 1. Calculate general solutions of differential equation:

a) $y'-y=e^x$;

b) $y'+\dfrac{1-2x}{x^2}\cdot y=1$;

c) $y' + \dfrac{y}{x+1} + y^2 = 0$;

d) $x\,dx = \left(\dfrac{x^2}{y} - y^3 \right) dy$;

e) $\left(3x^2y^2 + 7\right)dx + 2x^3y\,dy = 0$;

f) $\left(2y-3\right)dx + \left(2x + 3y^2\right)dy = 0$.

Exercise 2. Calculate particular solutions of differential equations under the initial conditions:

a) $\left(1-x\right)\left(y'+y\right) = e^{-x}$, $y\left(2\right) = 0$;

b) $y' = 3x^2y + x^5 + x^2$, $y\left(0\right) = 1$;

c) $t\left(1+t^2\right)dx = \left(x + xt^2 - t^2\right)dt$, $x\left(1\right) = -\dfrac{\pi}{4}$.

Practical lesson 3.

Differential equations of higher order. Equations of higher order that allow the decrease of order

I. Basic theoretical information

Definition 1. Differential equations which contain derivatives of higher order are called *differential equations of higher order.*

The order of the highest derivative of an unknown function which is included in a differential equation, is called *the order of this equation.*

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0 \tag{1}$$

is a differential equation of n-order, written in the form $\Phi(x, y, C) = 0$, where x is an independent variable, $y = y(x)$ is an unknown function, F is a known function.

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \tag{2}$$

is a normal differential equation of n-order, calculated with respect to derivative $y^{(n)}$.

Definition 2. *A solution of equation* (2) on a certain interval $(a;b)$ is called n times differentiated on this interval function $\phi(x)$, which under substitution into the given equation turns it into a congruence on x, that is $\forall x \in (a;b)$: $\phi^{(n)}(x) \equiv f\left(x, \phi(x), \phi'(x), \dots, \phi^{(n-1)}(x)\right)$.

The Cauchy task means finding among all the solutions of equation (2) such a solution $y = y(x)$, $x \in (a;b)$, which if $x = x_0 \in (a;b)$ satisfies the following initial conditions:

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad \dots, \quad y^{(n-1)}(x_0) = y_0^{(n-1)}. \qquad (3)$$

Theorem 1 (about existence and uniqueness of solution of the Cauchy task).

If function $f\left(x, y, y', \dots, y^{(n-1)}\right)$ and its partial derivatives upon arguments y, y', …, $y^{(n-1)}$ are continuous in a certain open area G of function, then for any point $\left(x_0, y_0, y_0', \dots, y_0^{(n-1)}\right) \in G$ there exists one unique solution $y = y(x)$ equation (2), which satisfies initial conditions (3).

A general solution of a differential equation of n-order (2) contains n arbitrary constants and has the following form:

$$y = \phi(x, C_1, C_2, \dots, C_n). \qquad (4)$$

If a general solution of equation is written in the form $\Phi(x, C_1, C_2, \dots, C_n) = 0$ (5), then it is called *a general integral* of equation (2).

A particular solution or *a particular integral* is calculated on the basis of a general one, if in (4) or (5) every arbitrary constant C_1, C_2, … , C_n is given a certain number value.

Equations of higher order that allow the decrease of order

1) *Equation of the form* $y^{(n)} = f(x)$, where $f(x)$ is a continuous function, solved by means of integrating it *n* times.

2) *Second-order equation* $F(x, y', y'') = 0$, which does not contain unknown function, turns into the first-order equation by means of substitution $y' = z$, where $z = z(x)$, hence $y'' = z'$.

3) Second-order equation $F(y, y', y'') = 0$, which does not contain independent variable *x*, allows the decrease of order by means of substitution $y' = z$, where $z = z(y)$, then $y'' = z'_y \cdot z$ ($y'' = \dfrac{d}{dx}(y') = \dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx} = z \cdot \dfrac{dz}{dy}$).

II. Exercises for practical lesson

Exercise 1. Calculate general solutions of differential equations:

a) $y''' = e^{2x}$;

b) $y''' = \dfrac{1}{x}$;

c) $y'' = \operatorname{arctg} x$;

d) $x^2 \cdot y''' = (y'')^2$;

e) $x \cdot y^{(5)} = y^{(4)}$.

Solution:

a) $y''' = e^{2x}$.

We integrate three times and calculate a general solution of equation:

$$y'' = \int e^{2x} dx = \frac{1}{2}e^{2x} + C_1 ; \qquad y' = \frac{1}{4}e^{2x} + C_1' x + C_2 ;$$

$$y = \frac{1}{8}e^{2x} + C_1 x^2 + C_2 x + C_3 \ \left(C_1 = \frac{C_1'}{2} \right).$$

b) $y''' = \dfrac{1}{x}$;

$$y'' = \ln|x| + C_1 ; \qquad y' = \int \ln|x|\, dx + C_1' x ;$$

$$\int \ln x\, dx = \begin{bmatrix} \ln x = u, & \frac{1}{x}dx = du, \\ dx = dv, & v = x \end{bmatrix} = x\ln x - \int dx = x\ln x - x + C ;$$

$$y' = x\ln x - x + C_1' \cdot \frac{x^2}{2} + C_2 ;$$

$$\int x\ln x\, dx = \begin{bmatrix} \ln x = u, & \frac{1}{x}dx = du, \\ x\,dx = dv, & v = \frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C;$$

$$y = \frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{x^2}{2} + C_1' \cdot \frac{x^2}{2} + C_2 x + C_3 = \underline{\frac{x^2}{2}\ln x + C_1 x^2 + C_2 x + C_3}$$

$$\left(C_1 = -\frac{3}{4} + \frac{C_1'}{2} \right).$$

c) $y'' = \operatorname{arctg} x$;

$$y' = \int \operatorname{arctg} x\,dx = \begin{bmatrix} \operatorname{arctg} x = u, & du = \dfrac{dx}{1+x^2}, \\ dx = dv, & v = x \end{bmatrix} = x \cdot \operatorname{arctg} x - \int \frac{x\,dx}{1+x^2} =$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2}\ln\left(1+x^2\right) + C_1;$$

$$I_1 = \int x \operatorname{arctg} x\,dx = \begin{bmatrix} \operatorname{arctg} x = u, & du = \dfrac{dx}{1+x^2}, \\ x\,dx = dv, & v = \dfrac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2 dx}{1+x^2} =$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\left(x - \operatorname{arctg} x\right) + C;$$

$$I_2 = \int \ln\left(1+x^2\right)dx = \begin{bmatrix} \ln\left(1+x^2\right) = u, & \dfrac{2x}{1+x^2}dx = du, \\ dx = dv, & v = x \end{bmatrix} =$$

$$= x \cdot \ln\left(1+x^2\right) - 2\int \frac{x^2 dx}{1+x^2} = x \cdot \ln\left(1+x^2\right) - 2\left(x - \operatorname{arctg} x\right) + C;$$

$$y = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\left(x - \operatorname{arctg} x\right) - \frac{x}{2}\ln\left(1+x^2\right) + \left(x - \operatorname{arctg} x\right) + C_1 x + C_2 =$$

$$= \underline{\frac{x^2}{2}\operatorname{arctg} x + \frac{x}{2} - \frac{1}{2}\operatorname{arctg} x - \frac{x}{2}\ln\left(1+x^2\right) + C_1 x + C_2}.$$

d) $x^2 \cdot y''' = \left(y''\right)^2$;

We label $y'' = u$, then $y''' = u' \Rightarrow x^2 \cdot u' = u^2$ is an equation with separable variables;

$$x^2 \cdot \frac{du}{dx} = u^2\,; \qquad \int \frac{du}{u^2} = \int \frac{dx}{x^2}\,;$$

$$\frac{1}{u} = \frac{1}{x} + C_1 \quad \Rightarrow \quad u = \frac{x}{1 + C_1 x}\,;$$

$y'' = \dfrac{x}{1 + C_1 x}$. We integrate the equation twice and calculate a general solution:

$$y' = \int \frac{x}{1 + C_1 x} dx = \frac{1}{C_1} \int \left(1 - \frac{1}{1 + C_1 x} \right) dx = \frac{1}{C_1} \left(x - \frac{1}{C_1} \ln\left|1 + C_1 x\right| \right) + C_2\,;$$

$$y = \frac{1}{C_1} \int \left(x - \frac{1}{C_1} \ln\left|1 + C_1 x\right| \right) dx + C_2 x\,;$$

$$\int \ln\left|1 + C_1 x\right| dx = \begin{bmatrix} \ln\left|1 + C_1 x\right| = u, & du = \dfrac{C_1}{1 + C_1 x}, \\ dx = dv, & v = x \end{bmatrix} = x \cdot \ln\left|1 + C_1 x\right| - \int \frac{C_1 x}{1 + C_1 x} dx =$$

$$= x \ln\left|1 + C_1 x\right| - \int \left(1 - \frac{1}{1 + C_1 x} \right) dx = x \ln\left|1 + C_1 x\right| - x + \frac{1}{C_1} \ln\left|1 + C_1 x\right|\,;$$

$$y = \frac{x^2}{2C_1} - \frac{1}{C_1} \left(x \ln\left|1 + C_1 x\right| - x + \frac{1}{C_1} \ln\left|1 + C_1 x\right| \right) + C_2 x + C_3 =$$

$$= \underline{\frac{x^2}{2C_1} + \left(\frac{1}{C_1} + C_2 \right) \cdot x + \left(\frac{1}{C_1} - \frac{x}{C_1} \right) \cdot \ln\left|1 + C_1 x\right| + C_3}\,.$$

e) $x \cdot y^{(5)} = y^{(4)}$.

We label $y^{(4)} = u$, $y^{(5)} = u'$; then $x \cdot u' = u$; $\displaystyle\int \frac{du}{u} = \int \frac{dx}{x}$; $\ln|u| = \ln|x| \cdot C_1' \Rightarrow u = C_1' \cdot x$;

$$y^{(4)} = C_1'x\,; \qquad y''' = C_1'\frac{x^2}{2} + C_2'\,; \qquad y'' = C_1' \cdot \frac{x^3}{6} + C_2' \cdot x + C_3'\,;$$

$$y' = C_1' \cdot \frac{x^4}{24} + C_2' \cdot \frac{x^2}{2} + C_3' \cdot x + C_4\,;$$

$$y = C_1' \cdot \frac{x^5}{120} + C_2' \cdot \frac{x^3}{6} + C_3' \cdot \frac{x^2}{2} + C_4 x + C_5\,;$$

$$y = C_1x^5 + C_2x^3 + C_3x^2 + C_4x + C_5\ (C_1 = \frac{C_1'}{120};\ \ C_2 = \frac{C_2'}{6};\ \ C_3 = \frac{C_3'}{2}).$$

Exercise 2. Calculate general solutions of equations:

a) $x \cdot y'' = y'$;

b) $y'' = y' + x$;

c) $xy'' = y' \cdot \ln\frac{y'}{x}$;

d) $1 + (y')^2 = 2y \cdot y''$;

e) $(y')^2 + 2y \cdot y'' = 0$;

f) $y \cdot y'' - (y')^2 = y^2 \cdot y'$.

Solution:

Equations a)-c) do not contain obvious variable *y*, substitution $y' = z$, $y'' = z'$; equations d)-f) do not contain obvious variable *x*, substitution $y' = z$, $y'' = z' \cdot z$.

a) $x \cdot y'' = y'$; $y' = z$, $y'' = z'$ $\Rightarrow$ $x \cdot z' = z$ – separable equations; $\int\frac{dz}{z} = \int\frac{dx}{x}$; $\ln|z| = \ln|x| \cdot C_1'$ $\Rightarrow$ $z = C_1'x$;

$$y' = C_1'x\,;$$

$$y = C_1' \cdot \frac{x^2}{2} + C_2 = \underline{C_1x^2 + C_2}\ \ (C_1 = \frac{C_1'}{2}).$$

b) $y'' = y' + x$; $y' = z$, $y'' = z'$ $\Rightarrow$ $z' = z + x$ or $z' - z = x$ is a linear equation, then $z = uv$, $z' = u'v + uv'$; $u'v + uv' - uv = x$;

$$v' = v; \quad \ln|v| = x \quad \Rightarrow \quad v = e^x; \quad u' \cdot e^x = x; \quad \int du = \int e^{-x} \cdot x\,dx;$$

$$\int e^{-x} \cdot x\,dx = \begin{bmatrix} x = t, & dx = dt, \\ e^{-x}dx = dv, & v = -e^{-x} \end{bmatrix} = -x \cdot e^{-x} + \int e^{-x}dx = -x \cdot e^{-x} - e^{-x} + C;$$

$$u = -e^{-x}\left(x+1\right) + C_1; \quad z = uv = \left(-e^{-x}\left(x+1\right) + C_1\right) \cdot e^x = -x - 1 + C_1e^x;$$

$$y' = -x - 1 + C_1e^x; \quad y = \int\left(-x - 1 + C_1e^x\right)dx = \underline{-\frac{x^2}{2} - x + C_1e^x + C_2}.$$

c) $xy'' = y' \cdot \ln\dfrac{y'}{x}$; $y' = z$, $y'' = z'$; $xz' = z\ln\dfrac{z}{x}$ or $z' = \dfrac{z}{x}\ln\dfrac{z}{x}$ is a homogeneous equation, then $z = ux$, $z' = u'x + u$;

$$u'x + u = \frac{ux}{x} \cdot \ln\frac{ux}{x}; \quad u'x + u = u \cdot \ln u; \quad u'x = u\ln u - u;$$

$$\frac{du}{dx} \cdot x = u\left(\ln u - 1\right); \quad \int\frac{du}{u\left(\ln u - 1\right)} = \int\frac{dx}{x};$$

$$\ln\left|\ln u - 1\right| = \ln x \cdot C_1 \quad \Rightarrow \quad \ln u - 1 = C_1x;$$

$$\ln u = C_1x + 1 \quad \Rightarrow \quad u = e^{C_1x+1};$$

$$\frac{z}{x} = e^{C_1x+1}; \quad z = xe^{C_1x+1}; \quad y' = xe^{C_1x+1}; \quad \int dy = \int xe^{C_1x+1}dx;$$

$$\int xe^{C_1x+1}dx = \begin{bmatrix} x = u, & dx = du, \\ e^{C_1x+1}dx = dv, & v = \dfrac{1}{C_1} \cdot e^{C_1x+1} \end{bmatrix} =$$

$$= \frac{x}{C_1}e^{C_1x+1} - \frac{1}{C_1}\int e^{C_1x+1}dx = \frac{x}{C_1}e^{C_1x+1} - \frac{1}{C_1^2}e^{C_2x+1};$$

$$y = \left(\frac{x}{C_1} - \frac{1}{C_1^2}\right) \cdot e^{C_1x+1} + C' = \underline{\left(Cx - C^2\right) \cdot e^{\frac{x}{C}+1} + C'} \ \left(C = \frac{1}{C_1}\right).$$

d) $1+\left(y'\right)^2=2y\cdot y''$; $y'=z$, $y''=z'\cdot z$; $1+z^2=2y\cdot z'\cdot z$ is a separable equation;

$$2yz\cdot\frac{dz}{dy}=1+z^2;\quad \int\frac{2z\,dz}{1+z^2}=\int\frac{dy}{y};$$

$$\ln\left(1+z^2\right)=\ln|y|\cdot C_1 \quad\Rightarrow\quad 1+z^2=C_1y;$$

$$z=\pm\sqrt{C_1y-1},\ \text{ hence }\ y'=\pm\sqrt{C_1y-1};\quad \pm\int\frac{dy}{\sqrt{C_1y-1}}=\int dx;$$

$$\pm\frac{2}{C_1}\sqrt{C_1y-1}=x+C_2 \quad\Rightarrow\quad \frac{4}{C_1^2}\left(C_1y-1\right)=\left(x+C_2\right)^2.$$

e) $\left(y'\right)^2+2y\cdot y''=0$; $y' \quad z$, $y''=z'\cdot z$; $z^2+2yz'\cdot z=0$;

$$2z'y=-z;\quad 2\int\frac{dz}{z}=-\int\frac{dy}{y};$$

$$2\ln|z|=-\ln|y|\cdot C \quad\Rightarrow\quad z=\frac{1}{\sqrt{Cy}};$$

$$z=\frac{C_1'}{\sqrt{y}}\quad (C_1'=\frac{1}{\sqrt{C}});\quad y'=\frac{C_1'}{\sqrt{y}};\quad \int\sqrt{y}\,dy=C_1'\int dx;$$

$$\frac{2}{3}y^{\frac{3}{2}}=C_1'x+C_2';\quad y^{\frac{3}{2}}=\frac{3}{2}C_1'x+\frac{3}{2}C_2';$$

$$y=\left(C_1x+C_2\right)^{\frac{2}{3}}\quad (C_1=\frac{3}{2}C_1',\ C_2=\frac{3}{2}C_2').$$

f) $y\cdot y''-\left(y'\right)^2=y^2\cdot y'$; $y'=z$, $y''=z'\cdot z$;

$y\cdot z'\cdot z-z^2=y^2z$ or $z'-\dfrac{z}{y}=y$ is a linear equation, then $z=uv$, $z'=u'v+uv'$;

$$u'v+uv'-\frac{uv}{y}=y;\quad v'=\frac{v}{y};\quad \int\frac{dv}{v}=\int\frac{dy}{y};$$

$$\ln|v| = \ln|y| \quad \Rightarrow \quad v = y;$$

$u' \cdot y = y; \quad u' = 1 \quad \Rightarrow \quad u = y + C_1; \quad z = y(y + C_1);$ thus

$$y' = y(y + C_1); \quad \int \frac{dy}{y(y + C_1)} = \int dx;$$

$$\frac{1}{y(y + C_1)} = \frac{A}{y} + \frac{B}{y + C_1} \quad \Rightarrow \quad A(y + C_1) + By \equiv 1;$$

we calculate coefficients A and B: if $y = 0$, then

$AC_1 = 1 \Rightarrow A = \dfrac{1}{C_1}$; if $y = -C_1$, then $-C_1 \cdot B = 1 \Rightarrow B = -\dfrac{1}{C_1}$;

$\displaystyle\int \frac{dy}{y(y + C_1)} = \int \left(\frac{1}{C_1 y} - \frac{1}{C_1} \cdot \frac{1}{y + C_1} \right) dy = \frac{1}{C_1} \ln\left| \frac{y}{y + C_1} \right|$; so

$$\frac{1}{C_1} \ln\left| \frac{y}{y + C_1} \right| = x + C_2.$$

Exercise 3. Calculate partial solutions of differential equations:

a) $y''(x^2 + 1) = 2xy'$, $y(0) = 1$, $y'(0) = 3$;

b) $xy'' + x \cdot (y')^2 - y' = 0$, $y(2) = 2$, $y'(2) = 1$;

c) $y \cdot y'' = (y')^2 - (y')^3$, $y(1) = 1$, $y'(1) = -1$;

d) $y^3 \cdot y'' = -1$, $y(1) = 1$, $y'(1) = 0$.

Solutions:

a) $y''(x^2 + 1) = 2xy'$; substitution $y' = z$, $y'' = z'$, then

$$z'(x^2 + 1) = 2xz; \quad \int \frac{dz}{z} = \int \frac{2x\,dx}{x^2 + 1};$$

$$\ln|z| = \ln\left(x^2+1\right)\cdot C_1 \quad \Rightarrow \quad z = C_1\left(x^2+1\right);$$

$$z = y' \quad \Rightarrow \quad y' = C_1\left(x^2+1\right).$$

We apply condition $y'(0)=3 \Rightarrow C_1=3$, then $y'=3\left(x^2+1\right)$; $y=3\int\left(x^2+1\right)dx$; $y=3\left(\frac{x^3}{3}+x\right)+C_2$; $y=x^3+3x+C_2$.

Under the condition $y(0)=1$ we calculate $C_2=1$, so $y=x^3+3x+1$.

b) $xy''+x\cdot\left(y'\right)^2-y'=0$; substitution $y'=z$, $y''=z'$, then $xz'+xz^2-z=0$; $z'-\frac{z}{x}=-z^2$ or $-\frac{z'}{z^2}+\frac{1}{zx}=1$ is a Bernoulli equation ($\alpha=2$), $p=z^{1-\alpha}=z^{-1}$, $p'=-\frac{1}{z^2}\cdot z'$.

Then, $p'+\frac{p}{x}=1$ is a linear equation, $p=uv$, $p'=u'v+uv'$; $u'v+uv'+\frac{uv}{x}=1$; $v'=-\frac{v}{x} \Rightarrow v=\frac{1}{x}$; $u'\cdot\frac{1}{x}=1 \Rightarrow u=\frac{x^2}{2}+C_1$; $p=uv=\frac{1}{x}\left(\frac{x^2}{2}+C_1\right)=\frac{x^2+2C_1}{2x}$; $z=\frac{1}{p}=\frac{2x}{x^2+2C_1}$; $z=y' \Rightarrow$ $y'=\frac{2x}{x^2+2C_1}$.

If $y'(2)=1$, then $1=\frac{4}{4+2C_1} \Rightarrow C_1=0$, and $y'=\frac{2x}{x^2}=\frac{2}{x}$;

$$y=\int\frac{2}{x}dx; \quad y=2\ln|x|+C_2; \quad y=\ln x^2+C_2.$$

Under the condition that $y(2)=2$: $C_2=2-\ln 4$.

So, $y=\ln x^2+2-\ln 4$ or $y=\ln\frac{x^2}{4}+2$.

c) $y \cdot y'' = (y')^2 - (y')^3$; substitution $y' = z$, $y'' = z' \cdot z$;
$y \cdot z' \cdot z = z^2 - z^3$;

$$\frac{z \cdot z'}{z^2(1-z)} = \frac{1}{y}; \qquad \int \frac{dz}{z(z-1)} = -\int \frac{dy}{y};$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}; \qquad A(z-1) + Bz \equiv 1;$$

$$z = 1 \quad \Rightarrow \quad B = 1, \quad z = 0 \quad \Rightarrow \quad A = -1;$$

$$\int \frac{dz}{z(z-1)} = -\int \frac{dz}{z} + \int \frac{dz}{z-1} = -\ln|z| + \ln|z-1| = \ln\left|\frac{z-1}{z}\right|;$$

$$\ln\left|1 - \frac{1}{z}\right| = -\ln|y| \cdot C_1 \quad \Rightarrow \quad 1 - \frac{1}{z} = \frac{1}{C_1 y}; \quad \frac{1}{z} = 1 - \frac{1}{C_1 y};$$

$$\frac{1}{z} = \frac{C_1 y - 1}{C_1 y} \quad \Rightarrow \quad z = \frac{C_1 y}{C_1 y - 1}; \quad z = y' \quad \Rightarrow \quad y' = \frac{C_1 y}{C_1 y - 1}.$$

Under the condition $y'(1) = -1$: $-1 = \dfrac{C_1}{C_1 - 1} \quad \Rightarrow \quad C_1 = \dfrac{1}{2}$.

So, $y' = \dfrac{\frac{1}{2} y}{\frac{1}{2} y - 1}$; $y' = \dfrac{y}{y-2}$; $\dfrac{y-2}{y} dy = dx$; $\displaystyle\int \left(1 - \frac{2}{y}\right) dy = \int dx$;
$y - 2\ln|y| = x + C_2$.

Since $y(1) = 1$, then $1 = 1 + C_2 \Rightarrow C_2 = 0$.

So, $y - 2\ln y = x$ or $y - x = 2\ln|y|$.

d) $y^3 \cdot y'' = -1$; substitution $y' = z$, $y'' = z' \cdot z$;

$y^3 \cdot z' \cdot z = -1$; $\displaystyle\int z\,dz = -\int \frac{1}{y^3} dy$; $\dfrac{z^2}{2} = \dfrac{1}{2y^2} + C_1$; $z^2 = \dfrac{1}{y^2} + 2C_1$;

$$z = y' \quad \Rightarrow \quad (y')^2 = \frac{1}{y^2} + 2C_1.$$

We apply condition $y'(1)=0 \Rightarrow C_1=-\frac{1}{2}$. So, $(y')^2=\frac{1}{y^2}-1$;

$y'=\sqrt{\frac{1}{y^2}-1}$; $\int\frac{y\,dy}{\sqrt{1-y^2}}=\int dx$; $-\sqrt{1-y^2}=x+C_2$.

According to $y(1)=1$ we calculate $C_2=-1$, then

$\sqrt{1-y^2}=1-x$; $1-y^2=(1-x)^2$ thus $y=\sqrt{2x-x^2}$.

III. Exercises for individual work

Exercise 1. Calculate general solutions of differential equations:

a) $y'''=60x^2$;

b) $y'''=\cos 2x$;

c) $xy''=y'$;

d) $(y'')^2=y'$;

e) $y''=\frac{y'}{x}+x$;

f) $2xy'\cdot y''=(y')^2+1$;

g) $y\cdot y''=(y')^2$;

h) $y(1-\ln y)\cdot y''+(1+\ln y)\cdot(y')^2=0$.

Exercise 2. Calculate particular solutions of differential equations under the initial conditions:

a) $y''=3x^2$, $y(0)=2$, $y'(0)=1$;

b) $y''=\frac{y'}{x}+\frac{x^2}{y'}$, $y(2)=0$, $y'(2)=4$;

c) $2(y')^2=y''\cdot(y-1)$, $y(1)=2$, $y'(1)=-1$.

Practical lesson 4.

Linear differential equations of higher order. Constant-coefficient equations. Nonhomogeneous equations with a special right-hand side

I. General theoretical information

Definition 1. Equation

$$b_0(x)\cdot y^{(n)}+b_1(x)\cdot y^{(n-1)}+\ldots+b_n(x)\cdot y=\phi(x), \tag{1}$$

where $b_0(x)$, $b_1(x)$, … , $b_n(x)$, $\phi(x)$ are the given functions, and the unknown function $y=y(x)$ and all its derivatives are only in first degree, is called *a linear differential equation of n-order.*

Functions $b_i(x)$, $i=0, 1, \ldots, n$ are called the coefficients of the equation. If $\phi(x)$ then equation (1) is called homogeneous; if $\phi(x)\equiv 0$, then it is nonhomogeneous.

Definition 2. Equations

$$y''+a_1(x)\cdot y'+a_2(x)\cdot y=0, \tag{2}$$

where $a_1(x)$, $a_2(x)$ are the given functions, are called *linear homogeneous differential equations of second order.*

Theorem 1. If functions $y_1(x)$ and $y_2(x)$ are the solutions of equation (2), then function $y = C_1 \cdot y_1(x) + C_2 \cdot y_2(x)$ (C_1, C_2 are arbitrary constants), is equally a solution of this equation.

Definition 3. Functions $y_1(x)$ and $y_2(x)$ are called *linearly independent on interval* $(a;b)$, if

$$\alpha_1 \cdot y_1(x) + \alpha_2 \cdot y_2(x) \equiv 0, \tag{3}$$

where α_1, α_2 are the real numbers, is valid only when $\alpha_1 = \alpha_2 = 0$. If at least one of the numbers α_1 or α_2 is not equal to zero and the congruence (3) takes place, then functions $y_1(x)$ and $y_2(x)$ are called *linearly independent on interval* $(a;b)$.

Functions $y_1(x)$ and $y_2(x)$ are only linearly dependent on interval $(a;b)$, when there is such a constant number l that for all $x \in (a;b)$ the valid equality is $\frac{y_1(x)}{y_2(x)} = \lambda$ or $y_1(x) = \lambda \cdot y_2(x)$.

Definition 4. If $y_1 = y_1(x)$ and $y_2 = y_2(x)$ the functions of x, then a determinant $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_1' \cdot y_2$ is called the Wronskian determinant or the Wronskian.

Theorem 2. If functions $y_1(x)$ and $y_2(x)$ are differentiated and linearly dependent on interval $(a;b)$, then the Wronskian $W(x) \equiv 0$ on $(a;b)$.

Theorem 3. If functions $y(x)$ and $y_2(x)$ are linearly independent solutions of equation (2) on interval $(a;b)$, then the Wronskian of these functions is not equal to zero in any of the points on this interval.

Now it is possible to formulate conditions, according to which function $y = C_1 \cdot y_1(x) + C_2 \cdot y_2(x)$ is a general solution of equation (2).

Theorem 4 (about the structure of a general solution of a homogeneous equation). If functions $y_1(x)$ and $y_2(x)$ are two linearly independent on interval $(a;b)$ solutions of equation (2), then function

$$y = C_1 \cdot y_1(x) + C_2 \cdot y_2(x), \qquad (4)$$

where C_1, C_2 are the arbitrary constants, is its general solution.

Definition 5. Equation

$$y'' + a_1(x) \cdot y' + a_2(x) \cdot y = f(x), \qquad (5)$$

where $a_1(x)$, $a_2(x)$, $f(x)$ are given and continuous on $(a;b)$ functions, is called linear nonhomogeneous differential equation of second order. The left-hand side of equation (5) is called a corresponding homogeneous equation.

Theorem 5 (about the structure of a solution of a nonhomogeneous equation). A general solution of equation (5) is a sum of its arbitrary particular solution y^* and general solution $\overline{y}$ of a corresponding homogeneous equation, that is $y = y^*(x) + \overline{y}(x)$.

Method of variation of arbitrary constants

We assume that $\overline{y}(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x)$ (6) is a general solution of a corresponding to (5) homogeneous equation. If in formula (6) we substitute arbitrary constants C_1, C_2 to unknown functions $C_1(x)$ and $C_2(x)$, then the function $y = C_1(x) \cdot y_1(x) + C_2(x) \cdot y_2(x)$ (7) is a particular solution of equation (5), when functions $C_1(x)$, $C_2(x)$ satisfy a system of conditions:

$$\begin{cases} C_1'(x)\cdot y_1(x)+C_2'(x)\cdot y_2(x)=0, \\ C_1'(x)\cdot y_1'(x)+C_2'(x)\cdot y_2'(x)=f(x). \end{cases} \tag{8}$$

Constant-coefficient linear differential equations

Definition 6. Equation of the form

$$y''+p\cdot y'+q\cdot y=0, \tag{9}$$

where p, q are real numbers, is called *a constant-coefficient linear homogeneous equation of second order.*

With the help of the Euler method, particular solutions of equation (9) are calculated in the form of functions $y=e^{kx}$, where k is a constant number, which is a root of a corresponding *characteristic equation*:

$$k^2+pk+q=0. \tag{10}$$

We label k_1, k_2 as roots of a characteristic equation (10).

Three cases are possible:

I. If k_1, k_2 are real different roots of equation (10), then a general solution of equation (9) is calculated by the formula:

$$y=C_1\cdot e^{k_1 x}+C_2\cdot e^{k_2 x}. \tag{11}$$

II. If $k_{1,2}=\alpha\pm\beta i$ are complex roots of equation (10), then a general solution of equation (9) is written as follows:

$$y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right). \tag{12}$$

III. If $k_1=k_2=k$ are real and equal roots of characteristic equation (10), then a general solution of equation (9) has the following form:

$$y=e^{kx}\left(C_1+C_2 x\right). \tag{13}$$

Nonhomogeneous differential equations with a special right-hand side

Definition 7. Equation of the form

$$y'' + p \cdot y' + q \cdot y = f(x), \tag{14}$$

where p, q are real numbers and $f(x) \not\equiv 0$ is given continuous on a certain interval (a;b) function, is called *a constant-coefficient linear honhomogeneous equation of second order.*

A general solution of equation (14) is a sum of a general solution $\overline{y}$ of a corresponding homogeneous equation, which is calculated by the formulas 3-5, and an arbitrary particular solution y^*, which is found with the method of variation of arbitrary constants.

However, there exists an easier way to calculate a particular solution of equations with a special right-hand side.

Let us consider the following cases:

I. If $f(x)$ is of a form: $f(x) = e^{\alpha x} \cdot P_n(x)$ (7), where α is a real number, $P_n(x)$ is a polynomial of degree n of x, then a particular solution of equation (14) is advisable to be calculated in the form

$$y^* = x^r \cdot e^{\alpha x} \cdot Q_n(x). \tag{16}$$

In formula (16): $Q_n(x)$ is a polynomial with undetermined coefficients of the same degree as $P_n(x)$, and r is a quantity of roots of a characteristic equation which are equal to α ($r = 0, 1, 2$).

II. Exercises for practical lesson

Exercise 1. Calculate general solutions of linear homogeneous equations:

a) $3y''-2y'-8y=0$;

b) $y''+4y'+4y=0$;

c) $4y''-8y'+5y=0$;

d) $y^{(4)}-16y=0$;

e) $y^{(4)}+2y'''+y''=0$.

Solution:

a) $3y''-2y'-8y=0$; $3k^2-2k-8=0$ is a characteristic equation, $k_1=-\dfrac{4}{3}$, $k_2=2$ are its real different roots, then $y=C_1\cdot e^{-\frac{4}{3}x}+C_2\cdot e^{2x}$ is a general solution of a differential equation.

b) $y''+4y'+4y=0$; $k^2+4k+4=0$ or $(k+2)^2=0 \Rightarrow$ $k_1=k_2=-2$ are real equal roots of a characteristic equation, so $y=e^{-2x}(C_1x+C_2)$ is a general solution of equation.

c) $4y''-8y'+5y=0$; $4k^2-8k+5=0$, $D=-16$, $k_{1,2}=1\pm\dfrac{1}{2}i$ are complex roots of a characteristic equation, that is why $y=e^x\left(C_1\cos\dfrac{x}{2}+C_2\sin\dfrac{x}{2}\right)$ is a general solution.

d) $y^{(4)}-16y=0$; $k^4-16=0$, $(k^4-4)(k^2+4)=0 \Rightarrow k_{1,2}=\pm2$, $k_{3,4}=\pm2i$; $y=C_1e^{-2x}+C_2e^{2x}+C_3\cos 2x+C_4\sin 2x$.

e) $y^{(4)}+2y'''+y''=0$; $k^4+2k^3+k^2=0$, $k^2(k^2+2k+1)$ or $k^2(k+1)^2=0 \Rightarrow k_1=k_2=0$, $k_3=k_4=-1$;

$y = C_1 x + C_2 + e^{-x}\left(C_3 x + C_4\right)$.

Exercise 2. Calculate particular solutions of homogeneous equations under the following initial conditions:

a) $y'' - 4y' + 3y = 0$, $y(0) = 6$, $y'(0) = 10$;

b) $4y'' + 4y' + y = 0$, $y(0) = 2$, $y'(0) = 0$;

c) $y'' + 4y' + 5y = 0$, $y(0) = -3$, $y'(0) = 0$.

Solution:

a) $y'' - 4y' + 3y = 0$; $k^2 - 4k + 3 = 0 \Rightarrow k_1 = 1$, $k_2 = 3$; $y = C_1 e^x + C_2 e^{3x}$ is a general solution; $y' = C_1 e^x + 3C_2 e^{3x}$. Under initial conditions $y(0) = 6$ and $y'(0) = 10$ we get a system to calculate C_1 and C_2:

$$\begin{cases} C_1 + C_2 = 6, \\ C_1 + 3C_2 = 10 \end{cases} \Rightarrow \begin{cases} C_1 = 4, \\ C_2 = 2. \end{cases}$$

So, $y = 4e^x + 2e^{3x}$ is a particular solution of the equation.

b) $4y'' + 4y' + y = 0$; $4k^2 + 4k + 1 = 0$ or $(2k+1)^2 = 0 \Rightarrow$

$k_1 = k_2 = -\dfrac{1}{2} \Rightarrow y = e^{-\frac{x}{2}}\left(C_1 x + C_2\right)$, $y' = -\dfrac{1}{2}e^{-\frac{x}{2}}\left(C_1 x + C_2\right) + C_1 e^{-\frac{x}{2}}$

is a general solution.

According to condition $y(0) = 2$ we calculate $C_2 = 2$; then out of condition $y'(0) = 0$ and $C_2 = 2$ we calculate $C_1 = 1$.

So, a particular solution of equation is $y = e^{-\frac{x}{2}}(x + 2)$.

c) $y'' + 4y' + 5y = 0$; $k^2 + 4k + 5 = 0$,

$k_{1,2} = -2 \pm i \Rightarrow y = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)$,

$y' = -2e^{-2x}\left(C_1 \cos x + C_2 \sin x\right) + e^{-2x}\left(-C_1 \sin x + C_2 \cos x\right)$.

Since $y(0)=-3$, then $C_1=-3$; then according to condition $y'(0)=0$ and $C_1=-3$ we calculate $C_2=-6$. So, $y=e^{-2x}\left(-3\cos x-6\sin x\right)$ is a particular solution.

Exercise 3. Calculate general solutions of nonhomogeneous equations:

a) $y''+y'-2y=6x^2$;

b) $y''+4y=5e^x$;

c) $y''+6y'+9y=10\sin x$;

d) $y''-2y=xe^{-x}$;

e) $2y''+5y'=0{,}1e^{-2{,}5x}-25\sin 2{,}5x$.

Solution:

a) $y''+y'-2y=6x^2$; $y''+y'-2y=0$ is a corresponding homogeneous equation, $k^2+k-2=0$ is its characteristic equation, the roots of which are $k_1=-2$, $k_2=1$, then $\overline{y}=C_1e^{-2x}+C_2e^x$ is a general solution of a corresponding homogeneous equation.

We calculate a particular solution of a given equation in a form of function $y^*=Ax^2+Bx+C$ ($r=0$); $\left(y^*\right)'=2Ax+B$, $\left(y^*\right)''=2A$, then $2A+2Ax+B-2Ax^2-2Bx-2C\equiv 6x^2$; $-2Ax^2+\left(2A-2B\right)x+2A+B-2C\equiv 6x^2$.

Comparing coefficients under the corresponding degrees of x, we make up a system to calculate A, B, C:

$$\begin{cases}-2A=6,\\ 2A-2B=0,\\ 2A+B-2C=0\end{cases}\qquad \begin{cases}A=-3,\\ B=-3,\\ C=-\dfrac{9}{2}.\end{cases}$$

In this case a particular solution is $y^* = -3x^2 - 3x - \dfrac{9}{2}$, and a general solution is: $y = \overline{y} + y^* = C_1 e^{-2x} + C_2 e^{x} - 3x^2 - 3x - \dfrac{9}{2}$.

b) $y'' + 4y = 5e^x$; $k^2 + 4 = 0$, $k^2 = -4$, $k_{1,2} = \pm 2i$ are the roots of a characteristic equation of a corresponding homogeneous equation; $\overline{y} = C_1 \cos 2x + C_2 \sin 2x$ is its general solution.

We calculate a particular solution: $y^* = Ae^x$ ($r = 0$); $\left(y^*\right)' = \left(y^*\right)'' = Ae^x$, $Ae^x + 4Ae^x = 5e^x \Rightarrow A = 1$, then $y^* = e^x$.

So, $y = \overline{y} + y^* = \underline{C_1 \cos 2x + C_2 \sin 2x + e^x}$ is a general solution.

c) $y'' + 6y' + 9y = 10\sin x$; $k^2 + 6k + 9 = 0$, $k_1 = k_2 = -3 \Rightarrow$ $\overline{y} = e^{-3x}\left(C_1 x + C_2\right)$ is a general solution; $y^* = A\cos x + B\sin x$ ($r = 0$); $\left(y^*\right)' = -A\sin x + B\cos x$, $\left(y^*\right)'' = -A\cos x - B\sin x$;
$-A\cos x - B\sin x - 6A\sin x + 6B\cos x + 9A\cos x + 9B\sin x \equiv 10\sin x$;

$$\left(8A + 6B\right)\cos x + \left(-6A + 8B\right)\sin x \equiv 10\sin x;$$

$$\begin{cases} 8A + 6B = 0, \\ -6A + 8B = 10 \end{cases} \qquad \begin{cases} 4A + 3B = 0, \\ -3A + 4B = 5 \end{cases} \qquad \begin{cases} A = -\dfrac{3}{5}, \\ B = \dfrac{4}{5}, \end{cases}$$

so $y^* = -\dfrac{3}{5}\cos x + \dfrac{4}{5}\sin x$ is a particular solution, then $y = \overline{y} + y^* = \underline{e^{-3x}\left(C_1 x + C_2\right) - \dfrac{3}{5}\cos x + \dfrac{4}{5}\sin x}$ is a general solution.

d) $y'' - 2y = xe^{-x}$; $k^2 - 2 = 0$, $k_{1,2} = \pm\sqrt{2} \Rightarrow$
$\overline{y} = C_1 e^{-\sqrt{2}x} + C_2 e^{\sqrt{2}x}$; $y^* = \left(Ax + B\right)\cdot e^{-x}$ ($r = 0$),

$(y^*)' = -e^{-x}(Ax+B) + Ae^{-x} = e^{-x}(-Ax + A - B)$,

$(y^*)'' = -e^{-x}(-Ax + A - B) - Ae^{-x} = e^{-x}(Ax + B - 2A)$;

$e^{-x}(Ax + B - 2A) - 2e^{-x}(Ax+B) \equiv x \cdot e^{-x}$;

$-Ax - B - 2A \equiv x$, thus $-A = 1$, $-B - 2A = 0$; $A = -1$, $B = 2$;

$y^* = (-x+2) \cdot e^{-x}$.

Thus, $y = \overline{y} + y^* = \underline{C_1 e^{-\sqrt{2}x} + C_2 e^{\sqrt{2}x} + (2-x) \cdot e^{-x}}$ is a general solution.

e) $2y'' + 5y' = 0{,}1e^{-2{,}5x} - 25\sin 2{,}5x$; $2k^2 + 5k = 0 \Rightarrow k_1 = 0$, $k_2 = -2{,}5$; $\overline{y} = C_1 + C_2 e^{-2{,}5x}$. To calculate a particular solution y^* we apply a theorem of overlapping solutions, that is $y^* = y_1^* + y_2^*$, where $y_1^* = Axe^{-2{,}5x}$ ($k_2 = -2{,}5$, $\alpha = -2{,}5 \Rightarrow r = 1$), $y_2^* = A\cos 2{,}5x + B\sin 2{,}5x$; $(y_1^*)' = Ae^{-2{,}5x} - 2{,}5Axe^{-2{,}5x}$,

$(y_1^*)'' = -2{,}5Ae^{-2{,}5x} - 2{,}5Ae^{-2{,}5x} + 6{,}25Axe^{-2{,}5x} =$
$= -5Ae^{-2{,}5x} + 6{,}25Axe^{-2{,}5x}$.

$$-10Ae^{-2{,}5x} + 12{,}5Axe^{-2{,}5x} + 5Ae^{-2{,}5x} - 12{,}5Axe^{-2{,}5x} \equiv 0{,}1e^{-2{,}5x};$$

$$5A \equiv 0{,}1 \Rightarrow A = 0{,}02, \text{ then } y_1^* = 0{,}02x \cdot e^{-2{,}5x}.$$

We calculate $(y_2^*)' = -2{,}5A\sin 2{,}5x + 2{,}5B\cos 2{,}5x$,

$(y_2^*)'' = -6{,}25A\cos 2{,}5x - 6{,}25B\sin 2{,}5x$;

$$-12{,}5A\cos 2{,}5x - 12{,}5B\sin 2{,}5x -$$
$$-12{,}5A\sin 2{,}5x + 12{,}5B\cos 2{,}5x \equiv -25\sin 2{,}5x,$$

$(-12{,}5A + 12{,}5B)\cos 2{,}5x + (-12{,}5A - 12{,}5B)\sin 2{,}5x \equiv -25\sin 2{,}5x$,

$$\begin{cases} -12{,}5A + 12{,}5B = 0, \\ -12{,}5A - 12{,}5B = -25, \end{cases}$$

thus $A = B = 1$, then $y_2^* = \cos 2{,}5x + \sin 2{,}5x$.

So,

$y=\overline{y}+y_1{}^*+y_2{}^*=\underline{C_1+C_2\cdot e^{-2,5x}+0,02x\cdot e^{-2,5x}+\cos 2,5x+\sin 2,5x}$.

Exercise 4. Calculate particular solutions of nonhomogeneous equations, which satisfy the initial conditions:

a) $y''+9y=15\sin 2x$, $y(0)=-7$, $y'(0)=0$;

b) $y''-3y'=3x+x^2$, $y(0)=0$, $y'(0)=\dfrac{70}{27}$;

c) $y''+y'=2x^2\cdot e^x$, $y(0)=5$, $y'(0)=0,5$.

Solutions:

a) $y''+9y=15\sin 2x$; $k^2+9=0$, $k_{1,2}=\pm 3i$ $\Rightarrow$ $\overline{y}=C_1\cos 3x+C_2\sin 3x$; $y^*=A\cos 2x+B\sin 2x$ ($r=0$), $(y^*)'=-2A\sin 2x+2B\cos 2x$, $(y^*)''=-4A\cos 2x-4B\sin 2x$;

$$-4A\cos 2x-4B\sin 2x+9A\cos 2x+9B\sin 2x\equiv 15\sin 2x,$$

$5A\cos 2x+5B\sin 2x\equiv 15\sin 2x$ $\Rightarrow$ $5A=0$, $A=0$; $5B=15$, $B=3$; $y^*=3\sin 2x$. So, $y=\overline{y}+y^*=C_1\cos 3x+C_2\sin 3x+3\sin 2x$ is a general solution of the equation.

Let us calculate a particular solution which satisfies the initial conditions: $y(0)=-7$ $\Rightarrow$ $C_1=-7$; $y'(x)=-3C_1\sin 3x+3C_2\cos 3x+6\cos 2x$, $y'(0)=0$ $\Rightarrow$ $3C_2+6=0$, $C_2=-2$.

Then $y=-7\cos 3x-2\sin 3x+3\sin 2x$ is a particular solution of the equation.

b) $y''-3y'=3x+x^2$; $k^2-3k=0$, $k_1=0$, $k_2=3$ $\Rightarrow$ $\overline{y}=C_1+C_2e^{3x}$; $y^*=\left(Ax^2+Bx+C\right)x=Ax^3+Bx^2+Cx$ ($k_1=0$, $\alpha=0$ $\Rightarrow$ $r=1$), $(y^*)'=3Ax^2+2Bx+C$, $(y^*)''=6Ax+2B$;

$$6Ax + 2B - 9Ax^2 - 6Bx - 3C \equiv 3x + x^2;$$

$$-9Ax^2 + \left(6A - 6B\right)x + 2B - 3C \equiv 3x + x^2;$$

$$\begin{cases} -9A = 1, \\ 6A - 6B = 3, \\ 2B - 3C = 0 \end{cases} \quad \begin{cases} A = -\dfrac{1}{9}, \\ B = -\dfrac{11}{18}, \\ C = -\dfrac{11}{27}; \end{cases} \quad y^* = -\frac{1}{9}x^3 - \frac{11}{18}x^2 - \frac{11}{27}x.$$

So, $y = \overline{y} + y^* = C_1 + C_2 e^{3x} - \frac{1}{9}x^3 - \frac{11}{18}x^2 - \frac{11}{27}x$ is a general solution of the equation.

We calculate a particular solution under the condition that $y\left(0\right) = 0$, then $C_1 + C_2 = 0$ and $y'\left(0\right) = \frac{70}{27}$, then $y'\left(x\right) = 3C_2 \cdot e^{3x} - \frac{1}{3}x^2 - \frac{11}{9}x - \frac{11}{27}$, that is $\frac{70}{27} = 3C_2 - \frac{11}{27} \Rightarrow$ $C_2 = 1$; $C_1 = -C_2 = -1$.

Thus $y = -1 + e^{3x} - \frac{1}{9}x^3 - \frac{11}{18}x^2 - \frac{11}{27}x$ is a particular solution of the equation.

c) $y'' + y' = 2x^2 \cdot e^x$; $k \ \ + k = \ \ $, $k_1 = 0$, $k_2 = -1 \Rightarrow$ $\overline{y} = C_1 + C_2 \cdot e^{-x}$;

$$y^* = \left(Ax^2 + Bx + C\right) \cdot e^x \ (r = 0);$$

$$\left(y^*\right)' = \left(2Ax + B\right) \cdot e^x + \left(Ax^2 + Bx + C\right) \cdot e^x = \left(Ax^2 + \left(2A + B\right)x + B + C\right) \cdot e^x;$$

$$\left(y^*\right)'' = \left(2Ax + 2A + B\right) \cdot e^x + \left(Ax^2 + \left(2A + B\right) \cdot x + B + C\right) \cdot e^x =$$

$$= \left(Ax^2 + \left(4A + B\right) \cdot x + 2A + 2B + C\right) \cdot e^x;$$

$$\left(Ax^2 + \left(4A + B\right)x + 2A + 2B + C\right) \cdot e^x + \left(Ax^2 + \left(2A + B\right)x + B + C\right) \equiv 2x^2 \cdot e^x;$$

$$2Ax^2 + \left(6A + 2B\right)x + 2A + 3B + 2C \equiv 2x^2;$$

$$\begin{cases} 2A = 2, \\ 6A + 2B = 0, \\ 2A + 3B + 2C = 0 \end{cases} \qquad \begin{cases} A = 1, \\ B = -3, \\ C = 3,5; \end{cases} \qquad y^* = \left(x^2 - 3x + 3,5\right) \cdot e^x .$$

So, $y = \overline{y} + y^* = C_1 + C_2 e^{-x} + \left(x^2 - 3x + 3,5\right) \cdot e^x$ is a particular solution.

Under initial conditions $y(0) = 5$ and $y'(0) = 0,5$ we calculate C_1 i C_2:

$$5 = C_1 + C_2 + 3,5 \quad \Rightarrow \quad C_1 + C_2 = 1,5 ;$$

$$y'(x) = -C_2 \cdot e^{-x} + (2x - 3) \cdot e^x + \left(x^2 - 3x + 3,5\right) \cdot e^x ,$$

$$0,5 = -C_2 - 3 + 3,5 \quad \Rightarrow \quad C_2 = 0 \text{, then } C_1 = 1,5 ;$$

$$y = 1,5 + \left(x^2 - 3x + 3,5\right) \cdot e^x \text{ is a particular solution.}$$

Exercise 5. Write a general solution of nonhomogeneous equations, having calculated their particular solutions by the method of variation of arbitrary constants:

1) $y'' + y + \operatorname{ctg}^2 x = 0$;

2) $y'' - y' = f(x)$, if

a) $f(x) = \dfrac{e^x}{1 + e^x}$;

b) $f(x) = e^{2x} \cdot \sqrt{1 - e^{2x}}$;

c) $f(x) = e^{2x} \cdot \cos e^x$.

Solution:

1) $y'' + y + \operatorname{ctg}^2 x = 0$ or $y'' + y = -\operatorname{ctg}^2 x$;

$k^2 + 1 = 0$, $k_{1,2} = \pm i$, $\overline{y} = C_1 \cos x + C_2 \sin x$ is a general solution of a corresponding homogeneous equation.

We calculate a particular solution in the form of function $y^*=C_1(x)\cdot y_1+C_2(x)\cdot y_2$, that is $y^*=C_1(x)\cdot\cos x+C_2(x)\cdot\sin x$, where unknown functions $C_1(x)$ and $C_2(x)$ are calculated out of the system

$$\begin{cases} C_1'(x)\cdot\cos x+C_2'(x)\cdot\sin x=0, \\ C_1'(x)\cdot(-\sin x)+C_2'(x)\cdot\cos x=-\mathrm{ctg}^2\,x; \end{cases}$$

$$\begin{cases} C_1'(x)=-C_2'(x)\cdot\mathrm{tg}\,x, \\ C_2'(x)\cdot\mathrm{tg}\,x\cdot\sin x+C_2'(x)\cdot\cos x=-\mathrm{ctg}^2\,x; \end{cases}$$

$$C_2'(x)\cdot\left(\frac{\sin^2 x}{\cos x}+\cos x\right)=-\mathrm{ctg}^2\,x; \qquad C_2'(x)\cdot\frac{1}{\cos x}=-\mathrm{ctg}^2\,x;$$

$$C_2'(x)=-\mathrm{ctg}^2\,x\cdot\cos x; \qquad C_2'(x)=-\frac{\cos^3 x}{\sin^2 x};$$

$$C_2(x)=-\int\frac{\cos x\left(1-\sin^2 x\right)}{\sin^2 x}dx=\int\left(1-\frac{1}{\sin^2 x}\right)d(\sin x)=\sin x+\frac{1}{\sin x};$$

$$C_1'(x)=-C_2'(x)\cdot\mathrm{tg}\,x=\frac{\cos^3 x}{\sin^2 x}\cdot\frac{\sin x}{\cos x}=\frac{\cos^2 x}{\sin x}=\frac{1-\sin^2 x}{\sin x}=\frac{1}{\sin x}-\sin x;$$

$$C_1(x)=\int\left(\frac{1}{\sin x}-\sin x\right)dx=\ln\left|\mathrm{tg}\frac{x}{2}\right|+\cos x;$$

$$y^*=C_1(x)\cdot\cos x+C_2(x)\cdot\sin x=$$

$$=\left(\ln\left|\mathrm{tg}\frac{x}{2}\right|+\cos x\right)\cdot\cos x+\left(\sin x+\frac{1}{\sin x}\right)\cdot\sin x=$$

$$=\cos x\cdot\ln\left|\mathrm{tg}\frac{x}{2}\right|+\cos^2 x+\sin^2 x+1=\cos x\cdot\ln\left|\mathrm{tg}\frac{x}{2}\right|+2;$$

$$y=\bar{y}+y^*=\underline{C_1\cos x+C_2\sin x+\cos x\cdot\ln\left|\mathrm{tg}\frac{x}{2}\right|+2}.$$

2) $y'' - y' = f(x)$; $k^2 - k = 0$; $k_1 = 0$, $k_2 = 1$; $\overline{y} = C_1 + C_2 e^x$ is a general solution of a corresponding homogeneous equation. We calculate a particular solution in the form $y^* = C_1(x) + C_2(x) \cdot e^x$.

a) $\begin{cases} C_1'(x) + C_2'(x) \cdot e^x = 0, \\ C_2'(x) \cdot e^x = \dfrac{e^x}{1+e^x}, \end{cases}$

$$C_2'(x) = \frac{1}{1+e^x};$$

$$C_2(x) = \int \frac{dx}{1+e^x} = \begin{bmatrix} 1+e^x = t, \ x = \ln|t-1|, \\ dx = \dfrac{dt}{t-1} \end{bmatrix} = \int \frac{dt}{t(t-1)} = -\int \frac{dt}{t} + \int \frac{dt}{t-1} =$$

$$= -\ln\left(1+e^x\right) + \ln e^x = x - \ln\left(1+e^x\right);$$

$$\left(\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} = \frac{A(t-1)+Bt}{t(t-1)}, \right.$$

$$A(t-1) + Bt \equiv 1, \ t = 0 \Rightarrow A = -1, \ t = 1 \Rightarrow B = 1;$$

$$\left. \frac{1}{t(t-1)} = -\frac{1}{t} + \frac{1}{t-1} \right);$$

$$C_1'(x) = -\frac{e^x}{1+e^x}; \quad C_1(x) = -\int \frac{d\left(1+e^x\right)}{1+e^x} = -\ln\left(1+e^x\right);$$

$$y^* = -\ln\left(1+e^x\right) + \left(x - \ln\left(1+e^x\right)\right) \cdot e^x =$$

$$= -\ln\left(1+e^x\right) + x \cdot e^x - e^x \cdot \ln\left(1+e^x\right) =$$

$$= xe^x - \left(e^x + 1\right) \cdot \ln\left(1+e^x\right);$$

$$y = \overline{y} + y^* = C_1 + C_2 e^x + xe^x - \left(e^x + 1\right) \ln\left(1+e^x\right) =$$

$$= C_1 + \left(C_2 + x\right) \cdot e^x - \left(e^x + 1\right) \ln\left(1+e^x\right).$$

b) $\begin{cases} C_1'(x) + C_2'(x)\cdot e^x = 0, \\ C_2'(x)\cdot e^x = e^{2x}\cdot\sqrt{1-e^{2x}}. \end{cases}$

$$C_2'(x) = e^x\cdot\sqrt{1-e^{2x}};$$

$$C_2(x) = \int e^x\cdot\sqrt{1-e^{2x}}\,dx = \begin{bmatrix} e^x = t,\ x = \ln t, \\ dx = \dfrac{dt}{t} \end{bmatrix} =$$

$$= \int t\cdot\sqrt{1-t^2}\cdot\frac{dt}{t} = \int\sqrt{1-t^2}\,dt = \begin{bmatrix} t = \sin z, \\ dt = \cos z\,dz \end{bmatrix} =$$

$$= \int\sqrt{1-\sin^2 z}\cdot\cos z\,dz = \int\cos^2 z\,dz =$$

$$= \frac{1}{2}\int(1+\cos 2z)\,dz = \frac{1}{2}\left(z + \frac{1}{2}\sin 2z\right) = \frac{1}{2}\left(z + \sin z\cdot\sqrt{1-\sin^2 z}\right) =$$

$$= \frac{1}{2}\left(\arcsin t + t\cdot\sqrt{1-t^2}\right) = \frac{1}{2}\left(\arcsin e^x + e^x\cdot\sqrt{1-e^{2x}}\right);$$

$$C_1'(x) = -C_2'(x)\cdot e^x = -e^{2x}\cdot\sqrt{1-e^{2x}};$$

$$C_1(x) = \frac{1}{2}\int\sqrt{1-e^{2x}}\,d\left(1-e^{2x}\right) = \frac{1}{3}\left(1-e^{2x}\right)\cdot\sqrt{1-e^{2x}};$$

$$y^* = \frac{1}{3}\left(1-e^{2x}\right)\cdot\sqrt{1-e^{2x}} + \frac{1}{2}e^x\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right);$$

$$y = \overline{y} + y^* =$$

$$\underline{= C_1 + C_2 x + \frac{1}{3}\left(1-e^{2x}\right)\sqrt{1-e^{2x}} + \frac{1}{2}e^x\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right).}$$

c) $\begin{cases} C_1'(x) + C_2'(x)\cdot e^x = 0, \\ C_2'(x)\cdot e^x = e^{2x}\cdot\cos e^x \end{cases}$

$$C_2'(x) = e^x \cdot \cos e^x \,; \quad C_2(x) = \int e^x \cdot \cos e^x \, dx = \int \cos e^x \, d\left(e^x\right) = \sin e^x \,;$$

$$C_1'(x) + e^{2x} \cdot \cos e^x = 0 \,; \quad C_1'(x) = -e^{2x} \cdot \cos e^x \,;$$

$$C_1(x) = -\int e^{2x} \cdot \cos e^x \, dx =$$

$$= \begin{bmatrix} e^x = t, \\ e^x dx = dt \end{bmatrix} = -\int t \cdot \cos t \, dt = \begin{bmatrix} t = u, & dt = du, \\ \cos t \, dt = dv, & v = \sin t \end{bmatrix} =$$

$$= -\left(t \sin t - \int \sin t \, dt\right) = -t \sin t - \cos t = -e^x \sin e^x - \cos e^x \,;$$

$$y^* = -e^x \cdot \sin e^x - \cos e^x + e^x \sin e^x = -\cos e^x \,;$$

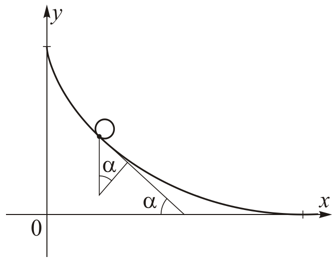
$$y = \overline{y} + y^* = \underline{C_1 + C_2 e^x - \cos e^x} \,.$$

Exercise 6. A ball rolls along a smooth gutter, curved by a cycloid (Drawing 4.1). Neglecting friction and air resistance, determine the dependence of the way, traced by a centre mass of a ball, upon time.

Solution:

If a solid with a mass m moves in an area or space, then the net force $\overline{F}$ and acceleration $\overline{\omega}$ of its mass centre are connected, according to Newton's second law, by equation $\overline{F} = m\overline{\omega}$.

Projecting vectors $\overline{F}$ and $\overline{\omega}$ on an arbitrary axis, for instance, ω_k, we can get a proportion which includes scalar values, that is $F_k = m\omega_k$.



Drawing 4.1

When we project the acceleration of a centre mass of a ball and the force acting upon it on the direction of a tangent to a cycloid, we get a differential equation of a movement of a centre mass of a ball: $m \cdot \dfrac{d^2 S}{dt^2} = mg \sin\alpha$, where $S = \overset{\smile}{AM}$ is a way, traced by a centre mass of a ball during time t, m is a mass of a ball, g is an acceleration of a gravity force.

Considering parametrical equations of a cycloid $x = a(\phi - \sin\phi)$, $y = a(1+\cos\phi)$, we express $\sin\alpha$ through S:

$$y' = \frac{dy}{dx} = \frac{-a\sin\phi \, d\phi}{a(1-\cos\phi)\, d\phi} = -\frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}} = -\operatorname{ctg}\frac{\phi}{2}.$$

Then a differential of an arc of cycloid is equal to $dS = \sqrt{1+(y')^2}\, dx = \sqrt{1+\operatorname{ctg}^2\frac{\phi}{2}} \cdot 2a\sin^2\frac{\phi}{2} d\phi = 2a\sin\frac{\phi}{2} d\phi$.

The length of its arc AM:

$S = \int_0^\phi 2a\sin\frac{\phi}{2} d\phi = 4a\cos\frac{\phi}{2}\Big|_\phi^0 = 4a\left(1-\cos\frac{\phi}{2}\right)$, thus $\cos\frac{\phi}{2} = 1 - \frac{S}{4a}$.

According to figure $y' = \operatorname{tg}(\pi - \alpha) = -\operatorname{tg}\alpha$, consequently

$$\sin\alpha = \operatorname{tg}\alpha \cdot \cos\alpha = \frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}} = -\frac{y'}{\sqrt{1+(y')^2}} = \frac{\operatorname{ctg}\frac{\phi}{2}}{\sqrt{1+\operatorname{ctg}^2\frac{\phi}{2}}} = \cos\frac{\phi}{2},$$

that is $\sin\alpha = 1 - \frac{S}{4a}$.

So, $\frac{d^2S}{dt^2} = g\left(1 - \frac{S}{4a}\right)$ or $4a\frac{d^2S}{dt^2} + gS = 4ag$ is a constant-coefficient linear nonhomogeneous equation. Its characteristic equation $4ak^2 + g = 0$ has roots $k_{1,2} = \pm i\sqrt{\frac{g}{4a}} = \pm ri$, then a general integral of a corresponding homogeneous equation $u = C_1\cos rt + C_2\sin rt$.

We write a particular integral S_1 of a nonhomogeneous equation in the form of a right-hand side: $S_1 = A$, then $gA \equiv 4ag \Rightarrow A = 4a$; $S_1 = 4a$.

A general solution: $S = u + S_1 = 4a + C_1 \cos rt + C_2 \sin rt$; we calculate the value of constants C_1 and C_2 according to the initial conditions $t = 0$, $S = 0$, $\frac{dS}{dt} = 0$; $0 = 4a + C_1 \Rightarrow C_1 = -4a$; $\frac{dS}{dt} = -C_1 r \sin rt + C_2 r \cos rt$, $0 = C_2 r \Rightarrow C_2 = 0$.

Thus, the dependence of way, traced by a centre mass of a ball, upon time is equal to $S = 4a\left(1 - \cos t \cdot \sqrt{\frac{g}{4a}}\right)$.

III. Exercises for individual work

Exercise 1. Calculate general solutions of differential equations:

a) $y'' - 5y' + 6y = 0$;

b) $y''' - 4y'' + 3y' = 0$;

c) $y''' - 3y'' + 3y' - y = 0$;

d) $y''' + 6y'' + 25y' = 0$;

e) $y'' - 2y' + 2y = 2x$;

f) $y'' + y = -8\cos 3x$;

g) $y'' - 2y' + y = xe^x$.

Exercise 2. Calculate particular solutions of differential equations under the given initial conditions:

a) $y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$;

b) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = 1$;

c) $y'' + y + \sin 2x = 0$, $y(\pi) = 1$, $y'(\pi) = 1$;

d) $y'' - 2y' + 10y = 10x^2 + 18x + 6$, $\quad y(0) = 1$, $\quad y'(0) = 3{,}2$;

e) $4y'' + 16y' + 15y = 4e^{-\frac{3}{2}x}$, $\quad y(0) = 3$, $\quad y'(0) = -5{,}5$;

f) $y'' - 2y' = e^x\left(x^2 + x - 3\right)$, $\quad y(0) = 2$, $\quad y'(0) = 2$.

Practical lesson 5.

Systems of differential equations

I. General theoretical information

Definition 1. *A normal system of differential equations* is called a system of the form:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1\left(t,\ x_1,\ x_2,\ \dots,\ x_n\right), \\ \dfrac{dx_2}{dt} = f_2\left(t,\ x_1,\ x_2,\ \dots,\ x_n\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dx_n}{dt} = f_n\left(t,\ x_1,\ x_2,\ \dots,\ x_n\right), \end{cases} \tag{1}$$

or $x_k' = f_k\left(t,\ x_1,\ x_2,\ \dots,\ x_n\right)$, $k = 1,\ 2,\ \dots,\ n$.

Definition 2. *A solution of system* (1) is a complex of functions $x_1\left(t\right)$, $x_2\left(t\right)$, … , $x_n\left(t\right)$, which under the substitution into each equation of the given system turns it into a congruence by x.

If a given equation of n-order $y^{(n)} = f\left(t,\ y,\ y',\ \dots,\ y^{(n-1)}\right)$, is solved by the highest derivative, then by the introduction of new variables $y = x_1$, $y' = x_2$, … , $y^{(n-1)} = x_n$, the given differential equation can be reduced to an equivalent normal system n of first-order equations:

$$x_i' = x_{i+1}, \quad i = 1, 2, \dots, n-1; \quad x_n' = f\left(t, x_1, x_2, \dots, x_n\right).$$

The reverse transition is also possible, that is if we differentiate by t any equation of a normal system (1) and replace into the resulting equality the value of derivatives x_1', x_2', … , x_n', taken from system (1), then we get the following system of equations:

$$\begin{cases} x_1' = f_1'\left(t, x_1, x_2, \dots, x_n\right), \\ x_1'' = F_2\left(t, x_1, x_2, \dots, x_n\right), \\ \dots\dots\dots\dots\dots\dots \\ x_1^{(n)} = F_n\left(t, x_1, x_2, \dots, x_n\right). \end{cases} \quad (2)$$

If from the first $n-1$ equations of system (2) we calculate, if possible, variables

$$\begin{aligned} x_2 &= \phi_2\left(t, x_1, x_1', \dots, x_1^{(n-1)}\right), \\ x_3 &= \phi_3\left(t, x_1, x_1', \dots, x_1^{(n-1)}\right), \\ &\dots\dots\dots\dots\dots\dots \\ x_n &= \phi_n\left(t, x_1, x_1', \dots, x_1^{(n-1)}\right) \end{aligned} \quad (3)$$

and substitute their values into the last equation, then we get an equation of n-order by variable x_1:

$$x_1^{(n)} = \Phi\left(t, x_1, x_1', \dots, x_1^{(n-1)}\right). \quad (4)$$

We assume that function $x_1 = \psi_1\left(t, C_1, C_2, \dots, C_n\right)$ (5), where C_1, C_2, … , C_n are arbitrary constants, is a solution of equation (4). Having differentiated it n-1 times and substituted the values of variables x_1', x_1'', … , $x_1^{(n-1)}$ into equation (3), we get a complexity of functions

$$x_2 = \psi_2\left(t,\, C_1, C_2, \ldots, C_n\right),$$
$$x_3 = \psi_3\left(t,\, C_1, C_2, \ldots, C_n\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = \psi_n\left(t,\, C_1, C_2, \ldots, C_n\right),$$

which along with function (5) is a general solution of system (2).

For a normal system (2) **the Cauchy theorem about existence and unity of solution** is valid: if in a certain domain G functions $f_k\left(t,\, x_1, x_2, \ldots, x_n\right)$ ($k = 1, 2, \ldots, n$) of system (2) are continuous along with all their derivatives $\frac{\partial f_k}{\partial x_i}$ ($i = 1, 2, \ldots, n$), then for any point $\left(t_0, x_{1_0}, x_{2_0}, \ldots, x_{n_0}\right) \in G$ exists one solution $x_1(t)$, $x_2(t)$, … , $x_n(t)$, which satisfies initial conditions: $x_1(t_0) = x_{1_0}$, $x_2(t_0) = x_{2_0}$, … , $x_n(t_0) = x_{n_0}$.

To integrate system of the form (2) we apply *the method of variable exception* by which this system is reduced to equation (4).

Systems of linear constant-coefficient differential equations

Consider a system of three linear constant-coefficient differential equations:

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + a_2 y + a_3 z, \\ \dfrac{dy}{dt} = b_1 x + b_2 y + b_3 z, \\ \dfrac{dz}{dt} = c_1 x + c_2 y + c_3 z, \end{cases} \tag{6}$$

where a_i, b_i, c_i are the constants. By the method of variable exception, this system can always be reduced to one homogeneous third-order linear constant-coefficient equation.

Consider another way of solving the system (6).

We calculate separate solutions of the system in the form:

$$x = \alpha \cdot e^{kt}, \qquad y = \beta \cdot e^{kt}, \qquad z = \gamma \cdot e^{kt}, \tag{7}$$

where α, β, γ, k are unknown constants which we need to calculate.

If we replace functions (7) into system (6) and divide it by $e^k \neq 0$, then we get:

$$\begin{cases} k \cdot \alpha = a_1\alpha + a_2\beta + a_3\gamma, \\ k \cdot \beta = b_1\alpha + b_2\beta + b_3\gamma, \\ k \cdot \gamma = c_1\alpha + c_2\beta + c_3\gamma, \end{cases} \quad \text{or} \quad \begin{cases} (a_1 - k)\alpha + a_2\beta + a_3\gamma = 0, \\ b_1\alpha + (b_2 - k)\beta + b_3\gamma = 0, \\ c_1\alpha + c_2\beta + (c_3 - k)\gamma = 0. \end{cases} \tag{8}$$

In order for the resulting algebraic homogeneous system of linear equations (8) to have non-zero solutions, it is necessary and sufficient for one determinant to be equal to zero:

$$\begin{vmatrix} a_1 - k & a_2 & a_3 \\ b_1 & b_2 - k & b_3 \\ c_1 & c_2 & c_3 - k \end{vmatrix} = 0. \tag{9}$$

Having expressed determinant (9), we get an algebraic equation of the third degree by a variable k, which is called a characteristic equation of system (6).

If equation (9) has three real different roots k_1, k_2, k_3, then for each of these roots we rewrite system (8) and calculate unknown α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3. Then a general solution of system (6) has the form:

$$x = C_1\alpha_1 e^{k_1 t} + C_2\alpha_2 e^{k_2 t} + C_3\alpha_3 e^{k_3 t},$$
$$y = C_1\beta_1 e^{k_1 t} + C_2\beta_2 e^{k_2 t} + C_3\beta_3 e^{k_3 t},$$
$$z = C_1\gamma_1 e^{k_1 t} + C_2\gamma_2 e^{k_2 t} + C_3\gamma_3 e^{k_3 t}. \qquad (10)$$

If equation (9) has complex roots, then the calculation of solutions of system is more complicated, thus we do not consider such cases.

A characteristic equation (9) of system (6) matches characteristic equation of third-order differential equation, to which this system can be reduced. Thus, if we know the roots of equation (9), then it is possible to calculate a general solution of a third-order equation, which system (6) is reduced to, and then the general solution of a system on the whole.

So, notwithstanding the structure of roots of a characteristic equation, system (6) can be solved only on the basis of these roots.

II. Exercises for practical lesson

Exercise 1. Calculate systems of equations:

a) $\begin{cases} \dfrac{dx}{dt} = 2x + y, \\ \dfrac{dy}{dt} = 3x + 4y \end{cases};$

b) $\begin{cases} \dfrac{dx}{dt} = x - 3y, \\ \dfrac{dy}{dt} = 3x + y \end{cases};$

c) $\begin{cases} \dfrac{dx}{dt} = x - 2y - z, \\ \dfrac{dy}{dt} = -x + y + z, \\ \dfrac{dz}{dt} = x - z; \end{cases}$

d) $\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = x + e^t + e^{-t}; \end{cases}$

e) $\begin{cases} \dfrac{dx}{dt} = 2y - 5x + e^t, \\ \dfrac{dy}{dt} = x - 6y + e^{-2t}. \end{cases}$

Solution:

a) We differentiate the first equation of system $x'' = 2x' + y'$, replace into the resulting equality the value of derivative $y' = 3x + 4y$ from the second equation of the system and then $y = x' - 2x$ from the first equation, and we get:

$x'' = 2x' + 3x + 4y$; $x'' = 2x' + 3x + 4(x' - 2x)$; $x'' - 6x' + 5x = 0$ – is a linear second-order constant-coefficient equation, $k^2 - 6k + 5 = 0$; $k_1 = 1$, $k_2 = 5$ – are the roots of characteristic equation, so $x = C_1 \cdot e^t + C_2 \cdot e^{5t}$ is a general solution.

Then $x' = C_1 \cdot e^t + 5C_2 \cdot e^{5t}$, hence

$y = x' - 2x = C_1 e^t + 5C_2 e^{5t} - 2C_1 e^t - 2C_2 e^{5t}$, $y = -C_1 e^t + 3C_2 e^{5t}$.

As a result we have: $x = C_1 e^t + C_2 e^{5t}$, $y = -C_1 e^t + 3C_2 e^{5t}$.

b) By analogy $x'' = x' - 3y'$; $x'' = x' - 3 \cdot (3x + y)$;

$x'' = x' - 9x - 3y$; $x'' = x' - 9x + x' - x$; $x'' - 2x' + 10x = 0$,

$k^2 - 2k + 10 = 0$, $k_{1,2} = \pm 3i$; $x = e^t\left(C_1\cos 3t + C_2\sin 3t\right)$;

$x' = e^t\left(C_1\cos 3t + C_2\sin 3t\right) + e^t\left(-3C_1\sin 3t + 3C_2\cos 3t\right)$;

$y = \frac{1}{3}\left(x - x'\right) = \frac{1}{3}e^t\left(3C_1\sin 3t - 3C_2\cos 3t\right) = e^t\left(C_1\sin 3t - C_2\cos 3t\right)$.

So, $x = e^t\left(C_1\cos 3t + C_2\sin 3t\right)$, $y = e^t\left(C_1\sin 3t - C_2\cos 3t\right)$.

c) We write a characteristic equation for the given system of linear constant-coefficient differential equations:

$$\begin{vmatrix} 1-k & -2 & -1 \\ -1 & 1-k & 1 \\ 1 & 0 & -1-k \end{vmatrix} = 0,$$

hence $k^3 - k^2 - 2k = 0$, $k_1 = 0$, $k_2 = -1$, $k_3 = 2$.

We calculate particular solutions of system in the form:

$$x_1 = \alpha_1 \cdot e^{k_1 t}, \quad x_2 = \alpha_2 \cdot e^{k_2 t}, \quad x_3 = \alpha_3 \cdot e^{k_3 t};$$

$$y_1 = \beta_1 \cdot e^{k_1 t}, \quad y_2 = \beta_2 \cdot e^{k_2 t}, \quad y_3 = \beta_3 \cdot e^{k_3 t};$$

$$z_1 = \gamma_1 \cdot e^{k_1 t}, \quad z_2 = \gamma_2 \cdot e^{k_2 t}, \quad z_3 = \gamma_3 \cdot e^{k_3 t}.$$

If $k_1 = 0$, then

$$\begin{cases} \alpha_1 - 2\beta_1 - \gamma_1 = 0, \\ -\alpha_2 + \beta_1 + \gamma_1 = 0, \\ \alpha_1 - \gamma_1 = 0, \end{cases}$$

Let us assume that $\alpha_1 = \gamma_1 = 1$, $\beta_1 = 0$, then $x_1 = e^0 = 1$, $y_1 = 0$, $z_1 = e^0 = 1$.

If $k_2 = -1$, then

$$\begin{cases} 2\alpha_2 - 2\beta_2 - \gamma_2 = 0, \\ -\alpha_2 + 2\beta_2 + \gamma_2 = 0, \\ \alpha_2 = 0, \end{cases}$$

hence $\alpha_2 = 0$.

Let us assume $\beta_2 = 1$, $\gamma_2 = -2$, then $x_2 = 0$, $y_2 = e^{-t}$, $z_2 = -2e^{-t}$. If $k_3 = 2$, then

$$\begin{cases} -\alpha_3 - 2\beta_3 - \gamma_3 = 0, \\ -\alpha_3 - \beta_3 + \gamma_3 = 0, \\ \alpha_3 - 3\gamma_3 = 0. \end{cases}$$

We assume that $\alpha_3 = 3$, $\beta_3 = -2$, $\gamma_3 = 1$, then $x_3 = 3e^{2t}$, $y_3 = 2e^{2t}$, $z_3 = e^{2t}$.

As a result we get:

$$x(t) = C_1 + 3C_3e^{2t}, \quad y(t) = C_2e^{-t} - 2C_3e^{2t},$$
$$z(t) = C_1 - 2C_2e^{-t} + C_3e^{2t}.$$

d) We differentiate the first equation of the system, we have: $x'' = y'$; since $y' = x + e^t + e^{-t}$, then $x'' = x + e^t + e^{-t}$ or $x'' - x = e^t + e^{\ t}$ is a linear nonhomogeneous second-order equation, $k^2 - 1 = 0$ is a characteristic equation of a corresponding homogeneous equation; $k_{1,2} = \pm 1 \Rightarrow x^* = C_1e^t + C_2e^{-t}$ is a general solution.

We calculate a particular solution in the form

$\overline{x} = Ate^t + Bte^{-t}$; $(\overline{x})' = \left(Ae^t - Be^{-t}\right)t + Ae^t + Be^{-t}$;

$(\overline{x})'' = \left(Ae^t + Be^{-t}\right)t + 2\left(Ae^t - Be^{-t}\right)$; $tAe^t + tBe^{-t} +$

$+2Ae^t - 2Be^{-t} - Ate^t + Bte^{-t} \equiv e^t + e^{-t} \quad \Rightarrow \quad A = \dfrac{1}{2},\ B = \dfrac{1}{2}$.

So, $x = x^* + \overline{x} = C_1e^t + C_2e^{-t} + t\cdot\dfrac{e^t + e^{-t}}{2} = \underline{C_1e^t + C_2e^{-t} + t\cdot \operatorname{sh} t}$;

$y = \dfrac{dx}{dt} = \underline{C_1e^t - C_2e^{-t} + \operatorname{sh} t + t\cdot\operatorname{ch} t}$.

e) By analogy $x'' = 2y' - 5x' + e^t$; consider that

$y' = x - 6y + e^{-2t}$ and $y = \dfrac{1}{2}\left(x' + 5x - e^t\right)$,

then $x'' = 2x - 6x' - 30x + 6e^t + 2e^{-2t} - 5x' + e^t$;

$x'' + 11x' + 28x = 2e^{-2t} + 7e^t$; $x'' + 11x + 28x = 0$ is a corresponding homogeneous equation, $k^2 + 11k + 28 = 0$, $k_1 = -4$, $k_2 = -7$ $\Rightarrow$ $y^* = C_1e^{-4t} + C_2e^{-7t}$ is a general solution; a particular solution is found in the form $\overline{x} = \overline{x}_1 + \overline{x}_2$; $\overline{x}_1 = Ae^{-2t}$, $\left(\overline{x}_1\right)' = -2Ae^{-2t}$,

$\left(\overline{x}_1\right)'' = 4Ae^{-2t}$, then $4Ae^{-2t} - 22Ae^{-2t} + 28Ae^{-2t} \equiv 2e^{-2t} \Rightarrow A = \dfrac{1}{5}$,

so $\overline{x}_1 = \dfrac{1}{5}e^{-2t}$; $\overline{x}_2 = Be^t$, $\left(\overline{x}_2\right)' = \left(\overline{x}_2\right)'' = Be^t$;

$Be^t + 11Be^t + 28Be^t \equiv 7e^t \Rightarrow B = \dfrac{7}{40}$, so $\overline{x}_2 = \dfrac{7}{40}e^t$;

$x = x^* + \overline{x}_1 + \overline{x}_2 = \underline{C_1e^{-4t} + C_2e^{-7t} + \dfrac{1}{5}e^{-2t} + \dfrac{7}{40}e^t}$; $x' = -4C_1e^{-4t} +$

$+7C_2e^{-7t} - \dfrac{2}{5}e^{-2t} + \dfrac{7}{40}e^t$; $\underline{y = \dfrac{1}{2}C_1e^{-4t} - C_2e^{-7t} + \dfrac{3}{10}e^{-2t} + \dfrac{1}{40}e^t}$.

Exercise 2. Calculate particular solutions of a system of differential equations under the following initial conditions:

a) $\begin{cases} \dfrac{dx}{dt} + 2x + y = \sin t, \\ \dfrac{dy}{dt} - 4x - 2y = \cos t, \quad x(\pi) = 1, \quad y(\pi) = 2; \end{cases}$

b) $\begin{cases} \dfrac{dx}{dt} = z + y - x, \\ \dfrac{dy}{dt} = z + x - y, \\ \dfrac{dz}{dt} = x + y + z, \quad x(0) = 1, \quad y(0) = z(0) = 0. \end{cases}$

Solution:

a) Let us calculate general solution of the system. We differentiate the first equation of the system and replace into the resulting equality expressions for $y'(t)$ and $x'(t)$: $x'' + 2x' + y' = \cos t$; $y'(t) = 4x + 2y + \cos t$, $y(t) = -x' - 2x + \sin t$, therefore $x'' + 2x' + 4x + 2(-x - 2x + \sin t) + \cos t = \cos t \Rightarrow$ $x'' + 2\sin t = 0$.

Having integrated the resulting equation twice, we get $x = 2\sin t + C_1 t + C_2$. We replace $x(t)$, $x'(t) = 2\cos t + C_1$ into the first equation of the system and calculate $y = -2\cos t - 3\sin t - C_1(2t+1) - 2C_2$.

The complexity $x(t)$, $y(t)$ makes a general solution of the system. We calculate a particular solution, which corresponds initial conditions $x(\pi) = 1$, $y(\pi) = 2$:

$$\begin{cases} C_1\pi + C_2 = 1, \\ 2 - C_1(1 + 2\pi) - 2C_2 = 2, \end{cases} \quad \begin{cases} C_1 = -2, \\ C_2 = 1 + 2\pi, \end{cases}$$

so $x = 2\sin t - 2t + 1 + 2\pi$,

$y = -2\cos t - 3\sin t + 2(2t+1) - 2(1+2\pi)$ is a particular solution.

b) We write and calculate characteristic equation of the system:

$$\begin{vmatrix} -1-k & 1 & 1 \\ 1 & -1-k & 1 \\ 1 & 1 & 1-k \end{vmatrix} = 0,$$

hence $k^3 + k^2 - 4k - 4 = 0$; $k_1 = -2$, $k_2 = -1$, $k_3 = 2$.

If $k_1 = -2$, then

$$\begin{cases} \alpha_1 + \beta_1 + \gamma_1 = 0, \\ \alpha_1 + \beta_1 + \gamma_1 = 0, \\ \alpha_1 + \beta_1 + 3\gamma_1 = 0, \end{cases}$$

therefore $\gamma_1 = 0$, $\alpha_1 = -\beta_1$.

If $\alpha_1 = 1$, $\beta_1 = -1$, $\gamma_1 = 0$: $x_1 = e^{-2t}$, $y_1 = -e^{-2t}$, $z_1 = 0$.

If $k_2 = -1$, then

$$\begin{cases} \beta_2 + \gamma_2 = 0, \\ \alpha_2 + \gamma_2 = 0, \\ \alpha_2 + \beta_2 + 2\gamma_2 = 0, \end{cases}$$

hence $\beta_2 = -\gamma_2$, $\alpha_2 = -\gamma_2$.

If $\beta_2 = 1$, $\gamma_2 = -1$, $\alpha_2 = 1$: $x_2 = e^{-t}$, $y_2 = e^{-t}$, $z_2 = -e^{-t}$.

If $k_3 = 2$, then

$$\begin{cases} -3\alpha_3 + \beta_3 + \gamma_3 = 0, \\ \alpha_3 - 3\beta_3 + \gamma_3 = 0, \\ \alpha_3 + \beta_3 - \gamma_3 = 0. \end{cases}$$

So, $\beta_3 = \dfrac{1}{2}\gamma_3$.

If $\beta_3 = 1$, $\gamma_3 = 2$, $\alpha_3 = 1$: $x_3 = e^{2t}$, $y_3 = e^{2t}$, $z_3 = 2e^{2t}$.

So, a general solution of the system:

$$\begin{cases} x(t) = C_1 e^{-2t} + C_2 e^{-t} + C_3 e^{2t}, \\ y(t) = -C_1 e^{-2t} + C_2 e^{-t} + C_3 e^{2t}, \\ z(t) = -C_2 e^{-t} + 2C_3 e^{2t}. \end{cases}$$

We apply initial conditions $x(0) = 1$, $y(0) = z(0) = 0$:

$$\begin{cases} C_1 + C_2 + C_3 = 1, \\ -C_1 + C_2 + C_3 = 0, \\ -C_2 + 2C_3 = 0, \end{cases} \qquad \begin{cases} C_1 = \frac{1}{2}, \\ C_2 = \frac{1}{3}, \\ C_3 = \frac{1}{6}. \end{cases}$$

Thus, $x(t) = \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t}$, $y(t) = -\frac{1}{2}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t}$, $z(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$ – is a particular solution of the system.

III. Exercises for individual work

Calculate systems of equation under the following initial conditions:

1) $\begin{cases} \frac{dx}{dt} + 3x + y = 0, \\ \frac{dy}{dt} - x + y = 0, \end{cases}$ $x(0) = 1$, $y(0) = -1$;

2) $\begin{cases} \frac{du}{dx} + 4v = \cos 2x, \\ \frac{dv}{dx} + 4u = \sin 2x, \end{cases}$ $u(0) = 0$, $v(0) = 0{,}1$.

Notes, solutions and keys to the exercises for individual work

Practical lesson 1

Exercise 1.

a) $y' = \dfrac{1}{2\sqrt{x}}$; $\quad 2\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}} = 1$; $\quad 1 = 1$.

b) $\dfrac{dS}{dt} = \left(-1 - \dfrac{1}{2} \cdot 2\cos 2t \right) = -1 - \cos 2t$; $\quad \dfrac{d^2 S}{dt^2} = 2\sin 2t$;

$$2\sin 2t + \operatorname{tg} t \cdot (-1 - \cos 2t) = \sin 2t, \qquad 2\sin 2t - 2\operatorname{tg} t \cdot \cos^2 t = \sin 2t,$$

$$2\sin 2t - 2\sin t \cdot \cos t = \sin 2t, \qquad 2\sin 2t - \sin 2t = \sin 2t,$$

$$\sin 2t = \sin 2t.$$

Exercise 2. Equations a)-c) are with separate variables; equations, d)-f) are homogeneous.

a) $(x+1)^3\, dy - (y-2)^2\, dx = 0$;

$$\int \frac{dy}{(y-2)^2} = \int \frac{dx}{(x+1)^3}; \quad -\frac{1}{y-2} = -\frac{1}{2(x+1)^2} + C_1;$$

$$\frac{1}{y-2} - \frac{1}{2(x+1)^2} = C,$$

where $C = -C_1$;

b) $y \cdot y' + x = 1$;

$$y \cdot \frac{dy}{dx} = 1 - x\ ; \quad \int y\,dy = \int (1 - x)\,dx\ ;$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C_1\ ; \quad y^2 = 2x + x^2 + C\ ,$$

where $C = 2C_1$;

c) $x^2(2yy' - 1) = 1$;

$$2yy' - 1 = \frac{1}{x^2}\ ; \quad 2y \cdot \frac{dy}{dx} = \frac{1}{x^2} + 1\ ;$$

$$\int 2y\,dy = \int \left(\frac{1}{x^2} + 1 \right) dx\ ; \quad y^2 = -\frac{1}{x} + x + C\ ;$$

d) $y = x\left(y' - \sqrt[x]{e^y} \right)$;

$$y' = \frac{y}{x} + \sqrt[x]{e^y}\ ; \quad y = ux\ , \quad y' = u'x + u\ ;$$

$$u'x + u = u + e^u\ ; \quad u'x = e^u\ ; \quad \frac{du}{dx} \cdot x = e^u\ ; \quad e^{-u}du = \frac{dx}{x}\ ;$$

$$-e^{-u} = \ln|x| + \ln C\ ; \quad -e^{-\frac{y}{x}} = \ln Cx\ ; \quad e^{-\frac{y}{x}} + \ln Cx = 0\ ;$$

e) $y^2 + x^2 y' = xy \cdot y'$;

$$\left(xy - x^2 \right) \cdot y' = y^2\ ; \quad y' = \frac{y^2}{xy - x^2}\ ; \quad y = ux\ , \quad y' = u'x + u\ ;$$

$$u'x + u = \frac{u^2 x^2}{x^2 u - x^2}\ ; \quad u'x + u = \frac{u^2}{u - 1}\ ; \quad u'x = \frac{u^2 - u^2 + u}{u - 1}\ ;$$

$$\frac{du}{dx} \cdot x = \frac{u}{u - 1}\ ; \quad \int \frac{u - 1}{u} du = \int \frac{dx}{x}\ ; \quad \int \left(1 - \frac{1}{u} \right) du = \int \frac{dx}{x}\ ;$$

$$u - \ln|u| = \ln|x| + \ln C\ ; \quad \frac{y}{x} = \ln \frac{y}{x} x \cdot C\ ; \quad \frac{y}{x} = \ln Cy\ ; \quad Cy = e^{\frac{y}{x}}\ ;$$

f) $xy' = y \cdot \ln\frac{y}{x}$;

$$y' = \frac{y}{x}\cdot\ln\frac{y}{x}; \quad y = ux, \quad y' = u'x + u; \quad u'x + u = u\ln u;$$

$$u'x = u\ln u - u; \quad \frac{du}{dx}\cdot x = u(\ln u - 1); \quad \int\frac{du}{u(\ln u - 1)} = \int\frac{dx}{x};$$

$$\ln|\ln u - 1| = \ln|x|\cdot C; \quad \ln u - 1 = Cx; \quad \ln u = Cx + 1;$$

$$u = e^{Cx+1}; \quad \frac{y}{x} = e^{Cx+1}; \quad y = x\cdot e^{Cx+1}.$$

Exercise 3.

a) $y^2 + x^2\cdot y' = 0, \quad y(-1) = 1$;

$$-x^2y' = y^2; \quad -\int\frac{dy}{y^2} = \int\frac{dx}{x^2}; \quad \frac{1}{y} = -\frac{1}{x} + C;$$

if $y(-1) = 1$, then:

$$1 = 1 + C \quad \Rightarrow \quad C = 0, \text{ therefore } \frac{1}{y} = -\frac{1}{x} \text{ or } x + y = 0;$$

b) $(1 + x^2)y^3dx - (y^2 - 1)x^3dy = 0, \quad y(1) = -1$;

$$\int\frac{y^2 - 1}{y^3}dy = \int\frac{1 + x^2}{x^3}dx; \quad \int\left(\frac{1}{y} - \frac{1}{y^3}\right)dy = \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx;$$

$$\ln|y| + \frac{1}{2y^2} = -\frac{1}{2x^2} + \ln|x| + C \quad \Rightarrow \quad C = \ln\left|\frac{y}{x}\right| + \frac{1}{2y^2} + \frac{1}{2x^2};$$

under the condition that $y(-1) = 1$ we have: $C = \ln|-1| + \frac{1}{2} + \frac{1}{2} = 1$,

hence $\ln\left|\frac{y}{x}\right| + \frac{1}{2y^2} + \frac{1}{2x^2} = 1$ or $x^{-2} + y^{-2} = 2\left(1 + \ln\left|\frac{x}{y}\right|\right)$.

Practical lesson 2

Exercise 1.

a) $y' - y = e^x$ is a linear equation, then $y = uv$, $y' = u'v + uv'$;

$$u'v + uv' - uv = e^x; \quad u'v + u\left(v' - v\right) = e^x; \quad v' - v = 0; \quad \int \frac{dv}{v} = \int dx;$$

$$\ln|v| = x \;\Rightarrow\; v = e^x; \quad u'e^x = e^x; \quad u' = 1; \quad \int du = \int dx; \quad u = x + C;$$

therefore $y = uv = \underline{\left(x + C\right) \cdot e^x}$.

b) $y' + \dfrac{1-2x}{x^2} y = 1$ is a linear equation, then $y = uv$,

$y' = u'v + uv'$;

$$u'v + uv' + \frac{1-2x}{x^2} uv = 1; \quad u'v + u\left(v' + \frac{1-2x}{x^2} v\right) = 1;$$

$$v' + \frac{1-2x}{x^2} v = 0; \quad \frac{dv}{dx} = \frac{2x-1}{x^2} v; \quad \int \frac{dv}{v} = \int \frac{2x-1}{x^2} dx;$$

$$\int \frac{dv}{v} = \int \left(\frac{2}{x} - \frac{1}{x^2}\right) dx;$$

$$\ln|v| = 2\ln|x| + \frac{1}{x}; \quad v = e^{\ln x^2 + \frac{1}{x}}; \quad v = x^2 \cdot e^{\frac{1}{x}};$$

$$u' \cdot x^2 \cdot e^{\frac{1}{x}} = 1; \quad \frac{du}{dx} = \frac{1}{x^2 \cdot e^{\frac{1}{x}}}; \quad \int du = \int \frac{e^{-\frac{1}{x}}}{x^2} dx;$$

$$\int du = \int e^{-\frac{1}{x}} d\left(-\frac{1}{x}\right); \; u = e^{-\frac{1}{x}} + C.$$

So, $y = uv = \left(e^{-\frac{1}{x}} + C\right) \cdot x^2 \cdot e^{\frac{1}{x}} = \underline{x^2 + Cx^2 e^{-\frac{1}{x}}}$.

c) $y' + \dfrac{y}{x+1} + y^2 = 0$; $\quad y' + \dfrac{y}{x+1} = -y^2$ or $-\dfrac{y'}{y^2} - \dfrac{1}{y} \cdot \dfrac{1}{x+1} = 1$ is the Bernoulli equation ($\alpha = 2$); substitution is $z = y^{1-\alpha} = y^{-1}$, $z' = -\dfrac{1}{y^2} \cdot y'$;

$z' - \dfrac{z}{x+1} = 1$ is a linear equation, then $z = uv$, $z' = u'v + uv'$;

$u'v + uv' - \dfrac{uv}{x+1} = 1$; $u'v + u\left(v' - \dfrac{v}{x+1}\right) = 1$;

$$v' - \frac{v}{x+1} = 0; \quad \frac{dv}{dx} = \frac{v}{x+1}; \quad \int \frac{dv}{v} = \int \frac{dx}{x+1};$$

$$\ln|v| = \ln|x+1| \quad \Rightarrow \quad v = x+1;$$

$$u'(x+1) = 1; \quad \int du = \int \frac{dx}{x+1}; \quad u = \ln|x+1| + C.$$

So, $z = uv = (x+1)\left(\ln|x+1| + C\right)$. Since $z = \dfrac{1}{y}$, then $y = \dfrac{1}{z}$, that is $y = \dfrac{1}{(x+1)\left(\ln|x+1| + C\right)}$.

d) $x\,dx = \left(\dfrac{x^2}{y} - y^3\right)dy$;

$x \cdot \dfrac{dx}{dy} = \dfrac{x^2}{y} - y^3$; $\quad x \cdot x' - \dfrac{x^2}{y} = -y^3$ or $x' - \dfrac{x}{y} = -y^3 \cdot x^{-1}$ is the Bernoulli equation ($x = x(y)$), $\alpha = -1$, $z = x^{1-\alpha} = x^2$, $z' = 2x \cdot x'$.

Then $\dfrac{z'}{2} - \dfrac{z}{y} = -y^3$ or $z' - \dfrac{2z}{y} = -2y^3$ is a linear equation;

$$z = uv, \quad z' = u'v + uv'; \quad u'v + uv' - \frac{2uv}{y} = -2y^3;$$

$$u'v + u\left(v' - \frac{2v}{y}\right) = -2y^3;$$

$$v' - \frac{2v}{y} = 0; \quad \frac{dv}{dy} = \frac{2v}{y}; \quad \frac{dv}{v} = 2\int\frac{dy}{y};$$

$$\ln|v| = 2\ln|y| \quad \Rightarrow \quad v = y^2;$$

$$u' \cdot y^2 = -2y^3; \quad u' = -2y; \quad \frac{du}{dy} = -2y;$$

$$\int du = -2\int y\,dy; \quad u = -y^2 + C.$$

Therefore, $z = uv = \left(C - y^2\right)\cdot y^2$. Since $z = x^2$, then $x^2 = y^2\left(C - y^2\right)$.

e) $\left(3x^2 \cdot y^2 + 7\right)dx + 2x^3y\,dy = 0$ is a total differential equation, since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 6x^2y$.

We calculate function

$u(x,y) = \int\left(3x^2y^2 + 7\right)dx + \phi(y) = x^3y^2 + 7x + \phi(y)$;

$\frac{\partial u}{\partial y} = 2x^3y + \phi'(y)$. Since $\frac{\partial u}{\partial y} = Q(x,y)$, that is $\frac{\partial u}{\partial y} = 2x^3y$, then

$2x^3y + \phi'(y) = 2x^3y \quad \Rightarrow \quad \phi'(y) = 0 \quad \Rightarrow \quad \phi(y) = C_1$.

Therefore, $u(x,y) = x^3y^2 + 7x + C_1$; $u(x,y) = C_2$, $x^3y^2 + 7x + C_1 = C_2$ or $x^3y^2 + 7x = C$.

f) $(2y - 3)dx + \left(2x + 3y^2\right)dy = 0$ is a total differential equation $\left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2\right)$.

Let us calculate $u(x,y)=\int(2y-3)\,dx+\phi(y)=2xy-3x+\phi(y)$;
$\frac{\partial u}{\partial y}=2x+\phi'(y)$. Since $\frac{\partial u}{\partial y}=2x+3y^2$, we calculate:

$$2x+\phi'(y)=2x+3y^2 \quad\Rightarrow\quad \phi'(y)=3y^2 \quad\Rightarrow\quad \phi(y)=y^3+C_1.$$

Then $u(x,y)=2xy-3x+y^3+C_1$; $u(x,y)=C_2$, so $2xy-3x+y^3=C$.

Exercise 2.

a) $(1-x)(y'+y)=e^{-x}$, $y(2)=0$; $y'+y=\frac{e^{-x}}{1-x}$ is a linear equation, then $y=uv$, $y'=u'v+uv'$; $u'v+uv'+uv=\frac{e^{-x}}{1-x}$;

$$v'+v=0\,;\quad \int\frac{dv}{v}=-\int dx\,;\quad \ln|v|=-x \quad\Rightarrow\quad v=e^{-x}\,;$$

$$u'\cdot e^{-x}=\frac{e^{-x}}{1-x}\,;\quad \int du=\int\frac{dx}{1-x}\,;\quad u=-\ln|1-x|+C\,;$$

$y=u\cdot v=\left(-\ln|1-x|+C\right)\cdot e^{-x}$ is a general solution.

Under the condition $y(2)=0$: $0=-\ln|-1|+C \quad\Rightarrow\quad C=0$, therefore $y=-e^{-x}\ln|1-x|$ is a particular solution of the equation.

b) $y'=3x^2y+x^5+x^2$, $y(0)=1$; $y'-3x^2y=x^5+x^2$ is a linear equation,

$$y=uv\,,\quad y'=u'v+uv'\,;\quad u'v+uv'-\ x^2uv=x^5+x^2\,;$$

$$v'=3x^2v\,;\quad \int\frac{dv}{v}=\int 3x^2\,dx\,;\quad \ln|v|=x^3 \quad\Rightarrow\quad v=e^{x^3}\,;$$

$$u'\cdot e^{x^3}=x^5+x^2\,;\quad \int du=\int e^{-x^3}\left(x^5+x^3\right)dx\,;$$

$$u=\int e^{-x^3}x^5dx+\int e^{-x^3}x^2\,dx\,;$$

$$I_1 = \frac{1}{3}\int e^{-x^3}\left(-x^3\right)\cdot\left(-3x^2\right)dx = \begin{bmatrix} -x^3 = t, \\ -3x^2\,dx = dt \end{bmatrix} = \frac{1}{3}\int e^t \cdot t\;dt =$$

$$= \begin{bmatrix} u = t, & du = dt, \\ dv = e^t\,dt, & v = e^t \end{bmatrix} = \frac{1}{3}\left(te^t - e^t\right) = \frac{1}{3}e^{-x^3}\left(-x^3 - 1\right);$$

$$I_2 = \int e^{-x^3}x^2\,dx = -\frac{1}{3}\int e^{-x^3}\left(-3x^2\right)dx = -\frac{1}{3}\int e^{-x^3}d\left(-x^3\right) = -\frac{1}{3}e^{-x^3}.$$

So, $u = I_1 + I_2 = -\frac{1}{3}e^{-x^3}\left(x^3+1\right) - \frac{1}{3}e^{-x^3} + C$, then $y = uv =$ $=\left(-\frac{1}{3}e^{-x^3}\left(x^3+2\right)+C\right)\cdot e^{x^3} = -\frac{1}{3}\left(x^3+2\right) + C\cdot e^{x^3}$ is a general solution. If $y(0)=1$, then: $1 = -\frac{2}{3} + C \quad \Rightarrow \quad C = \frac{5}{3}$, hence $y = -\frac{1}{3}\left(x^3+2\right) + \frac{5}{3}e^{x^3}$ is a particular solution of the equation.

c) $t\left(1+t^2\right)dx = \left(x + xt^2 - t^2\right)dt$, $x(1) = -\frac{\pi}{4}$;

having divided both sides of the equation by $t\left(1+t^2\right)dt$, we get: $\frac{dx}{dt} = \frac{x\left(1+t^2\right)}{t\left(1+t^2\right)} - \frac{t^2}{t\left(1+t^2\right)}$ or $x' - \frac{x}{t} = -\frac{t}{1+t^2}$ is a linear equation;

$$x = uv, \quad x' = u'v + uv', \quad u'v + uv' - \frac{uv}{t} = -\frac{t}{1+t^2};$$

$$v' = \frac{v}{t}; \quad \int\frac{dv}{v} = \int\frac{dt}{t};$$

$$\ln|v| = \ln|t| \quad \Rightarrow \quad v = t;$$

$$u'\cdot t = -\frac{t}{1+t^2}; \quad \int du = -\int\frac{dt}{1+t^2}; \quad u = -\operatorname{arctg} t + C.$$

As a result, $x = u \cdot v = \left(-\operatorname{arctg} t + C\right)t$ is a general solution of the equation. We calculate a particular solution of the equation under the condition that $x(1) = -\dfrac{\pi}{4}$:

$$-\frac{\pi}{4} = -\operatorname{arctg} 1 + C \quad \Rightarrow \quad C = 0\,, \quad y = -t \cdot \operatorname{arctg} t\,.$$

Practical lesson 3

Exercise 1.

a) $y''' = 60x^2$; having integrated three times, we calculate the solution of equation: $y'' = 20x^3 + C$; $y' = 5x^4 + Cx + C_2$; $y = x^5 + \dfrac{Cx^2}{2} + C_2 x + C_3$; $y = x^5 + C_1 x^2 + C_2 x + C_3$ ($C_1 = \dfrac{C}{2}$).

b) $y''' = \cos 2x$; $y'' = \dfrac{1}{2}\sin 2x + C$; $y' = -\dfrac{1}{4}\cos 2x + Cx + C_2$; $y = -\dfrac{1}{8}\sin 2x + C_1 x^2 + C_2 x + C_3$ ($C_1 = \dfrac{C}{2}$).

c) $xy'' = y'$; we label $y' = u$, $y'' = u'$, then $x \cdot u' = u$; $\displaystyle\int \frac{du}{u} = \int \frac{dx}{x}$; $\ln|u| = \ln|x| \cdot C_1' \Rightarrow u = C_1' x$; $y' = C_1' x$; $y = C_1 x^2 + C_2$ ($C_1 = \dfrac{C_1'}{2}$).

d) $\left(y''\right)^2 = y'$; we label $y' = u$, $y'' = u'$, then $\left(u'\right)^2 = u \Rightarrow$ $u' = \pm\sqrt{u}$; $\pm\displaystyle\int \frac{du}{\sqrt{u}} = \int dx$; $\pm 2\sqrt{u} = x + C_1 \Rightarrow 4u = \left(x + C_1\right)^2 \Rightarrow$ $u = \dfrac{1}{4}\left(x + C_1\right)^2$; $u = y' \Rightarrow y' = \dfrac{1}{4}\left(x + C_1\right)^2$; $y = \dfrac{1}{12}\left(x + C_1\right)^3 + C_2$.

e) $y'' = \dfrac{y'}{x} + x$; substitution is $y' = z$, $y'' = z'$, then $z' = \dfrac{z}{x} + x$ or $z' - \dfrac{z}{x} = x$ is a linear equation, $z = u \cdot v$, $z' = u'v + uv'$;

$u'v-uv'-\dfrac{uv}{x}=x$; $v'=\dfrac{v}{x}$; $\int\dfrac{dv}{v}=\int\dfrac{dx}{x}$; $\ln|v|=\ln|x| \Rightarrow v=x$;

$u'\cdot x=x$; $u'=1$; $u=x+C_1'$; so $z=u\cdot z=\left(x+C_1'\right)\cdot x=x^2+C_1'x$;

$z=y' \Rightarrow y'=x^2+C_1'x$; $y=\dfrac{x^3}{3}+C_1x^2+C_2$ ($C_1=\dfrac{C_1'}{2}$).

f) $2xy'\cdot y''=\left(y'\right)^2+1$; substitution is $y'=z$, $y''=z'$, then $2xz\cdot z'=z^2+1$; $\int\dfrac{2z}{z^2+1}dz=\int\dfrac{dx}{x}$; $\ln\left(z^2+1\right)=\ln x\cdot C_1 \Rightarrow$

$z=\sqrt{C_1x-1}$; $y'=\sqrt{C_1x-1}$; $y=\dfrac{2}{3C_1}(C_1x-1)^{\frac{3}{2}}+C_2$.

g) $y\cdot y''=\left(y'\right)^2$; substitution is $y'=z$, $y''=z'\cdot z$, then

$y\cdot z'\cdot z=z^2$; $\int\dfrac{dz}{z}=\int\dfrac{dy}{y}$; $\ln|z|=\ln|y|\cdot C_1 \Rightarrow z=C_1y$; $y'=C_1y$;

$\int\dfrac{dy}{y}=C_1\int dx$; $\ln|y|=C_1x+C_2' \Rightarrow y=e^{C_1x+C_2'} \Rightarrow$

$y=C_2\cdot e^{C_1x}$ ($C_2=e^{C_2'}$).

h) $y\left(1-\ln y\right)\cdot y''+\left(1+\ln y\right)\cdot\left(y'\right)^2=0$; substitution is $y'=z$, $y''=z'\cdot z$;

$$y\left(1-\ln y\right)\cdot z'\cdot z+\left(1+\ln y\right)\cdot z^2=0;\quad \int\frac{dz}{z}=\int\frac{\left(1+\ln y\right)dy}{y\cdot\left(\ln y-1\right)};$$

$$\int\frac{\left(1+\ln y\right)dy}{y\left(\ln y-1\right)}=\begin{bmatrix}\ln y=u,\\ \dfrac{dy}{y}=du\end{bmatrix}=\int\frac{u+1}{u-1}du=\int\left(1+\frac{2}{u-1}\right)du=$$

$$=u+2\ln|u-1|=\ln y+2\ln|\ln y-1|;$$

$$\ln|z|=\ln y+2\ln|\ln y-1|+C \quad\Rightarrow\quad z=y\cdot\left(\ln y-1\right)^2\cdot C_1;$$

$$y' = y(\ln y - 1)^2 \cdot C_1\,;\quad \int \frac{dy}{y(\ln y - 1)^2} = C_1 \int dx\,;$$

$$\int \frac{d(\ln y - 1)}{(\ln y - 1)^2} = -\frac{1}{\ln y - 1} = \frac{1}{1 - \ln y}\,;$$

$$\frac{1}{1 - \ln y} = C_1 x + C_2 \quad \Rightarrow \quad (1 - \ln y)(C_1 x + C_2) = 1\,.$$

Exercise 2.

a) $y'' = 3x^2$, $y(0) = 2$, $y'(0) = 1$.

$y' = x^3 + C_1$; $y'(0) = 1 \quad \Rightarrow \quad C_1 = 1$, so $y' = x^3 + 1$; $y = \dfrac{x^4}{4} + x + C_2$;

$y(0) = 2 \quad \Rightarrow \quad C_2 = 2$; $y = \dfrac{x^4}{4} + x + 2$.

b) $y'' = \dfrac{y'}{x} + \dfrac{x^2}{y'}$, $y(2) = 0$, $y'(2) = 4$.

The substitution is $y' = z$, $y'' = z'$; $z' - \dfrac{z}{x} = x^2 z^{-1}$ or $2z \cdot z' - \dfrac{2z^2}{x} = 2x^2$ is the Bernoulli equation ($\alpha = -1$), then $p = z^{1-\alpha} = z^2$, $p' = 2z \cdot z'$; so $p' - \dfrac{2p}{x} = 2x^2$ is a linear equation, $p = uv$, $p' = u'v + uv'$; $u'v + uv' - \dfrac{2uv}{x} = x^2$; $v' = \dfrac{2v}{x} \quad \Rightarrow \quad v = x^2$; $u'x^2 = 2x^2 \quad \Rightarrow \quad u = 2x + C_1$; then $p = uv = (2x + C_1) \cdot x^2$; $z = \pm\sqrt{p} = \pm x\sqrt{2x + C_1}$.

Under the condition that $y'(2) = 4$ we have: $4 = \pm 2\sqrt{4 + C_1} \quad \Rightarrow \quad C_1 = 0$, then $z = x\sqrt{2x}$; $y' = x\sqrt{2x}$;

$$y = \sqrt{2}\int x^{\frac{3}{2}}\,dx = \frac{2\sqrt{2}}{5} x^{\frac{5}{2}} + C_2 = \frac{2\sqrt{2}}{5} x^2 \sqrt{x} + C_2\,.$$

Since $y(2)=0$ we calculate $C_2=-\frac{16}{5}$, then $y=\frac{2}{5}x^2\sqrt{2x}-\frac{16}{5}$.

c) $2(y')^2=y''(y-1)$, $y(1)=2$, $y'(1)=-1$.

The substitution is $y'=z$, $y''=z'\cdot z$; $2z^2=z'\cdot z\cdot(y-1)$ or $\frac{z'}{z}=\frac{2}{y-1}$; $\int\frac{dz}{z}=2\int\frac{dy}{y-1}$; $\ln|z|=2\ln|y-1|+\ln C_1$ $\Rightarrow$ $z=(y-1)^2\cdot C_1$; $y'=C_1(y-1)^2$.

We apply initial condition $y=2$, $y'=-1$ $\Rightarrow$ $C_1=-1$, then $y'=-(y-1)^2$; $-\int\frac{dy}{(y-1)^2}=\int dx$; $\frac{1}{y-1}=x+C_2$ $\Rightarrow$ $y=\frac{1}{x+C_2}+1$.

We calculate C_2 under the condition that $y(1)=2$: $2=\frac{1}{C_2+1}+1$ $\Rightarrow$ $C_2=0$, therefore $y=\frac{1}{x}+1=\frac{x+1}{x}$.

Practical lesson 4

Exercise 1. Equations a)-d) are linear constant-coefficient homogeneous equations, e)-g) are nonhomogenenous equations.

a) $y''-5y'+6y=0$; $k^2-5k+6=0$; $k_1=2$, $k_2=3$ $\Rightarrow$ $y=C_1e^{2x}+C_2e^{3x}$.

b) $y'''-4y''+3y'=0$; $k^3-4k^2+3k=0$ or $k(k^2-4k+3)=0$; $k_1=0$, $k_2=1$, $k_3=3$; $y=C_1+C_2e^{x}+C_3e^{3x}$.

c) $y'''-3y''+3y'-y=0$; $k^3-3k^2+3k-1=0$ or $(k-1)^3=0$; $k_1=k_2=k_3=1$; $y=e^{x}\left(C_1+C_2x+C_3x^2\right)$.

d) $y'''+6y''+25y'=0$; $k^3+6k^2+25k=0$ or $k(k^2+6k+25)=0$; $k_1=0$, $k_{2,3}=-3\pm 4i$; $y=C_1+e^{-3x}\left(C_2\cos 4x+C_3\sin 4x\right)$.

e) $y'' - 2y' + 2y = 2x$; $k^2 - 2k + 2 = 0$ is a characteristic equation of a corresponding homogeneous equation; $k_{1,2} = 1 \pm i$; $\overline{y} = e^x\left(C_1 \cos x + C_2 \sin x\right)$; $y^* = Ax + B$ ($r = 0$), $\left(y^*\right)' = A$, $\left(y^*\right)'' = 0$; $-2A + 2Ax + 2B \equiv 2x$; $2A = 2 \Rightarrow A = 1$, $-2A + 2B = 0$ $\Rightarrow B = 1$; $y^* = x + 1$; $y = \overline{y} + y^* = \underline{e^x\left(C_1 \cos x + C_2 \sin x\right) + x + 1}$.

f) $y'' + y = -8\cos 3x$; $k^2 + 1 = 0$; $k_{1,2} = \pm i$; $\overline{y} = C_1 \cos x + C_2 \sin x$; $y^* = A\cos 3x + B\sin 3x$ ($r = 0$); $\left(y^*\right)' = -3A\sin 3x + 3B\cos 3x$; $\left(y^*\right)'' = -9A\cos 3x - 9B\sin 3x$;

$$-9A\cos 3x - 9B\sin 3x + A\cos 3x + B\sin 3x \equiv -8\cos 3x;$$

$$-8A\cos 3x - 8B\sin 3x \equiv -8\cos 3x;$$

$$A\cos 3x + B\sin 3x \equiv \cos 3x \Rightarrow A = 1,\ B = 0;\ y^* = \cos 3x;$$

$$y = \overline{y} + y^* = \underline{C_1 \cos x + C_2 \sin x + \cos 3x}.$$

g) $y'' - 2y' + y = xe^x$; $k^2 - 2k + 1 = 0$ or $\left(k - 1\right)^2 = 0$, $k_1 = k_2 = 1$ $\Rightarrow \overline{y} = e^x\left(C_1 + C_2 x\right)$;

$$y^* = \left(Ax + B\right)\cdot e^x \cdot x^2 = \left(Ax^3 + Bx^2\right)\cdot e^x \quad (k_1 = k_2 = 1,\ \alpha = 1 \Rightarrow r = 2);$$

$$\left(y^*\right)' = \left(3Ax^2 + 2Bx\right)\cdot e^x + \left(Ax^3 + Bx^2\right)\cdot e^x =$$

$$= \left(Ax^3 + \left(3A + B\right)x^2 + 2Bx\right)\cdot e^x;$$

$$\left(y^*\right)'' = \left(3Ax^2 + \left(6A + 2B\right)x + 2B\right)\cdot e^x + \left(Ax^3 + \left(3A + B\right)x^2 + 2Bx\right)\cdot e^x;$$

$$3Ax^2 + 6Ax + 2Bx + 2B + Ax^3 + 3Ax^2 + Bx^2 + 2Bx - 2Ax^3 - 6Ax^2 -$$

$$-2Bx^2 - 4Bx + + Ax^3 + Bx^2 \equiv x;\ 6Ax + 2B \equiv x \Rightarrow 6A = 1,\ A = \frac{1}{6},$$

$$B = 0;\ y^* = \frac{1}{6}x^3 \cdot e^x;\ y = \overline{y} + y^* = \underline{e^x\left(C_1 + C_2 x\right) + \frac{1}{6}x^3 \cdot e^x}.$$

Exercise 2.

a) $y'' - y = 0$; $k^2 - 1 = 0$, $k = \pm 1 \Rightarrow y = C_1 e^{-x} + C_2 e^x$ is a general solution of equation; $y(0) = 0$: $C_1 + C_2 = 0$; $y'(x) = -C_1 e^{-x} + C_2 e^x$; $y'(0) = 1$: $-C_1 + C_2 = 1$;

$$\begin{cases} C_1 + C_2 = 0, \\ -C_1 + C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = -\frac{1}{2}, \\ C_2 = \frac{1}{2}, \end{cases}$$ so $y = -\frac{1}{2}e^{-x} + \frac{1}{2}e^x$ is a particular solution.

b) $y'' + 2y' + 2y = 0$; $k^2 + 2k + 2 = 0$, $k_{1,2} = -1 \pm i \Rightarrow$ $y = e^{-x}\left(C_1 \cos x + C_2 \sin x\right)$ – is a general solution;

$$y(0) = 1:\ C_1 = 1;$$

$$y'(x) = -e^{-x}\left(C_1 \cos x + C_2 \sin x\right) + e^{-x}\left(-C_1 \sin x + C_2 \cos x\right);$$

$$y'(0) = 1,\ C_1 = 1 \Rightarrow -1 + C_2 = 1 \Rightarrow C_2 = 2,$$

so $y = e^{-x}\left(\cos x + 2\sin x\right)$ is a particular solution.

c) $y'' + y + \sin 2x = 0$ or $y'' + y = -\sin 2x$ is a nonhomogeneous equation;

$y'' + y = 0$ is a corresponding homogeneous equation, $k^2 + 1$ is its characteristic equation, $k_{1,2} = \pm i \Rightarrow \overline{y} = C_1 \cos x + C_2 \sin x$;

$y^* = A\cos 2x + B\sin 2x$, $\left(y^*\right)' = -2A\sin 2x + 2B\cos 2x$,

$\left(y^*\right)'' = -4A\cos 2x - 4B\sin 2x$; $-4A\cos 2x - 4B\sin 2x + A\cos 2x +$ $+ B\sin 2x \equiv -\sin 2x$; $-3A\cos 2x - 3B\sin 2x \equiv -\sin 2x \Rightarrow A = 0$,

$B = \frac{1}{3}$, $y^* = \frac{1}{3}\sin 2x$, so $y = \overline{y} + y^* = C_1 \cos x + C_2 \sin x + \frac{1}{3}\sin 2x$ is a general solution of the equation.

We apply initial conditions $y(\pi)=y'(\pi)=1$: $C_1=-1$; $y'(x)=-C_1\sin x+C_2\cos x+\frac{2}{3}\cos 2x$; $1=-C_2+\frac{2}{3}$ $\Rightarrow$ $C_2=-\frac{1}{3}$; $y=-\cos x-\frac{1}{3}\sin x+\frac{1}{3}\sin 2x$ is a particular solution.

d) $y''-2y'+10y=10x^2+18x+6$; $y''-2y'+10y=0$, $k^2-2k+10=0$, $k_{1,2}=1\pm 3i$ $\Rightarrow$ $\overline{y}=e^x\left(C_1\cos 3x+C_2\sin 3x\right)$ is a general solution of a characteristic equation of a corresponding homogeneous equation; we calculate a particular solution in the form $y^*=Ax^2+Bx+C$, $(y^*)'=2Ax+B$, $(y^*)''=2A$;

$$2A-4Ax-2B+10Ax^2+10Bx+10C\equiv 10x^2+18x+6;$$

$$10Ax^2+(-4A+10B)x+(2A-2B+10C)\equiv 10x^2+18x+6;$$

$$\begin{cases}10A=10,\\ -4A+10B=18,\\ 2A-2B+10C=6\end{cases}\qquad \begin{cases}A=1,\\ B=2,2,\\ C=0,84,\end{cases}$$

then $y^*=x^2+2,2x+0,84$, so $y=e^x\left(C_1\cos 3x+C_2\sin 3x\right)+x^2+$ $+2,2x+0,84$ is a general solution of the equation.

We calculate a particular solution of the equation under initial conditions $y(0)=1$, therefore $C_1+0,84=1$, $C_1=0,16$ and $y'(0)=3,2$, that is

$$y'(x)=e^x\left(C_1\cos 3x+C_2\sin 3x\right)+e^x\left(-3C_1\sin 3x+3C_2\cos 3x\right)+2x+2,2,$$

$$3,2=C_1+3C_2+2,2,\ C_1=0,16 \ \Rightarrow\ 3C_2=1-0,16\ \Rightarrow\ C_2=0,28.$$

Therefore, $y=e^x\left(0,16\cos 3x+0,28\sin 3x\right)+x^2+2,2x+0,84$ – is a particular solution, which satisfies the initial conditions.

e) $4y''+16y'+15y=4e^{-\frac{3}{2}x}$; $4k^2+16k+15=0$, $k_1=-\frac{3}{2}$, $k_2=-\frac{5}{2}$ $\Rightarrow$ $\overline{y}=C_1e^{-\frac{3}{2}x}+C_2e^{-\frac{5}{2}x}$; $y^*=Axe^{-\frac{3}{2}x}$ ($k_1=-\frac{3}{2}$, $\alpha=-\frac{3}{2}$ $\Rightarrow$ $r=1$), $(y^*)'=Ae^{-\frac{3}{2}x}-\frac{3}{2}Axe^{-\frac{3}{2}x}=\left(A-\frac{3}{2}Ax\right)\cdot e^{-\frac{3}{2}x}$,

$(y^*)''=-\frac{3}{2}Ae^{-\frac{3}{2}x}-\frac{3}{2}\left(A-\frac{3}{2}Ax\right)\cdot e^{-\frac{3}{2}x}$;

$$-6Ae^{-\frac{3}{2}x}-6Ae^{-\frac{3}{2}x}+9Axe^{-\frac{3}{2}x}+16Ae^{-\frac{3}{2}x}-24Axe^{-\frac{3}{2}x}+15Axe^{-\frac{3}{2}x}\equiv 4e^{-\frac{3}{2}x};$$

$$4A=4 \Rightarrow A=1, \text{ then } y^*=x\cdot e^{-\frac{3}{2}x},$$

$$y=\overline{y}+y^*=C_1e^{-\frac{3}{2}x}+C_2e^{-\frac{5}{2}x}+x\cdot e^{-\frac{3}{2}x}\equiv(C_1+x)\cdot e^{-\frac{3}{2}x}+C_2\cdot e^{-\frac{5}{2}x}$$

is a general solution of the equation.

Under the initial conditions $y(0)=3$ and $y'(0)=-5{,}5$ we calculate C_1 and C_2:

$$y(0)=3:\ C_1+C_2=3;\ y'(x)=e^{-\frac{3}{2}x}-\frac{3}{2}(C_1+x)\cdot e^{-\frac{3}{2}x}-\frac{5}{2}C_2\cdot e^{-\frac{5}{2}x};$$

$$y'(0)=-5{,}5:\ -\frac{11}{2}=1-\frac{3}{2}C_1-\frac{5}{2}C_2 \Rightarrow 3C_1+5C_2=13;$$

$$\begin{cases} C_1+C_2=3, \\ 3C_1+5C_2=13, \end{cases} \quad \begin{cases} C_1=1, \\ C_2=2. \end{cases}$$

So, $y=(1+x)e^{-\frac{3}{2}x}+2\cdot e^{-\frac{5}{2}x}$ is a particular solution of the equation.

f) $y''-2y'=e^x\left(x^2+x-3\right)$; $k^2-2k=0$, $k_1=0$, $k_2=2$ $\Rightarrow$ $\overline{y}=C_1+C_2e^{2x}$; $y^*=\left(Ax^2+Bx+C\right)e^x$;

$$\left(y^*\right)' = \left(2Ax + B\right)\cdot e^x + \left(Ax^2 + Bx + C\right)\cdot e^x;$$

$$\left(y^*\right)'' = 2Ae^x + \left(2Ax + B\right)e^x + \left(2Ax + B\right)e^x +$$

$$+\left(Ax^2 + Bx + C\right)e^x = 2Ae^x + \left(4Ax + 2B\right)e^x + \left(Ax^2 + Bx + C\right)e^x;$$

$$2Ae^x + \left(4Ax + 2B\right)e^x + \left(Ax^2 + Bx + C\right)e^x -$$

$$-\left(4Ax + 2B\right)e^x - 2\left(Ax^2 + Bx + C\right)e^x \equiv e^x\left(x^2 + x - 3\right);$$

$$2A - \left(Ax^2 + Bx + C\right) \equiv x^2 + x - 3;$$

$$-Ax^2 - Bx + 2A - C \equiv x^2 + x - 3 \Rightarrow A = -1,\ B = -1,\ C = 1,$$

$$\text{then } y^* = \left(-x^2 - x + 1\right)e^x;$$

$y = \overline{y} + y^* = C_1 + C_2e^{2x} + \left(-x^2 - x + 1\right)e^x$ is a general solution of the equation.

We apply initial conditions:

$y(0) = 2 \Rightarrow 2 = C_1 + C_2 + 1,\ C_1 + C_2 = 1;$

$y'(x) = 2C_2e^{2x} + \left(-2x - 1\right)e^x + \left(-x^2 - x + 1\right)e^x;$

$y'(0) = 2 \Rightarrow 2 = 2C_2 - 1 + 1 \Rightarrow C_2 = 1$, then $C_1 = 0$.

As a result, $y = e^{2x} + \left(-x^2 - x + 1\right)e^x$ is a particular solution.

Practical lesson 5

1) $\begin{cases} \dfrac{dx}{dt} + 3x + y = 0, \\ \dfrac{dy}{dt} - x + y = 0. \end{cases}$

We differentiate the first equation of the system $x'' + 3x' + y' = 0$; from the second equation $y' = x - y$, so $x'' + 3x' + x - y = 0$; having substituted from the first equation of the system $y=-x'-3x$, we get $x''+3x'+x-\left(x'-3x\right)=0$ or $x''+4x'+4x=0$ is a linear homogeneous second-order equation, $k^2+4k+4=0$ is a characteristic equation, $k_1=k_2=-2$ are its roots, therefore $x\left(t\right)=e^{-2t}\left(C_1t+C_2\right)$.

We calculate $x'\left(t\right)=-2e^{-2t}\left(C_1t+C_2\right)+C_1\cdot e^{-2t}$, then

$$y=-x'-3x=-\left(-2e^{-2t}\left(C_1t+C_2\right)+C_1e^{-2t}\right)-3e^{-2t}\left(C_1t+C_2\right)=$$

$=-e^{-2t}\left(C_1t+C_2\right)-C_1e^{-2t}$, so $y\left(t\right)=-e^{-2t}\left(C_1t+C_2-C_1\right)$.

We apply initial conditions $x(0)=1$, $x\left(0\right)=1$, $y\left(0\right)=-1$ and calculate from the first equality $C_2=1$; from the second equality: $-1=-\left(C_2-C_1\right)$, so $C_1=0$, and $x\left(t\right)=e^{-2t}$, $y\left(t\right)=-e^{-2t}$ is a particular solution of the system.

$$2)\ \begin{cases}\dfrac{du}{dx}+4v=\cos 2x,\\[2mm] \dfrac{dv}{dx}+4u=\sin 2x.\end{cases}$$

From the first equation of the system we calculate $u''+4v'=\cos 2x$; having considered, that $v'=-4u+\sin 2x$, we get $u''+4\left(-4u+\sin 2x\right)=-2\sin 2x$ or $u''-16u=-6\sin 2x$ is a linear homogeneous constant-coefficient equation. We calculate general solution in the form $u=\overline{u}+u^*$.

We write a characteristic equation $k^2-16=0$, $k_{1,2}=\pm 4$ is a solution of characteristic equation of a corresponding homogeneous equation; $\overline{u}=C_1e^{-4x}+C_2e^{4x}$ is a general solution of a corresponding homogeneous equation. We calculate a particular solution in the form $u^*=A\cos 2x+B\sin 2x$, then

$(u^*)' = -2A\sin 2x + 2B\cos 2x$, $(u^*)'' = -4A\cos 2x - 4B\sin 2x$;

$-4A\cos 2x - 4B\sin 2x - 16A\cos 2x - 16B\sin 2x \equiv -6\sin 2x$, and

$-20A = 0$, $-20B = -6$, so $A = 0$, $B = 0{,}3$, then $u^* = 0{,}3\sin 2x$;

$u = \overline{u} + u^* = C_1 e^{-4x} + C_2 e^{4x} + 0{,}3\sin 2x$.

We calculate v' from the second equation of the system: $v' = -4u + \sin 2x$; $v' = -4C_1 e^{-4x} - 4C_2 e^{4x} - 1{,}2\sin 2x + \sin 2x$, from which $v(x) = -4\left(-\frac{1}{4}\right)C_1 e^{-4x} - 4\cdot\frac{1}{4}C_2 e^{4x} + 0{,}2\cdot\frac{1}{2}\cos 2x$, as a result, we get $v(x) = C_1 e^{-4x} - C_2 e^{4x} + 0{,}1\cos 2x$.

Under the initial conditions $u(0) = 0$ and $v(0) = 0{,}1$ we calculate a particular solution of the system of equations:

$$\begin{cases} u(0) = 0: & C_1 + C_2 = 0, \\ v(0) = 0{,}1: & C_1 - C_2 + 0{,}1 = 0{,}1, \end{cases}$$

from which $C_1 = C_2 = 0$, then $u(x) = 0{,}3\sin 2x$, $v(x) = 0{,}1\cos 2x$.

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