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MULTIPLE, CURVILINEAR, SURFACE INTEGRALS

КРАТНІ, КРИВОЛІНІЙНІ, ПОВЕРХНЕВІ ІНТЕГРАЛИ

Tutorial / Навчальний посібник

Рекомендовано Методичною радою НУК

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К77 **Кратні**, криволінійні, поверхневі інтеграли : навчальний посібник / Н. О. Романчук, М. В. Ушкац, О. В. Майборода, І. В. Пєтков. – Миколаїв : НУК, 2021. – 128 с.

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Представлено навчальний посібник для проведення лекційних та практичних занять з розділу «Кратні, криволінійні, поверхневі інтеграли». Матеріал подано у вигляді семи лекцій, кожна з яких містить основні означення, властивості, теореми та приклади розв'язування задач. Наведено приклади розв'язків більш ніж ста задач з даного розділу.

Призначено студентам, аспірантам та викладачам для підготовки до лекційних і практичних занять. Також буде корисним студентам, яким зручно вивчати вищу математику англійською мовою.

The tutorial is developed as material for lectures and practical lessons from the section "Multiple, curvilinear, surface integrals". The teaching material is presented in seven lectures, each containing key definitions, properties, theorems and examples of calculations as well as the examples of solving more than 100 tasks from this section.

The tutorial is designed for students, postgraduates and university teachers to prepare for lectures and practical lessons. The tutorial will be useful for students who are comfortable studying higher mathematics in English.

УДК 517.37(075.8)

INTRODUCTION

The tutorial is developed for lectures from the section "Multiple, Curvilinear, Surface Integrals". The teaching material is given in the form of seven lectures. The examples of solving of more than 100 tasks from this section are represented.

The provided tutorial is developed to meet the current requirements for an essential increase in the level of basic mathematical training and to strengthen the applied orientation of the course and aims to help students in self-mastery with theoretical knowledge and necessary practical skills. The tutorial will be useful for students who are comfortable studying higher mathematics in English.

The given number of examples of solving tasks not only satisfies the needs of students in the practical training of the corresponding topics of the section, but also gives the teacher the opportunity to diversify the choice of tasks within the presented section.

ВСТУП

Навчальний посібник розроблено для проведення лекцій з розділу «Кратні, криволінійні, поверхневі інтеграли». Навчальний матеріал подано у вигляді семи лекцій. Наведено приклади розв'язування більш ніж ста задач з даного розділу.

Представлений навчальний посібник створено з огляду на сучасні вимоги щодо істотного підвищення рівня фундаментальної математичної підготовки і посилення прикладної спрямованості курсу та має на меті допомогти студентам у самостійному оволодінні теоретичними знаннями та необхідними практичними навичками. Посібник буде корисним студентам, яким зручно вивчати вищу математику англійською мовою.

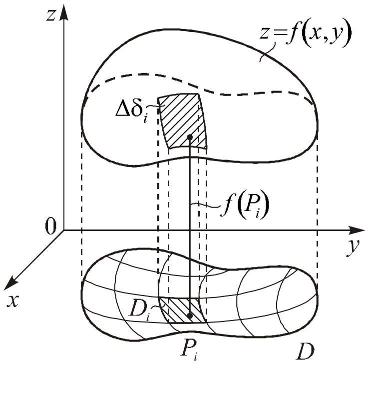
Наведена кількість прикладів розв'язування задач не лише задовольняє потреби студентів у практичному закріпленні відповідних тем розділу, а й надає можливість викладачеві урізноманітнити вибір задач в межах представленого розділу.

Lecture 1. DOUBLE INTEGRAL: DEFINITION, PROPERTIES, CALCULATION

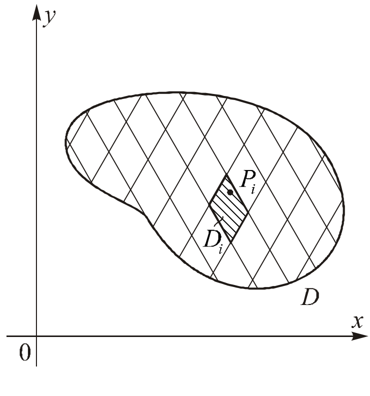
Tasks leading to the definition of double integral

Task 1. (About the volume of a cylindrical solid)

The given solid is bounded by the area $z = f(x, y) \geq 0$ above, by the closed bounded domain $D \subset Oxy$ below, and by the cylindrical area on the sides, the generatrix of this cylindrical area which are parallel to the axis Oz (Drawing 1).



Drawing 1



Drawing 2

We arbitrarily divide domain D into n parts D_i that do not have any common inner points, and the areas of which are ΔS_i ($i = 1, 2, ..., n$). We choose an arbitrary point $P_i(\xi_i; \eta_i) \in D_i$, the function in this point is $f(P_i) = f(\xi_i;\eta_i)$. We calculate the product $f(\xi_i;\eta_i)\cdot\Delta S_i$, which is equal to the volume of cylindrical column with generatrix parallel to the axis *Oz,* to the base D_i and to the altitude $f(P_i)$. The general number of these columns is *n*, and the sum of their volumes is approximately equal to the volume of a cylindrical solid:

$$V \approx V_n = \sum_{i=1}^{n} f(\xi_i;\eta_i)\cdot\Delta S_i\ . \qquad (1)$$

We designate the biggest diameter of domains D_i as $\lambda = \max\limits_{1\le i\le n} d(D_i)$; a diameter of a domain is its biggest chord. The volume of the given cylindrical solid is calculated as limit of the sum (1) when $\lambda \to 0$:

$$V = \lim_{\lambda\to 0} V_n = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i;\eta_i)\cdot\Delta S_i\ . \qquad (2)$$

Task 2. (About the mass of a material plate)

We have a flat inhomogeneous material plate in the shape of domain *D* (Drawing 2). Continuous function $\gamma = \gamma(x, y)$ in domain *D* determines the density of the plate at the point $(x; y)$.

We arbitrarily divide domain *D* into parts D_i that do not have any common inner points, and the areas of which are equal to ΔS_i ($i = 1, 2, ..., n$). We choose an arbitrary point $P_i(\xi_i;\eta_i) \in D_i$, and calculate density in this point $\gamma(P_i) = \gamma(\xi_i;\eta_i)$. Then the mass of the part D_i is equal to the product $\gamma(P_i)\cdot\Delta S_i$, and the mass *m* of the whole material plate is approximately equal to the sum of:

$$m \approx m_n = \sum_{i=1}^{n} \gamma(\xi_i;\eta_i)\cdot\Delta S_i\ . \qquad (3)$$

We calculate the exact value of the mass as equal to the limit of the sum (3) when $\lambda \to 0$, i.e.:

$$m = \lim_{\lambda \to 0} m_n = \lim_{\lambda \to 0} \sum_{i=1}^{n} \gamma(\xi_i ; \eta_i) \cdot \Delta S_i \ . \tag{4}$$

So, different tasks lead to calculating limits of the same kind. Each of such limits is called double integral. In such case occurs the need to learn properties of these limits, regardless the content of the task.

Definition of double integral

Let us assume that function $f(x, y)$ is determined in a closed bounded domain $D \subset R^2$. We arbitrarily divide domain *D* into *n* parts D_i, the areas of which are equal to ΔS_i ($i = 1, 2, \ldots, n$); $\lambda = \max\limits_{1 \le i \le n} d(D_i)$ is the biggest diameter of domains D_i. We choose an arbitrary point $P_i(\xi_i ; \eta_i)$ in each domain D_i and calculate the sum

$$I_n = \sum_{i=1}^{n} f(\xi_i ; \eta_i) \cdot \Delta S_i \ , \tag{5}$$

which we will call the integral sum for function $f(x, y)$ in domain *D*.

If $\lambda \to 0$, and the integral sum (5) has the finite limit, that depends neither on the way we divide domain *D* into parts D_i, nor on the choice of the points P_i, then this limit is called double integral of function $f(x, y)$ in domain *D*:

$$\iint_D f(x, y)\, dxdy = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i ; \eta_i) \cdot \Delta S_i \ . \tag{6}$$

Function $f(x, y)$ is called integrable in domain *D*; *D* is an integrating domain; *x,y* are integrating variables; *dxdy* is an element of an area.

Geometrical meaning of double integral:

$V = \iint\limits_D f(x, y)\, dxdy$ – volume of a cylindrical solid.

Mechanical meaning of double integral:

$$m = \iint_D \gamma(x, y)\, dxdy - \text{ mass of a material plate.}$$

Theorem (sufficient condition of integrability of a function)

If function $f(x, y)$ is continuous in a closed bounded domain *D*, then it is integrable in this domain.

Later on we consider the function $f(x, y)$ that is being integrated in the domain *D* as a continuous one.

Properties of double integral

1. $\iint_D C \cdot f(x, y)\, dxdy = C \cdot \iint_D f(x, y)\, dxdy$, $C - \text{const}$.

2. $\iint_D (f(x, y) \pm g(x, y))\, dxdy = \iint_D f(x, y)\, dxdy \pm \iint_D g(x, y)\, dxdy$.

The property is valid for a finite number of functions.

3. $\iint_D f(x, y)\, dxdy \geq 0$, if $f(x, y) \geq 0$ in domain *D*.

4. $\iint_D f(x, y)\, dxdy \geq \iint_D g(x, y)\, dxdy$ if $f(x, y) \geq g(x, y)$ in domain *D*.

5. $\iint_D f(x, y)\, dxdy = \iint_{D_1} f(x, y)\, dxdy + \iint_{D_2} f(x, y)\, dxdy$,

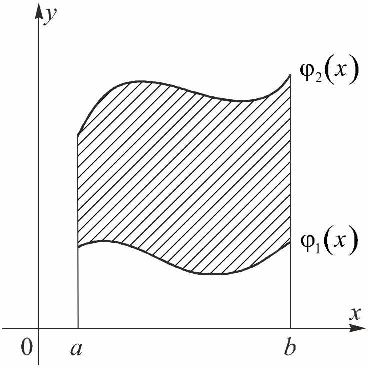
where domains D_1, D_2 are the parts of the divided integrating domain *D*.

6. $mS \leq \iint_D f(x, y)\, dxdy \leq MS$,

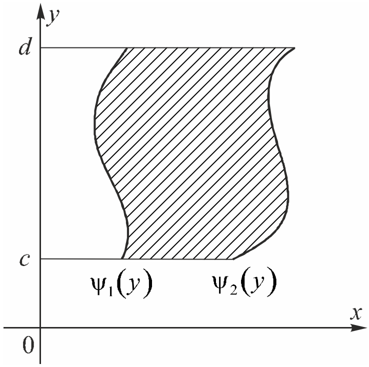
where *S* is the area of domain *D*; *m*, *M* are respectively the least and the biggest values of the function that is being integrated in domain *D*.

Calculation of double integrals

1) Integrating domain D is bounded by the lines $x = a$ and $x = b$ ($a < b$) on the left and on the right, and continuous curves $y = \varphi_1(x)$ and $y = \varphi_2(x)$ ($\varphi_1(x) \le \varphi_2(x)$) below and above (Drawing 3).



Drawing 3



Drawing 4

In this case calculation of the integral (6) means calculating repeated integral according to the formula:

$$\iint_D f(x,y)\,dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,. \tag{7}$$

2) Integrating domain D is bounded by lines $y = c$ and $y = d$ ($c < d$) below and above, and by continuous curves $x = \psi_1(y)$ and $x = \psi_2(y)$ ($\psi_1(y) \le \psi_2(y)$) on the left and on the right (Drawing 4). For such domain double integral (6) is calculated according to the formula:

$$\iint_D f(x,y)\,dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,. \tag{8}$$

Task 1. Calculate double integrals of rectangular domains.

a) $\iint\limits_D e^{x+y}\,dxdy$, D: $0 \le x \le 1$, $0 \le y \le 1$;

$$\iint\limits_D e^{x+y}\,dxdy = \iint\limits_D e^{x}\cdot e^{y}\,dxdy = \int_0^1 e^x dx \int_0^1 e^y dy = e^x\Big|_0^1 \cdot e^y\Big|_0^1 = (e-1)^2.$$

b) $\iint\limits_D x^2 y\cdot\cos\left(xy^2\right)dxdy$, D: $0 \le x \le \dfrac{\pi}{2}$, $0 \le y \le 2$;

$$\iint\limits_D x^2 y\cdot\cos\left(xy^2\right)dxdy = \frac{1}{2}\int_0^{\pi/2} x^2\,dx \cdot \int_0^2 2y\cos\left(xy^2\right)dy =$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2\,dx \cdot \frac{1}{x}\sin\left(xy^2\right)\Big|_0^2 =$$

$$= \frac{1}{2}\int_0^{\pi/2} x\cdot\sin 4x\;dx = \begin{bmatrix} x = u, & dx = du, \\ \sin 4x\;dx = dv, & v = -\dfrac{1}{4}\cos 4x \end{bmatrix} =$$

$$= \frac{1}{2}\left(-\frac{x}{4}\cos 4x\Big|_0^{\pi/2} + \frac{1}{4}\int_0^{\pi/2}\cos 4x\;dx\right) =$$

$$= \frac{1}{2}\left(-\frac{\pi}{8}\cos 2\pi + \frac{1}{16}\cos 4x\Big|_0^{\frac{\pi}{2}}\right) =$$

$$= \frac{1}{2}\left(-\frac{\pi}{8} + \frac{1}{16}\left(\cos 2\pi - \cos 0\right)\right) = -\frac{\pi}{16}.$$

Task 2. Calculate repeated integrals.

a) $$\int_0^1 dx\int_x^{2x}(x-y+1)\,dy=\int_0^1 dx\left((x+1)\cdot y-\frac{y^2}{2}\right)\Bigg|_x^{2x}=$$

$$=\int_0^1\left(2x^2+2x-2x^2-\left(x^2+x-\frac{x^2}{2}\right)\right)dx=\int_0^1\left(-\frac{x^2}{2}+x\right)dx=$$

$$=\left(-\frac{x^3}{6}+\frac{x^2}{2}\right)\Bigg|_0^1=-\frac{1}{6}+\frac{1}{2}=\frac{1}{3}.$$

b) $$\int_1^2 dx\int_{1/x}^{x}\frac{x^2}{y^2}\,dy=\int_1^2 x^2\,dx\cdot\left(-\frac{1}{y}\right)\Bigg|_{\frac{1}{x}}^{x}=-\int_1^2 x^2\,dx\cdot\left(\frac{1}{x}-x\right)=$$

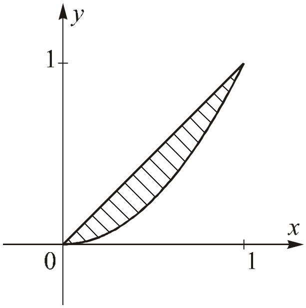
$$=\int_1^2\left(x^3-x\right)dx=\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Bigg|_1^2=4-2-\left(\frac{1}{4}-\frac{1}{2}\right)=2\frac{1}{4}.$$

c) $$\int_2^4 dy\int_0^y\frac{y^3}{x^2+y^2}\,dx=\int_2^4 y^3\,dy\int_0^y\frac{dx}{x^2+y^2}=\int_2^4 y^3\,dy\cdot\frac{1}{y}\operatorname{arctg}\frac{x}{y}\Bigg|_0^y=$$

$$=\int_2^4 y^2\left(\operatorname{arctg}1-\operatorname{arctg}0\right)dy=\frac{\pi}{4}\cdot\frac{y^3}{3}\Bigg|_2^4=\frac{\pi}{12}\left(4^3-2^3\right)=\frac{14\pi}{3}.$$

Task 3. Change the order of integration of the following integrals:

a) $I_1=\int_0^1 dy\int_y^{\sqrt{y}} f(x,y)\,dx$.

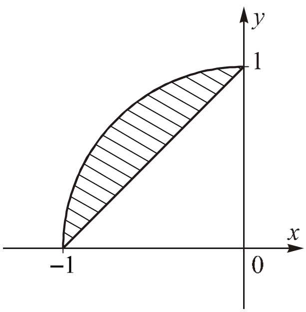


Drawing 5

We represent the integrating domain determined by the lines $y=0$, $y=1$, $x=y$, $x=\sqrt{y}$ (Drawing 5). Having changed the order of integration, we get new limits:

$x=0$, $x=1$, $y=x^2$, $y=x$, i.e.

$$I_1=\int_0^1 dx\int_{x^2}^{x} f(x,y)\,dy.$$



Drawing 6

b) $I_2=\int_{-1}^{0} dx\int_{x+1}^{\sqrt{1-x^2}} f(x,y)\,dy.$

The initial limits are $x=-1$, $x=0$, $y=x+1$, $y=\sqrt{1-x^2}$ (Drawing 6); we express the new limits of integration: $x=y-1$, $x=-\sqrt{1-y^2}$, $y=0$, $y=1$, consequently

$$I_2=\int_0^1 dy\int_{-\sqrt{1-y^2}}^{y-1} f(x,y)\,dx.$$

c) $I_3=\int_1^e dx\int_0^{\ln x} f(x,y)\,dy.$

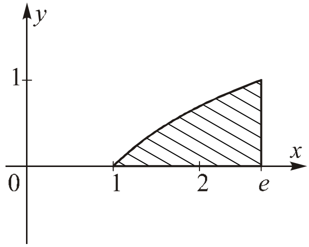
$y=0$, $y=1$, $x=e^y$, $x=e$, that is why $I_3=\int_0^1 dy\int_{e^y}^{e} f(x,y)\,dx.$

Integrating domain is represented on Drawing 7, out of which we find new limits:

$$y = 0\,, y = 1\,, x = e^{y}\,, x = e\,,$$

that is why

$$I_3 = \int_0^1 dy \int_{e^y}^{e} f(x,y)\,dx\,.$$



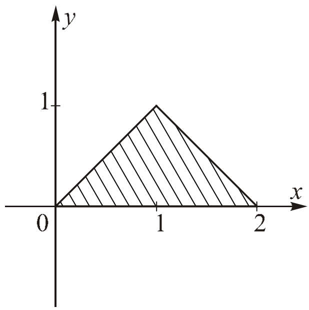
Drawing 7

Task 4. Changing the integrating order, rewrite the given expression in the form of twofold integral.

a) $I_1 = \int_0^1 dx \int_0^x f(x,y)\,dx + \int_1^2 dx \int_0^{2-x} f(x,y)\,dy\,.$

Integrating domain is represented on Drawing 8. We change the order of integration and rewrite the given integral as one twofold.

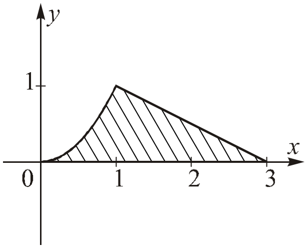
$$I_1 = \int_1^2 dy \int_y^{2-y} f(x,y)\,dx\,.$$



Drawing 8

b) $I_2 = \int_0^1 dx \int_0^{x^2} f(x,y)\,dy + \int_1^3 dx \int_0^{\frac{3-x}{2}} f(x,y)\,dy\,.$

The given integrating limits $0 \le x \le 1$, $0 \le y \le x^2$ and $0 \le x \le 3$, $0 \le y \le \dfrac{3-x}{2}$ (Drawing 9) may be represented in the following way: $\sqrt{y} \le x \le 3-2y$, $0 \le y \le 1$, so

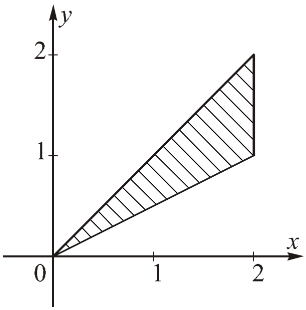


Drawing 9

$$I_2 = \int_0^1 dy \int_{\sqrt{y}}^{3-2y} f(x, y)\, dx .$$

Task 5. Calculate double integrals in the given domains:

a) $\iint_D \frac{x\, dxdy}{x^2 + y^2}$; $D:\ x = 2,\ y = x,$ $x = 2y$ (Drawing 10).



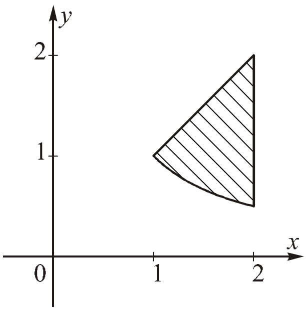
Drawing 10

$$\iint_{D_1} \frac{x\, dxdy}{x^2 + y^2} = \int_0^2 x\, dx \int_{x/2}^{x} \frac{dy}{x^2 + y^2} =$$

$$= \int_0^2 x\, dx \cdot \frac{1}{x} \operatorname{arctg} \frac{y}{x} \Bigg|_{\frac{x}{2}}^{x} =$$

$$= \int_0^2 \left(\operatorname{arctg} 1 - \operatorname{arctg} \frac{1}{2} \right) dx = \left(\frac{\pi}{4} - \operatorname{arctg} \frac{1}{2} \right) \cdot x \Bigg|_0^2 = \frac{\pi}{2} - 2 \operatorname{arctg} \frac{1}{2};$$

b) $\iint_D \frac{x^2}{y^2}\, dxdy$; $D:\ x = 2,\ y = x,$ $xy = 1$ (Drawing 11).



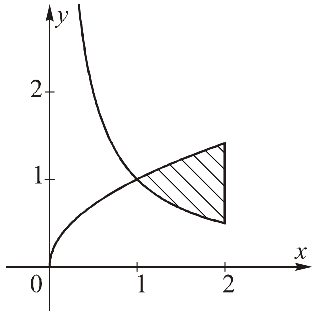
Drawing 11

$$\iint_D \frac{x^2}{y^2}\, dxdy = \int_1^2 x^2\, dx \int_{1/x}^{x} \frac{dy}{y^2} =$$

$$=\int_1^2 x^2\,dx\cdot\left(-\frac{1}{y}\right)\Bigg|_{\frac{1}{x}}^{x}=\int_1^2 x^2\,dx\cdot\left(x-\frac{1}{x}\right)=\int_1^2\left(x^3-x\right)dx=$$

$$=\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Bigg|_1^2=4-2-\frac{1}{4}+\frac{1}{2}=2\frac{1}{4}.$$

c) $\iint\limits_D y\ln x\,dxdy$; D : $xy=1$, $y=\sqrt{x}$, $x=2$ (Drawing 12).



Drawing 12

$$\iint\limits_D y\ln x\;dxdy=\int_1^2\ln x\;dx\int_{1/x}^{\sqrt{x}} y\;dy=$$

$$=\int_1^2\ln x\;dx\cdot\frac{y^2}{2}\Bigg|_{\frac{1}{x}}^{\sqrt{x}}=$$

$$=\frac{1}{2}\int_1^2\ln x\cdot\left(x-\frac{1}{x^2}\right)dx=\begin{bmatrix}\ln x=u, & du=\dfrac{1}{x}dx,\\ \left(x-\dfrac{1}{x^2}\right)dx=dv, & v=\dfrac{x^2}{2}+\dfrac{1}{x}\end{bmatrix}=$$

$$=\frac{1}{2}\left(\left(\frac{x^2}{2}+\frac{1}{x}\right)\ln x\Bigg|_1^2-\int_1^2\left(\frac{x^2}{2}+\frac{1}{x}\right)\cdot\frac{1}{x}\;dx\right)=$$

$$= \frac{1}{2}\left(\frac{5}{2}\ln 2 - \int_1^2 \left(\frac{x}{2} + \frac{1}{x^2}\right) dx\right) = \frac{1}{2}\left(\frac{5}{2}\ln 2 - \left.\left(\frac{x^2}{4} - \frac{1}{x}\right)\right|_1^2\right) =$$

$$= \frac{1}{2}\left(\frac{5}{2}\ln 2 - \frac{5}{4}\right) = \frac{5}{8}\left(2\ln 2 - 1\right).$$

Lecture 2. CHANGE OF VARIABLES IN DOUBLE INTEGRAL. POLAR COORDINATES. APPLICATION OF DOUBLE INTEGRAL TO GEOMETRICAL AND MECHANICAL TASKS

If the given function $f(x, y)$ is continuous in closed and bounded domain *D*, then there is a double integral

$$I = \iint_D f(x, y)\, dxdy\,. \tag{1}$$

Let us assume that in the integral (1) we can make a transition to new variables *u* and *v* with the help of formulas

$$x = x(u, v), \quad y = y(u, v), \tag{2}$$

out of which the variables *u* and *v* are uniquely expressed:

$$u = u(x, y), \quad v = v(x, y). \tag{3}$$

In this case every point $M(x, y) \in D$ corresponds to the point $M^*(u, v)$. The set of all such points $M^*(u, v)$ forms closed and bounded domain D^*.

Theorem. If the change (3) turns closed bounded domain *D* into closed bounded domain D^* and is one-to-one correspondent, and if functions (2) in domain D^* have continuous partial first-order derivatives and nonzero determinant

$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}, \tag{4}$$

and if function $f(x,y)$ is continuous in domain D, then the following formula of the change of variables is valid:

$$\iint\limits_{D} f(x,y)\,dxdy=\iint\limits_{D^*} f(x(u,v),y(u,v))\cdot|J(u,v)|\,dudv\,. \tag{5}$$

Functional determinant (4) is called the Jacobian determinant or Jacobian.

Let us consider the change of Cartesian coordinates x, y into the polar ones – ρ,φ – connected by relations $x=\rho\cos\varphi$, $y=\rho\sin\varphi$.

Since Jacobian $J(\rho,\varphi)=\begin{vmatrix}\cos\varphi & -\rho\sin\varphi\\ \sin\varphi & \rho\cos\varphi\end{vmatrix}=\rho$, $|J(\rho,\varphi)|=\rho$,

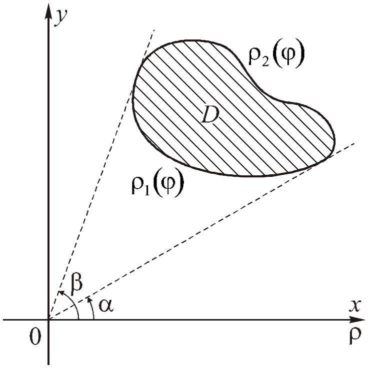
then formula (5) turns into:

$$\iint\limits_{D} f(x,y)\,dxdy=\iint\limits_{D^*} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,d\rho d\varphi\,, \tag{6}$$

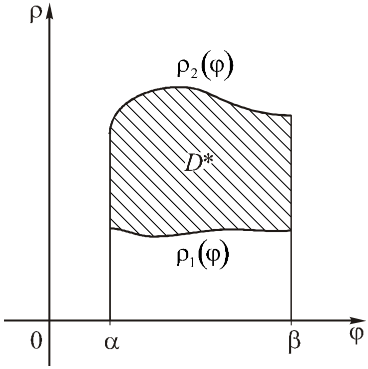
where domain D is set in Cartesian coordinate system Oxy, and D^* is its corresponding domain in polar coordinate system.

If domain D (Drawing 13,a) is bounded by rays, that make with polar axis angles α and β ($\alpha<\beta$), and by curves $\rho=\rho_1(\varphi)$ and $\rho=\rho_2(\varphi)$ ($\rho_1(\varphi)\le\rho_2(\varphi)$), then polar coordinates of domain D^* change in limits $\rho_1(\varphi)\le\rho\le\rho_2(\varphi)$, $\alpha\le\varphi\le\beta$ (Drawing 13,b). Then formula (6) may be presented as follows:

$$\iint_D f(x,y)\,dxdy = \int_{\alpha}^{\beta} d\varphi \int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho\,. \qquad (7)$$



Drawing 13,a



Drawing 13,b

Tasks. Make a transition to polar coordinates and calculate double integrals:

a) $I_1 = \iint_D \sqrt{x^2+y^2}\,dxdy$; D is the first quarter of a circle $x^2+y^2 \le a^2$;

$$I_1 = \iint_{D^*} \sqrt{\rho^2}\,\rho\,d\rho d\varphi = \iint_{D^*} \rho^2\,d\rho d\varphi = \int_0^{\pi/2} d\varphi \int_0^a \rho^2 d\rho =$$

$$= \varphi\Big|_0^{\pi/2} \cdot \frac{\rho^3}{3}\Big|_0^a = \frac{\pi}{2}\cdot\frac{a^3}{3} = \frac{\pi a^3}{6}\,.$$

b) $I_2 = \iint_D \sqrt{4-x^2-y^2}\,dxdy$; $\quad D:\ x^2+y^2 \le 4$;

$$I_2 = \iint_{D^*} \rho\sqrt{4-\rho^2}\, d\rho d\varphi = \int_0^{2\pi} d\varphi \int_0^2 \rho\sqrt{4-\rho^2}\, d\rho =$$

$$= -\frac{1}{2}\int_0^{2\pi} d\varphi \int_0^2 \sqrt{4-\rho^2}\, d\left(4-\rho^2\right) =$$

$$= -\frac{1}{2}\varphi\Big|_0^{2\pi} \cdot \frac{2}{3}\left(4-\rho^2\right)^{3/2}\Big|_0^2 = -\pi\cdot\frac{2}{3}\left(0-4^{3/2}\right) = \frac{16\pi}{3}.$$

c) $I_3 = \iint_D \frac{dxdy}{x^2+y^2+1}$; $D:\ y=\sqrt{1-x^2}$, $y=0$ (Drawing 14);

$$I_3 = \iint_{D^*} \frac{\rho\, d\rho d\varphi}{\rho^2+1} = \frac{1}{2}\int_0^{\pi} d\varphi \int_0^1 \frac{d\left(\rho^2+1\right)}{\rho^2+1} = \frac{1}{2}\varphi\Big|_0^{\pi} \cdot \left(\ln\left(\rho^2+1\right)\right)\Big|_0^1 =$$

$$= \frac{\pi}{2}\left(\ln 2 - \ln 1\right) = \frac{\pi \ln 2}{2}.$$

d) $I_4 = \iint_D \sqrt{x^2+y^2}\, dxdy$; $\quad D:\ x^2+y^2=2x$, $x^2+y^2=4x$ (Drawing 15);

$$D^*:\ \rho^2 = 2\rho\cos\varphi \ \Rightarrow\ \rho = 2\cos\varphi;$$

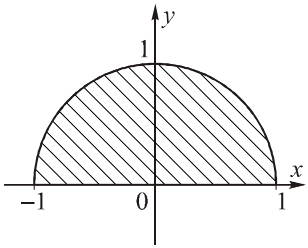
$$\rho^2 = 4\rho\cos\varphi \ \Rightarrow\ \rho = 4\cos\varphi;$$

$$I_4 = \int_{-\pi/2}^{\pi/2} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} \rho^2\, d\rho = \frac{1}{3}\int_{-\pi/2}^{\pi/2} d\varphi \cdot \rho^3\Big|_{2\cos\varphi}^{4\cos\varphi} =$$

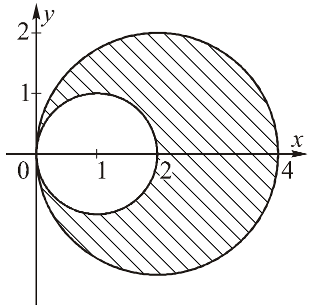
$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2} \left(64\cos^3\varphi - 8\cos^3\varphi\right) d\varphi =$$

$$= \frac{56}{3} \int_{-\pi/2}^{\pi/2} \left(1 - \sin^2 \varphi\right) d(\sin \varphi) = \frac{56}{3} \left(\sin \varphi - \frac{\sin^3 \varphi}{3} \right) \Bigg|_{-\pi/2}^{\pi/2} =$$

$$= \frac{56}{3} \left(\sin \frac{\pi}{2} - \sin\left(-\frac{\pi}{2} \right) - \frac{1}{3} \left(\sin^3 \frac{\pi}{2} - \sin^3 \left(-\frac{\pi}{2} \right) \right) \right) = \frac{56}{3} \left(2 - \frac{2}{3} \right) = \frac{224}{9}.$$



Drawing 14

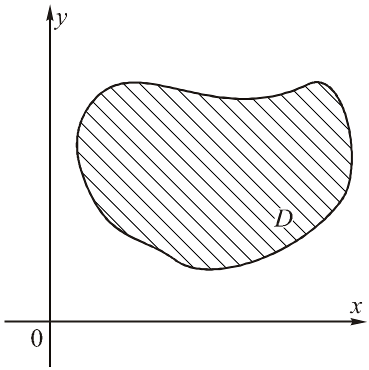


Drawing 15

Application of double integral to geometrical and mechanical tasks

In the plane *Oxy* there is a material plate in the shape of closed bounded domain *D* (Drawing 16), in every point of which density is determined by continuous function $\gamma = \gamma(x, y)$. Then *the area of the figure*, that has the shape of domain *D*, is calculated according to the formula:

$$S = \iint_D dxdy. \qquad (8)$$



Drawing 16

Mass of a material plate, bounded by domain D, with density γ is calculated with the formula:

$$m = \iint_D \gamma(x, y)\, dxdy\,. \tag{9}$$

Parameters

$$M_x = \iint_D y \cdot \gamma(x, y)\, dxdy \text{ and } M_y = \iint_D x \cdot \gamma(x, y)\, dxdy \tag{10}$$

are respectively called *static moments of a plate* about axes *Ox* and *Oy*.

Coordinates of center of mass of a plate are calculated with the formulas:

$$x_c = \frac{M_y}{m}, \qquad y_c = \frac{M_x}{m}\,. \tag{11}$$

Parameters

$$I_x = \iint_D y^2 \gamma(x, y)\, dxdy\,, \qquad I_y = \iint_D x^2 \gamma(x, y)\, dxdy\,,$$

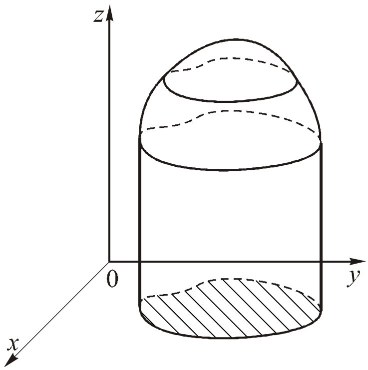
$$I_0 = \iint_D \left(x^2 + y^2\right)\gamma(x, y)\, dxdy \tag{12}$$

are respectively called *moments of inertia of a plate* about axes *Ox*, *Oy* and origin of coordinates.

The volume of the cylindrical solid, generatrix of which are parallel to axis *Oz* and which is bounded by domain *D* of the plane *Oxy* below, and by surface $z = f(x, y)$ above, where function $f(x, y)$ is continuous and nonnegative in domain *D* (Drawing 17), is calculated with the formula:

$$V = \iint_D f(x, y)\, dxdy\,. \tag{13}$$

If surface σ, represented by equation $z = f(x, y)$, is projected on the plane Oxy in domain D, and if functions $f(x, y)$, $f'_x(x, y)$, $f'_y(x, y)$ are continuous in this domain, then area Q of surface σ is calculated according to the formula:



Drawing 17

$$Q = \iint_D \sqrt{1 + \left(f'_x(x, y)\right)^2 + \left(f'_y(x, y)\right)^2}\, dxdy. \qquad (14)$$

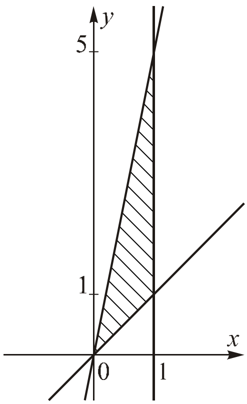
Task 1. Calculate areas of figures, bounded by the lines:

a) $y = x$, $y = 5x$, $x = 1$.

The given figure is represented on Drawing 18. We calculate the required area with the formula (8):

$$S = \iint_D dxdy = \int_0^1 dx \int_x^{5x} dy = \int_0^1 dx \cdot y\Big|_x^{5x} =$$

$$= \int_0^1 4x\ dx = 2x^2\Big|_0^1 = 2 \text{ (sq.un.)}$$



Drawing 18

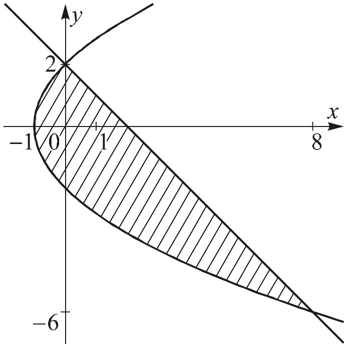
b) $y = 2 - x$, $y^2 = 4x + 4$.

We calculate common points of the lines:

$$\begin{cases} y = 2 - x, \\ y^2 = 4x + 4 \end{cases} \Rightarrow \ (2 - x)^2 = 4x + 4,$$

$$x^2 - 8x = 0 \;\Rightarrow\; (0;2),\; (8;-6);$$

and make a sketch (Drawing 19).



Drawing 19

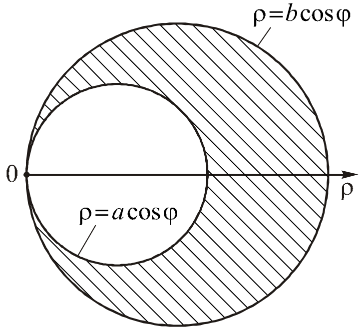
$$S = \iint_D dxdy = \int_{-6}^{2} dy \int_{\frac{y^2}{4}-1}^{2-y} dx =$$

$$= \int_{-6}^{2}\left(2 - y - \frac{y^2}{4} + 1\right)dy =$$

$$= \left(3y - \frac{y^2}{2} - \frac{y^3}{12}\right)\Bigg|_{-6}^{2} = 4 - \frac{2}{3} + 18 = 21\frac{1}{3}\,\text{(sq.un.)}$$

c) $\rho = a\cos\varphi$, $\rho = b\cos\varphi$, $0 < a < b$ (Drawing 20).

Taking into consideration symmetry of the figure, bounded by the given lines, its area is equal to:



Drawing 20

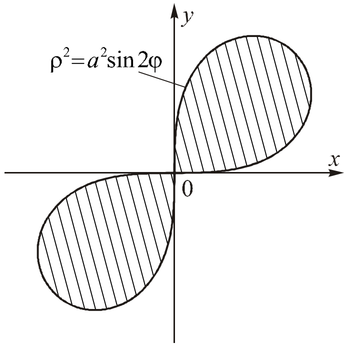
$$S = 2\iint_{D_1^*} \rho\, d\rho d\varphi =$$

$$= 2\int_0^{\pi/2} d\varphi \int_{a\cos\varphi}^{b\cos\varphi} \rho\, d\rho =$$

$$= \left(b^2 - a^2\right)\int_0^{\pi/2} \cos^2\varphi\, d\varphi = \frac{b^2 - a^2}{2}\int_0^{\pi/2}(1 + \cos 2\varphi)\, d\varphi =$$

$$= \frac{b^2 - a^2}{2}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)\Bigg|_0^{\pi/2} = \frac{\left(b^2 - a^2\right)\pi}{4}\text{. (sq.un.)}$$

d) $\left(x^2 + y^2\right)^2 = 2a^2xy$.



Drawing 21

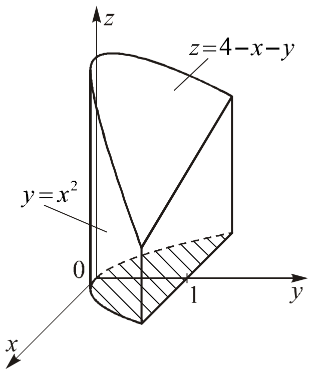
To calculate the area of the figure, bounded by a closed line (lemniscate), we apply formula (6) of transition to polar coordinates and take into consideration its symmetry (Drawing 21), then

$$S = 2\int_0^{\pi/2} d\varphi \int_0^{a\sqrt{\sin 2\varphi}} \rho\, d\rho = a^2 \int_0^{\pi/2} \sin 2\varphi\, d\varphi =$$

$$= -\frac{a^2}{2}\cos 2\varphi\Bigg|_0^{\pi/2} = a^2 \text{ (sq.un.)}$$

Task 2. Calculate volumes of solids, bounded by the given surfaces:

a) $y = x^2$, $y = 1$, $x + y + z = 4$, $z = 0$.



Drawing 22

The given solid is a vertical cylinder $y = x^2$, bounded by plane $z = 4 - x - y$ above, and by a part of plane $z = 0$ below (Drawing 22); its projection on the plane Oxy is a figure, bounded by parabola $y = x^2$ and line $y = 1$. Volume of the solid is calculated with the formula (13):

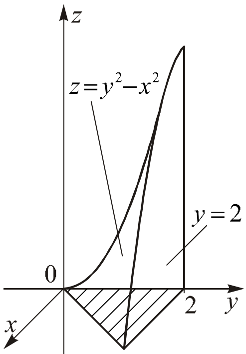
$$V=\iint_D f(x,y)\,dxdy=\int_{-1}^{1}dx\int_{x^2}^{1}(4-x-y)\,dy=\int_{-1}^{1}dx\left((4-x)y-\frac{y^2}{2}\right)\Bigg|_{x^2}^{1}=$$

$$=\int_{-1}^{1}\left(3\frac{1}{2}-x-4x^2+x^3+\frac{x^4}{2}\right)dx=\left(\frac{x^5}{10}+\frac{x^4}{4}-\frac{4x^3}{3}-\frac{x^2}{2}+3\frac{1}{2}x\right)\Bigg|_{-1}^{1}=$$

$$=2\left(\frac{1}{10}-\frac{4}{3}+3\frac{1}{2}\right)=2\cdot\frac{34}{15}=\frac{68}{15}\ \text{(c. un.)}$$

b) $z=y^2-x^2$, $z=0$, $y=\pm 2$.

Hyperbolic paraboloid $z=y^2-x^2$ crosses plane Oxy ($z=0$) on the lines $y=\pm x$. Along with the planes $z=0$, $y=\pm 2$, this paraboloid bounds a solid, symmetric to planes *Oxz* and *Oyz*. Taking into account the symmetry, volume V_1 of a quarter of the solid, situated in the first octant (Drawing 23), is equal to:



Drawing 23

$$V_1=\iint_D f(x,y)\,dxdy=\int_0^2 dy\int_0^y\left(y^2-x^2\right)dx=$$

$$=\int_0^2 dy\left(y^2x-\frac{x^3}{3}\right)\Bigg|_0^y=$$

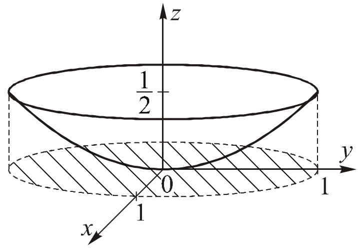
$$=\frac{2}{3}\int_0^2 y^3\,dy=\frac{2}{3}\cdot\frac{y^4}{4}\Bigg|_0^2=\frac{8}{3}\ \text{(c. un.)}$$

Then the volume of the given solid $V=4V_1=\dfrac{32}{3}$ (c. un.).

Task 3. Calculate area of a part of a surface of paraboloid $2z = x^2 + y^2$, cut out by cylinder $x^2 + y^2 = 1$ (Drawing 24).

Out of equation of paraboloid $z = \frac{1}{2}\left(x^2 + y^2\right)$ we express $z'_x = x$, $z'_y = y$.

Integrating domain D is a circle $x^2 + y^2 = 1$. According to the formula (14) we calculate:



Drawing 24

$$Q = \iint_D \sqrt{1 + x^2 + y^2}\, dxdy =$$

$$= \iint_{D^*} \sqrt{1 + \rho^2}\, \rho\, d\rho d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^1 \sqrt{1 + \rho^2}\, d\left(1 + \rho^2\right) = \frac{1}{2}\varphi\Big|_0^{2\pi} \cdot \frac{\left(1 + \rho^2\right)^{3/2}}{3/2}\Bigg|_0^1 =$$

$$= \frac{2}{3}\pi\left(2\sqrt{2} - 1\right) \text{ (sq. un.)}$$

Task 4. Calculate the coordinates of center of mass of upper half of homogeneous ($\gamma = 1$) circle $x^2 + y^2 = 9$.

Due to the symmetry of a figure about axis Oy $x_c = 0$. To calculate y_c we apply formula (11), so $y_c = \frac{M_x}{m}$, where according to the formulas (9) and (10):

$$M_x = \iint_D y\gamma\, dxdy = \iint_{D^*} \rho^2 \sin\varphi\, d\rho d\varphi = \int_0^{\pi} \sin\varphi\, d\varphi \int_0^3 \rho^2\, d\rho =$$

$$= \left(-\cos\varphi\right)\Big|_0^{\pi} \cdot \frac{\rho^3}{3}\Big|_0^3 = \left(\cos 0 - \cos\pi\right)\cdot 9 = 18,$$

$$m = \iint\limits_D \gamma\; dxdy = \iint\limits_{D^*} \rho\; d\rho d\varphi = \int\limits_0^{\pi} d\varphi \int\limits_0^3 \rho\, d\rho = \varphi\Big|_0^{\pi} \cdot \frac{\rho^2}{2}\Big|_0^3 = \frac{9\pi}{2},\ \text{then}$$

$$y_c = \frac{18}{9\pi/2} = \frac{4}{\pi}.$$

Task 5. Calculate moment of inertia about the origin of coordinates of a homogeneous figure, bounded by the lines $\frac{x}{a} + \frac{y}{b} = 1$, $x = 0$, $y = 0$.

According to the formulas (12) we calculate:

$$I_0 = \iint\limits_D \left(x^2 + y^2\right)\gamma\, dxdy = \int\limits_0^a dx \int\limits_0^{\frac{b}{a}(a-x)} \left(x^2 + y^2\right) dy =$$

$$= \int\limits_0^a dx \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_0^{\frac{b}{a}(a-x)} =$$

$$= \int\limits_0^a \left(\frac{b}{a} x^2 (a - x) + \frac{1}{3}\cdot\frac{b^3}{a^3}\left(a - x^3\right)\right) dx =$$

$$= \left(\frac{b}{3} x^3 - \frac{b}{4a} x^4 - \frac{b^3}{3a^3}\cdot\frac{(a-x)^4}{4} \right)\Bigg|_0^a = \frac{ab\left(a^2 + b^2\right)}{12}.$$

Task 6. Calculate mass of a circular ring, built on circles with radiuses r_1 and r_2 ($r_1 < r_2$), if in every point surface density is inversely proportional to the square of its distance to the center of a ring.

According to the task, density $\gamma(x, y) = \dfrac{k}{x^2 + y^2}$, where k is a proportionality coefficient, so by the formula (9) the mass of the ring will be equal to:

$$m = \iint_D \gamma(x, y)dxdy = \iint_D \frac{k}{x^2 + y^2}\, dxdy = k\iint_{D^*} \frac{\rho\, d\rho d\varphi}{\rho^2} = k\int_0^{2\pi} d\varphi \int_{r_1}^{r_2} \frac{d\rho}{\rho} =$$

$$= k \cdot \varphi\Big|_0^{2\pi} \cdot \ln\rho\Big|_{r_1}^{r_2} = 2\pi k(\ln r_2 - \ln r_1) = 2\pi k \ln\frac{r_2}{r_1}.$$

Task 7. Calculate moment of inertia of homogeneous ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ($a > b$) about its bigger axis.

Since $a > b$, we calculate:

$$I_x = \iint_D y^2\, dxdy = 4\int_0^a dx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} y^2\, dy = 4\int_0^a dx \cdot \frac{y^3}{3}\Bigg|_0^{\frac{b}{a}\sqrt{a^2-x^2}} = \frac{4}{3}\int_0^a \frac{b^3}{a^3}\left(\sqrt{a^2 - x^2}\right)^3 dx =$$

$$= \begin{bmatrix} x = a\sin z, \\ dx = a\cos z\ dz, \\ z = \arcsin\dfrac{x}{a}, \\ z_1 = 0,\ z_2 = \dfrac{\pi}{2} \end{bmatrix} = \frac{4b^3}{3a^3}\int_0^{\pi/2} \left(\sqrt{a^2 - a^2\sin^2 z}\right)^3 \cdot a\cos z\, dz = \frac{4b^3}{3a^3}\int_0^{\pi/2} a^4 \cos^4 z\ dz =$$

$$= \frac{4}{3} ab^3 \int_0^{\pi/2} \left(\frac{1 + \cos 2z}{2} \right)^2 dz = \frac{ab^3}{3} \int_0^{\pi/2} \left(1 + 2\cos 2z + \cos^2 2z\right) dz =$$

$$= \frac{ab^3}{3} \int_0^{\pi/2} \left(1 + 2\cos 2z + \frac{1 + \cos 4z}{2} \right) dz = \frac{ab^3}{3} \left(z + \sin 2z + \frac{1}{2} \left(z + \frac{1}{4} \sin 4z \right) \right) \Bigg|_0^{\pi/2} =$$

$$= \frac{ab^3}{3} \left(\frac{\pi}{2} + \sin \pi + \frac{1}{2} \left(\frac{\pi}{2} + \frac{1}{4} \sin 2\pi \right) \right) = \frac{ab^3}{3} \left(\frac{\pi}{2} + \frac{\pi}{4} \right) = \frac{ab^3}{3} \cdot \frac{3\pi}{4} = \frac{\pi a b^3}{4} \; .$$

Lecture 3. DEFINITION AND CALCULATION OF TRIPLE INTEGRALS

Function $u = f(x, y, z)$ is determined in a closed bounded domain $G \subset R^3$. We divide domain *G* by a set of surfaces into *n* parts G_i, that do not have any common inner points and the volumes of which are ΔV_i, $i = 1, 2, \ldots, n$. In each part G_i we take an arbitrary point $P_i(\xi_i; \eta_i; \mu_i)$ and calculate the sum

$$\sum_{i=1}^{n} f(\xi_i; \eta_i; \mu_i) \cdot \Delta V_i, \qquad (1)$$

which is called an integral sum for function $f(x, y, z)$ in domain G. $\lambda = \max\limits_{1 \le i \le n} d(G_i)$ is the biggest diameter among domains G_i.

Definition. If $\lambda \to 0$, and integral sum (1) has a finite limit, that depends neither on the way we divide domain *G* into parts G_i, nor on the choice of points P_i, then this limit is called triple integral of function $f(x, y, z)$ in domain *G*:

$$\iiint\limits_{G} f(x, y, z)\, dxdydz = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i; \eta_i; \mu_i) \cdot \Delta V_i, \qquad (2)$$

where $f(x, y, z)$ is a function, integrable in domain *G*; *G* is an integrating domain; *x*, *y*, *z* are integrating variables; *dxdydz* (or *dV*) is an element of volume.

Mechanical meaning of triple integral. If mass with bulk density $\gamma = \gamma(x, y, z)$ in the point $(x, y, z) \in G$ is spread over the solid *G*, then mass *m* of this solid is calculated with the formula:

$$m = \iiint_G \gamma(x, y, z)\, dxdydz\,. \tag{3}$$

Geometrical meaning. If everywhere in domain *G* $\gamma(x, y, z) \equiv 1$, then the formula (3) is followed by the formula of calculation volume *V* of solid *G*:

$$V = \iiint_G dxdydz\,. \tag{4}$$

Theorem (sufficient condition of integrability of function). If function $f(x, y, z)$ is continuous in a closed bounded domain *G*, then it is integrable in this domain.

Properties of triple integrals

1. Constant factor may be factorized out of the sign of triple integral

$$\iiint_G C \cdot f(x, y, z)\, dV = C \cdot \iiint_G f(x, y, z)\, dV\,.$$

2. Triple integral of sum of integrable functions is equal to the sum of triple integrals of addends:

$$\iiint_G \left(f(x, y, z) \pm \varphi(x, y, z)\right) dV = \iiint_G f(x, y, z)\, dV \pm \iiint_G \varphi(x, y, z)\, dV$$

3. If function $f(x, y, z) \geq 0$ in domain *G*, then

$$\iiint_G f(x, y, z)\, dV \geq 0\,.$$

4. If functions $f(x, y, z)$ and $\varphi(x, y, z)$ are determined in domain G and if $f(x, y, z) \geq \varphi(x, y, z)$, then

$$\iiint_G f(x, y, z)\, dV \geq \iiint_G \varphi(x, y, z)\, dV .$$

5. (Additivity of triple integral). If we divide integrating domain G into parts G_1 and G_2, that do not have common inner points, then

$$\iiint_G f(x, y, z)\, dV = \iiint_{G_1} f(x, y, z)\, dV + \iiint_{G_2} f(x, y, z)\, dV .$$

6. (Triple integral evaluation). If function $f(x, y, z)$ is continuous in a closed bounded domain G that has volume V, then

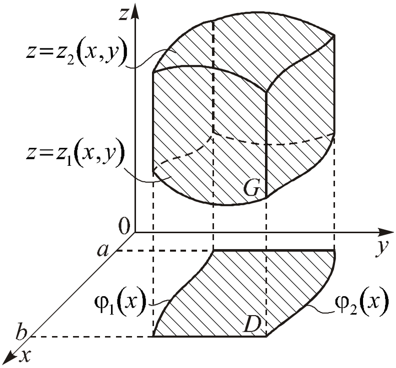
$$mV \leq \iiint_G f(x, y, z)\, dV \leq MV ,$$

where m and M are respectively the minimal and the maximal values of function $f(x, y, z)$ in domain G.

7. (Mean value of a function). If function $f(x, y, z)$ is continuous in a closed bounded domain G that has volume V, then a certain point $(x_0; y_0; z_0)$ exists in this domain:

$$\iiint_G f(x, y, z)\, dV = f(x_0; y_0; z_0) \cdot V .$$

Parameter $f(x_0; y_0; z_0) = \dfrac{1}{V} \iiint_G f(x, y, z)\, dV$ is called a mean value of function $f(x, y, z)$ in domain G.



Drawing 25

Calculation of triple integrals

Integrating domain G is bounded by surfaces $z = z_1(x, y)$ and $z = z_2(x, y)$ below and above, and on the sides by a cylindrical area, the generatrix of which are parallel to the axis Oz (Drawing 25). Domain G is called regular in the direction of the axis Oz, if domain D (which is a projection of G on the area Oxy, bounded by curves $y = \varphi_1(x)$ and $y = \varphi_2(x)$ and lines $x = a$ and $x = b$) is regular. Then calculation of triple integral means calculating three determined integrals, i.e.:

$$\iiint\limits_G f(x, y, z)\, dxdydz = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} dy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\,. \qquad (5)$$

The order of integration may also be different, for example, if domain G is regular in the direction of the axis Ox, i.e.

$$G = \{x_1(y, z) \le x \le x_2(y, z),\ \psi_1(y) \le z \le \psi_2(y),\ c \le y \le d\},$$

where $x_1(y, z)$, $x_2(y, z)$, $\psi_1(y)$, $\psi_2(y)$ are continuous functions, then

$$\iiint\limits_G f(x, y, z)\, dxdydz = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} dz \int\limits_{x_1(y,z)}^{x_2(y,z)} f(x, y, z)\, dx$$

Task 1. Calculate triple integrals:

a) $$\int_0^a y\,dy\int_0^h dx\int_0^{a-y} dz=\int_0^a y\,dy\int_0^h dx\cdot z\Big|_0^{a-y}=\int_0^a y(a-y)dy\int_0^h dx=$$

$$=\left(a\cdot\frac{y^2}{2}-\frac{y^3}{3}\right)\Bigg|_0^a\cdot x\Big|_0^h=\left(\frac{a^3}{2}-\frac{a^3}{3}\right)\cdot h=\frac{a^3\cdot h}{6}.$$

b) $$\int_0^1 dx\int_0^{\sqrt{x}} y\,dy\int_{1-x}^{2-2x} dz=\int_0^1 dx\int_0^{\sqrt{x}} y\,dy\cdot z\Big|_{1-x}^{2-2x}=\int_0^1(1-x)dx\int_0^{\sqrt{x}} y\,dy=$$

$$=\int_0^1(1-x)dx\cdot\frac{y^2}{2}\Bigg|_0^{\sqrt{x}}=$$

$$=\frac{1}{2}\int_0^1\left(x-x^2\right)dx=\frac{1}{2}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)=\frac{1}{2}\cdot\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{12}.$$

c) $$\int_0^1 dx\int_0^1 dy\int_0^1\frac{dz}{\sqrt{x+y+z+1}}=2\int_0^1 dx\int_0^1 dy\cdot\sqrt{x+y+z+1}\Bigg|_0^1=$$

$$=2\int_0^1 dx\int_0^1\left(\sqrt{x+y+2}-\sqrt{x+y+1}\right)dy=$$

$$=2\int_0^1 dx\left(\frac{(x+y+2)^{3/2}}{3/2}-\frac{(x+y+1)^{3/2}}{3/2}\right)\Bigg|_0^1=$$

$$=\frac{4}{3}\int_0^1\left((x+3)^{3/2}-(x+2)^{3/2}-(x+2)^{3/2}+(x+1)^{3/2}\right)dx=$$

$$=\frac{4}{3}\left(\frac{(x+3)^{5/2}}{5/2}-\frac{2\cdot(x+2)^{5/2}}{5/2}+\frac{(x+1)^{5/2}}{5/2}\right)\Bigg|_0^1=$$

$$=\frac{8}{15}\left(4^{5/2}-2\cdot3^{5/2}+2^{5/2}-3^{5/2}+2\cdot2^{5/2}-1\right)=$$

$$=\frac{8}{15}\left(32-3\cdot3^{5/2}+3\cdot2^{5/2}-1\right)=\frac{8}{15}\left(31-27\sqrt{3}+12\sqrt{2}\right).$$

d) $$\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}xyz\,dz=\int_0^1 x\,dx\int_0^{1-x}y\,dy\cdot\frac{z^2}{2}\Bigg|_0^{1-x-y}=$$

$$=\frac{1}{2}\int_0^1 x\,dx\int_0^{1-x}y(1-x-y)^2\,dy=$$

$$=\frac{1}{2}\int_0^1 x\,dx\int_0^{1-x}y\left((1-x)^2-2y(1-x)+y^2\right)dy=$$

$$=\frac{1}{2}\int_0^1 x\,dx\int_0^{1-x}\left((1-x)^2y-2y^2(1-x)+y^3\right)dy=$$

$$=\frac{1}{2}\int_0^1 x\,dx\cdot\left((1-x)^2\cdot\frac{y^2}{2}-\frac{2y^3}{3}\cdot(1-x)+\frac{y^4}{4}\right)\Bigg|_0^{1-x}=$$

$$=\frac{1}{2}\int_0^1 x\left(\frac{(1-x)^4}{2}-\frac{2}{3}(1-x)^4+\frac{(1-x)^4}{4}\right)dx=$$

$$=\frac{1}{2}\int_0^1 x\cdot\frac{(1-x)^4}{12}dx=\frac{1}{24}\int_0^1 x\left(1-4x+6x^2-4x^3+x^4\right)dx=$$

$$=\frac{1}{24}\int_0^1\left(x-4x^2+6x^3-4x^4+x^5\right)dx=$$

$$=\frac{1}{24}\left(\frac{x^2}{2}-\frac{4x^3}{3}+\frac{3x^4}{2}-\frac{4x^5}{5}+\frac{x^6}{6}\right)\Bigg|_0^1=$$

$$=\frac{1}{24}\left(\frac{1}{2}-\frac{4}{3}+\frac{3}{2}-\frac{4}{5}+\frac{1}{6}\right)=\frac{1}{24}\cdot\frac{1}{30}=\frac{1}{720}.$$

e) $$\int_0^{e-1}dx\int_0^{e-x-1}dy\int_e^{x+y+e}\frac{\ln(z-x-y)}{(x-e)(x+y-e)}dz=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\int_0^{e-x-1}\frac{dy}{x+y-e}\int_e^{x+y+e}\ln(z-x-y)\,dz=$$

$$=\left[\int\ln t\,dt=t(\ln t-1)+C\right]=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\int_0^{e-x-1}\frac{dy}{x+y-e}\cdot(z-x-y)(\ln(z-x-y)-1)\Bigg|_e^{x+y+e}=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\int_0^{e-x-1}\frac{dy}{x+y-e}\cdot\left(e\left(\ln e-1\right)-\left(e-x-y\right)\left(\ln\left(e-x-y\right)-1\right)\right)=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\int_0^{e-x-1}\frac{dy}{x+y-e}\cdot\left(x+y-e\right)\left(\ln\left(e-x-y\right)-1\right)=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\int_0^{e-x-1}\left(\ln\left(e-x-y\right)-1\right)dy=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\cdot\left(\left(e-x-y\right)\left(\ln\left(e-x-y\right)-1\right)-y\right)\Big|_0^{e-x-1}=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\cdot\left(-\left(\ln 1-1\right)-e+x+1+\left(e-x\right)\left(\ln\left(e-x\right)-1\right)\right)=$$

$$=\int_0^{e-1}\frac{dx}{x-e}\cdot\left(1-e+x+1+\left(e-x\right)\left(\ln\left(e-x\right)-1\right)\right)=$$

$$=\int_0^{e-1}\frac{2-e+x}{x-e}dx-\int_0^{e-1}\left(\ln\left(e-x\right)-1\right)dx=$$

$$=\int_0^{e-1}\left(\frac{2}{x-e}+1\right)dx-\int_0^{e-1}\ln\left(e-x\right)dx+\int_0^{e-1}dx=$$

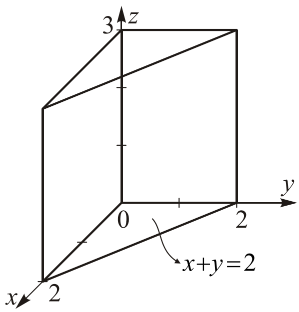
$$=\left(2\ln\left|x-e\right|+2x+\left(e-x\right)\left(\ln\left(e-x\right)-1\right)\right)\Big|_0^{e-1}=$$

$$=2\ln 1+2e-2+\ln 1-1-\left(2\ln e+0+e\left(\ln e-1\right)\right)=2e-2-1-2=2e-5\,.$$

Example 2. Calculate triple integrals:

a) $I_1 = \iiint\limits_G (2x + 3y - z)\, dxdydz$;

G : $x = 0$, $y = 0$, $z = 0$, $z = 3$, $x + y = 2$ (Drawing 26).



Drawing 26

$$I_1 = \int_0^2 dx \int_0^{2-x} dy \int_0^3 (2x + 3y - z)\, dz =$$

$$= \int_0^2 dx \int_0^{2-x} dy \left((2x+3y)z - \frac{z^2}{2} \right)\Bigg|_0^3 =$$

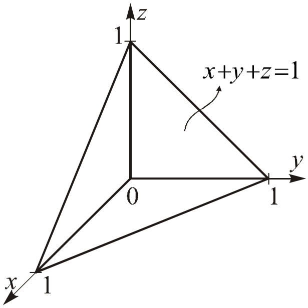
$$= \int_0^2 dx \int_0^{2-x} \left(6x + 9y - \frac{9}{2} \right) dy = \int_0^2 dx \left(\left(6x - \frac{9}{2} \right) y + \frac{9}{2} y^2 \right)\Bigg|_0^{2-x} =$$

$$= \int_0^2 \left(\left(6x - \frac{9}{2} \right)(2 - x) + \frac{9}{2}(2 - x)^2 \right) dx =$$

$$= \int_0^2 \left(12x - 6x^2 - 9 + \frac{9}{2}x + 18 - 18x + \frac{9}{2}x^2 \right) dx =$$

$$= \int_0^2 \left(-\frac{3}{2}x^2 - \frac{3}{2}x + 9 \right) dx = \left(-\frac{x^3}{2} - \frac{3x^2}{4} + 9x \right)\Bigg|_0^2 =$$

$$= -\frac{8}{2} - 3 + 18 = 11.$$



Drawing 27

b) $I_2 = \iiint\limits_G \frac{dxdydz}{(x+y+z+1)^3}$;

$G:\ x=0,\ y=0,\ z=0,\ x+y+z=1$ (Drawing 27).

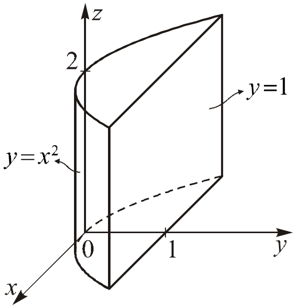
$$I_2 = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} =$$

$$= -\frac{1}{2}\int_0^1 dx \int_0^{1-x} dy \cdot \frac{1}{(x+y+z+1)^2}\Bigg|_0^{1-x-y} =$$

$$= -\frac{1}{2}\int_0^1 dx \int_0^{1-x} \left(\frac{1}{4} - \frac{1}{(x+y+1)^2}\right) dy = -\frac{1}{2}\int_0^1 dx \left(\frac{1}{4}y + \frac{1}{x+y+1}\right)\Bigg|_0^{1-x} =$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{1}{4} - \frac{1}{4}x + \frac{1}{2} - \frac{1}{x+1}\right) dx = -\frac{1}{2}\left(\frac{3}{4}x - \frac{x^2}{8} - \ln|x+1|\right)\Bigg|_0^1 =$$

$$= -\frac{1}{2}\left(\frac{3}{4} - \frac{1}{8} - \ln 2\right) = \frac{1}{2}\left(\ln 2 - \frac{5}{8}\right).$$



Drawing 28

c) $I_3 = \iiint\limits_G (4+z)\, dxdydz$; $G:\ y = x^2$,

$y=1,\ z=0,\ z=2$ (Drawing 28).

$$I_3 = \int_{-1}^1 dx \int_{x^2}^1 dy \int_0^2 (4+z)\, dz =$$

$$= \int_{-1}^{1} dx \int_{x^2}^{1} dy \left(4z + \frac{z^2}{2} \right)\Bigg|_0^2 = 10 \int_{-1}^{1} dx \int_{x^2}^{1} dy = 10 \int_{-1}^{1} \left(1 - x^2\right) dx =$$

$$= 10\left(x - \frac{x^3}{3} \right)\Bigg|_{-1}^{1} = 10\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3} \right) \right) = 20\left(1 - \frac{1}{3} \right) = \frac{40}{3}.$$

d) $I_4 = \iiint\limits_G xy\, dxdydz$; $G:\ z = xy,\ x + y = 1,\ z = 0$.

$$I_4 = \int_0^1 x\, dx \int_0^{1-x} y\, dy \int_0^{xy} dz = \int_0^1 x^2\, dx \int_0^{1-x} y^2\, dy = \frac{1}{3} \int_0^1 x^2\, dx \cdot (1 - x)^3 =$$

$$= \frac{1}{3} \int_0^1 x^2 \left(1 - 3x + 3x^2 - x^3\right) dx = \frac{1}{3} \int_0^1 \left(x^2 - 3x^3 + 3x^4 - x^5\right) dx =$$

$$= \frac{1}{3} \left(\frac{x^3}{3} - \frac{3x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{6} \right)\Bigg|_0^1 =$$

$$= \frac{1}{3} \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right) = \frac{1}{3} \cdot \frac{1}{60} = \frac{1}{180}.$$

e) $I_5 = \iiint\limits_G y \cos(z + x)\, dxdydz$; $G:\ y = \sqrt{x},\ y = 0,\ z = 0,$
$x + z = \frac{\pi}{2}$.

$$I_5 = \int_0^{\pi/2} dx \int_0^{\sqrt{x}} y\, dy \int_0^{\pi/2 - x} \cos(z + x)\, dz = \int_0^{\pi/2} dx \int_0^{\sqrt{x}} y\, dy \cdot \sin(z + x)\Bigg|_0^{\pi/2 - x} =$$

$$= \int_0^{\pi/2} dx \int_0^{\sqrt{x}} y\,dy \cdot \left(\sin\frac{\pi}{2} - \sin x \right) = \int_0^{\pi/2} (1 - \sin x)\,dx \int_0^{\sqrt{x}} y\,dy =$$

$$= \int_0^{\pi/2} (1 - \sin x)\,dx \cdot \left. \frac{y^2}{2} \right|_0^{\sqrt{x}} =$$

$$= \frac{1}{2} \int_0^{\pi/2} (x - x\sin x)\,dx = \begin{bmatrix} x = u, & dx = du, \\ \sin x\,dx = dv, & v = -\cos x \end{bmatrix} =$$

$$= \frac{1}{2} \left. \left(\frac{x^2}{2} + x \cdot \cos x \right) \right|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \cos x\,dx =$$

$$= \frac{1}{2} \left(\frac{\pi^2}{8} + \frac{\pi}{2} \cos\frac{\pi}{2} \right) - \frac{1}{2} \sin x \Big|_0^{\pi/2} =$$

$$= \frac{1}{2} \cdot \frac{\pi^2}{8} - \frac{1}{2} \left(\sin\frac{\pi}{2} - \sin 0 \right) = \frac{\pi^2}{16} - \frac{1}{2}.$$

Lecture 4. CYLINDRICAL AND SPHERICAL COORDINATES. APPLICATION OF TRIPLE INTEGRAL

Closed bounded domain G is one-to-one mapped onto domain G^* with the help of continuously differentiable functions $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, Jacobian J is nonzero in domain G^*:

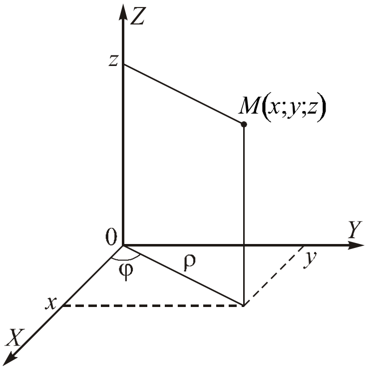
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0,$$

and if $f(x, y, z)$ is the function, continuous in G, then the following formula is valid:

$$\iiint_G f(x, y, z)\, dxdydz =$$

$$= \iiint_{G^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \cdot |J|\, dudvdw. \quad (1)$$

While making a transition from rectangular coordinates x, y, z to cylindrical ρ, φ, z (Drawing 29), connected with x, y, z by relations



Drawing 29

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi,$$

$$z = z;$$

$$0 \le \rho < +\infty, \quad 0 \le \varphi < 2\pi,$$

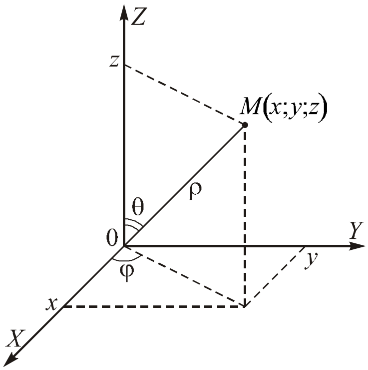
$$-\infty < z < +\infty.$$

Jacobian of transition:

$$J = \begin{vmatrix} -\rho\sin\varphi & \cos\varphi & 0 \\ \rho\cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\rho.$$

Then out of formula (1) we express triple integral in cylindrical coordinates:

$$\iiint\limits_G f(x, y, z)\, dxdydz = \iiint\limits_{G^*} f(\rho\cos\varphi, \rho\sin\varphi, z)\cdot\rho\, d\rho d\varphi dz. \quad (2)$$



Drawing 30

While making a transition from rectangular coordinates x, y, z to spherical ρ, φ, θ (Drawing 30), connected with x, y, z by formulas

$$x = \rho\sin\theta\cos\varphi,$$

$$y = \rho\sin\theta\sin\varphi, \quad z = \rho\cos\theta;$$

$$0 \le \rho < +\infty, \quad 0 \le \varphi < 2\pi,$$

$$0 \le \theta < \pi.$$

Jacobian of transition

$$J = \begin{vmatrix} \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix} = \rho^2\sin\theta.$$

Out of formula (1) we calculate triple integral in spherical coordinates:

$$\iiint\limits_G f(x,y,z)\,dxdydz =$$

$$= \iiint\limits_{G^*} f(\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta)\cdot\rho^2\sin\theta\; d\rho d\varphi d\theta\,. \quad (3)$$

Task 1. Calculate $I = \iiint\limits_G \left(x^2 + y^2\right) dxdydz$, where domain G is bounded by areas $z = \frac{1}{2}\left(x^2 + y^2\right)$ and $z = 2$.

The projection of integrating domain G on the area Oxy is a circle $x^2 + y^2 = 4$, the equation of which in polar coordinate system is $\rho = 2$. We apply formula (2) of transition to cylindrical coordinates, then

$$I = \iiint\limits_{G^*} \rho^3\, d\rho d\varphi dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^2 \rho^3\, d\rho \int\limits_{\rho^2/2}^{2} dz = \varphi\Big|_0^{2\pi} \cdot \int\limits_0^2 \rho^3 \left(2 - \frac{\rho^2}{2}\right) d\rho =$$

$$= 2\pi \int\limits_0^2 \left(2\rho^3 - \frac{\rho^5}{2}\right) d\rho = 2\pi \left(\frac{\rho^4}{2} - \frac{\rho^6}{12}\right)\Bigg|_0^2 =$$

$$= 2\pi\left(8 - \frac{64}{12}\right) = 2\pi\left(8 - \frac{16}{3}\right) = 2\pi\cdot\frac{8}{3} = \frac{16\pi}{3}\,.$$

Task 2. Calculate $I = \iiint\limits_G \frac{xy}{\sqrt{z}}\, dxdydz$, where domain G is set in the first octant and is bounded by cone $4z^2 = x^2 + y^2$ and areas $x = 0$, $y = 0$, $z = 1$.

$$I = \iiint\limits_{G^*} \frac{\rho^3 \sin\varphi \cos\varphi}{\sqrt{z}}\, d\rho d\varphi dz = \int\limits_0^{\pi/2} \sin\varphi \cos\varphi\ d\varphi \int\limits_0^2 \rho^3\ d\rho \int\limits_{\rho/2}^1 \frac{dz}{\sqrt{z}} =$$

$$= \frac{1}{2}\int\limits_0^{\pi/2} \sin 2\varphi\ d\varphi \int\limits_0^2 \rho^3\ d\rho \cdot 2\sqrt{z}\Big|_{\rho/2}^1 = \int\limits_0^{\pi/2} \sin 2\varphi\ d\varphi \int\limits_0^2 \rho^3\ d\rho\left(1 - \sqrt{\frac{\rho}{2}}\right) =$$

$$= -\frac{1}{2}\cos 2\varphi\Big|_0^{\pi/2} \cdot \int\limits_0^2 \left(\rho^3 - \frac{1}{\sqrt{2}}\rho^{7/2}\right) d\rho =$$

$$= -\frac{1}{2}(\cos\pi - \cos 0) \cdot \left(\frac{\rho^4}{4} - \frac{2}{9\sqrt{2}}\rho^{9/2}\right)\Bigg|_0^2 =$$

$$= -\frac{1}{2} \cdot (-2)\left(4 - \frac{\sqrt{2}}{9} \cdot \sqrt{2^9}\right) = 4 - \frac{32}{9} = \frac{4}{9}.$$

Task 3. Calculate $I = \iiint\limits_G dxdydz$, where domain G is a sphere $x^2 + y^2 + z^2 \le R^2$.

We apply formula (3) of transition to spherical coordinates:

$$I = \iiint\limits_{G^*} \rho^2 \sin\theta\ d\rho d\varphi d\theta = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} \sin\theta\ d\theta \int\limits_0^R \rho^2\ d\rho =$$

$$= \varphi\Big|_0^{2\pi} \cdot (-\cos\theta)\Big|_0^{\pi} \cdot \frac{\rho^3}{3}\Big|_0^R =$$

$$= 2\pi(\cos 0 - \cos\pi)\cdot\frac{R^3}{3} = 4\pi\cdot\frac{R^3}{3} = \frac{4}{3}\pi R^3 .$$

Task 4. Calculate $I = \iiint\limits_G \left(x^2 + y^2\right) dxdydz$, G is determined by inequalities $z \geq 0$, $r^2 \leq x^2 + y^2 + z^2 \leq R^2$.

$$I = \iiint\limits_{G^*} \left(\rho^2 \sin^2\theta\cos^2\varphi + \rho^2\sin^2\theta\sin^2\varphi\right)\rho^2 \sin\theta \; d\rho d\varphi d\theta =$$

$$= \iiint\limits_{G^*} \rho^4 \sin^3\theta \; d\rho d\varphi d\theta =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin^3\theta \; d\theta \int_r^R \rho^4 \, d\rho = 2\pi \int_0^{\pi/2} \sin\theta\left(1 - \cos^2\theta\right) d\theta \cdot \frac{\rho^5}{5}\Big|_r^R =$$

$$= \frac{2\pi}{5}\left(R^5 - r^5\right)\int_0^{\pi/2}\left(\cos^2\theta - 1\right) d\left(\cos\theta\right) =$$

$$= \frac{2\pi}{5}\left(R^5 - r^5\right)\cdot\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi/2} =$$

$$= \frac{2\pi}{5}\left(R^5 - r^5\right)\cdot\left(0 - \left(\frac{1}{3} - 1\right)\right) = \frac{2\pi}{5}\left(R^5 - r^5\right)\cdot\frac{2}{3} = \frac{4\pi}{15}\left(R^5 - r^5\right).$$

Application of triple integral to geometrical and mechanical tasks

If a given solid is a closed bounded domain *G*, which has volume *V*, then according to geometrical meaning of triple integral:

$$V = \iiint\limits_G dxdydz\,. \qquad (4)$$

If *G* is a closed bounded domain of space R_3, occupied by material solid with density $\gamma = \gamma(x, y, z)$, where $\gamma(x, y, z)$ is a function, continuous in domain *G*, then according to mechanical meaning of triple integral, the mass of this solid is equal to:

$$m = \iiint\limits_G \gamma\, dV\,. \qquad (5)$$

Moments of inertia of a solid about axes *Ox*, *Oy*, *Oz* are respectively equal to:

$$I_x = \iiint\limits_G \left(y^2 + z^2\right)\gamma\ dV\,; \qquad I_y = \iiint\limits_G \left(x^2 + z^2\right)\gamma\ dV\,;$$

$$I_z = \iiint\limits_G \left(x^2 + y^2\right)\gamma\ dV\,. \qquad (6)$$

Moments of inertia about coordinate planes *Oxy*, *Oxz*, *Oyz* are respectively calculated with the formulas

$$I_{xy} = \iiint\limits_G z^2\gamma\ dV\,; \quad I_{xz} = \iiint\limits_G y^2\gamma\ dV\,; \quad I_{yz} = \iiint\limits_G x^2\gamma\ dV\,. \qquad (7)$$

Moment of inertia of a solid about origin of coordinates:

$$I_0 = \iiint\limits_G \left(x^2 + y^2 + z^2\right)\gamma\ dV\,. \qquad (8)$$

Coordinates of a center of mass of a solid are calculated with the formulas:

$$x_C = \frac{M_{yz}}{m}, \quad y_C = \frac{M_{xz}}{m}, \quad z_C = \frac{M_{xy}}{m}, \tag{9}$$

Where

$$M_{xy} = \iiint\limits_G z\gamma\, dV, \quad M_{xz} = \iiint\limits_G y\gamma\, dV, \quad M_{yz} = \iiint\limits_G x\gamma\, dV \tag{10}$$

are respectively static moments of a solid about coordinate planes *Oxy*, *Oxz*, *Oyz*.

Task 1. Calculate volume of a solid, bounded by cylinders $z = 4 - y^2$, $z = y^2 + 2$ and areas $x = -1$, $x = 2$.

Cylinders $z = 4 - y^2$ and $z = y^2 + 2$, generatrix of which are parallel to the axis *Ox*, cross on the lines: $4 - y^2 = y^2 + 2 \Rightarrow y^2 = 1$ $\Rightarrow y = \pm 1$. Then the required volume is found with the formula (4):

$$V = \iiint\limits_G dxdydz = \int_{-1}^{2} dx \int_{-1}^{1} dy \int_{y^2+2}^{4-y^2} dz = x\Big|_{-1}^{2} \cdot \int_{-1}^{1} \left(2 - 2y^2\right) dy =$$

$$= 3 \cdot 2\left(y - \frac{y^3}{3}\right)\Bigg|_{-1}^{1} = 6 \cdot 2 \cdot \left(1 - \frac{1}{3}\right) = 8 \text{ (c.un.)}$$

Task 2. Calculate volume of a solid, bounded by paraboloids $z = x^2 + y^2$, $z = x^2 + 2y^2$ and areas $y = x$, $y = 2x$, $x = 1$.

Volume of a given body, according to the formula (4), will be equal to:

$$V=\iiint\limits_G dxdydz=\int_0^1 dx\int_x^{2x}dy\int_{x^2+y^2}^{x^2+2y^2}dz=\int_0^1 dx\int_x^{2x}y^2dy=\int_0^1 dx\cdot\left.\frac{y^3}{3}\right|_x^{2x}=$$

$$=\frac{1}{3}\int_0^1\left(8x^3-x^3\right)dx=\frac{7}{3}\cdot\left.\frac{x^4}{4}\right|_0^1=\frac{7}{12}\ (\text{c.un.}).$$

Task 3. Calculate volume of a solid, bounded by areas $z=\sqrt{x^2+y^2}$, $z=x^2+y^2$.

The given solid is bounded by cone $z=\sqrt{x^2+y^2}$ below and by paraboloid $z=x^2+y^2$ above, they cross on the area, which projection on area Oxy is circle $x^2+y^2=1$ ($\rho=1$ is a polar equation). Let us calculate volume of a solid according to the formula (4), making a transition to the polar coordinates:

$$V=\iiint\limits_G dxdydz=\iiint\limits_{G^*}\rho\, d\rho d\varphi dz=$$

$$=\int_0^{2\pi}d\varphi\int_0^1\rho\, d\rho\int_{\rho^2}^{\rho}dz=\varphi\Big|_0^{2\pi}\cdot\int_0^1\rho\left(\rho-\rho^2\right)d\rho=$$

$$=2\pi\cdot\left.\left(\frac{\rho^3}{3}-\frac{\rho^4}{4}\right)\right|_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=2\pi\cdot\frac{1}{12}=\frac{\pi}{6}\quad(\text{c.un.}).$$

Task 4. Calculate mass of a cube $0\le x\le a$, $0\le y\le a$, $0\le z\le a$, if its density in each point is $\gamma(x,y,z)=x+y+z$.

$$m = \iiint\limits_G \gamma(x, y, z)\,dxdydz = \iiint\limits_G (x + y + z)\,dxdydz =$$

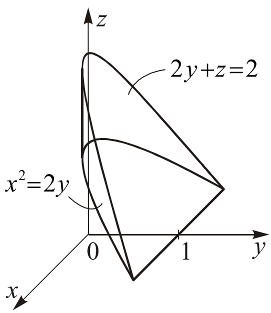
$$= \int_0^a dx \int_0^a dy \int_0^a (x + y + z)\,dz =$$

$$= \int_0^a dx \int_0^a dy \cdot \left((x + y)z + \frac{z^2}{2} \right)\Bigg|_0^a = \int_0^a dx \int_0^a \left(ay + ax + \frac{a^2}{2} \right) dy =$$

$$= \int_0^a dx \cdot \left(\frac{ay^2}{2} + \left(ax + \frac{a^2}{2} \right) \cdot y \right)\Bigg|_0^a =$$

$$= \int_0^a \left(\frac{a^3}{2} + a^2 x + \frac{a^3}{2} \right) dx = \left(\frac{a^2 x^2}{2} + a^3 x \right)\Bigg|_0^a = \frac{a^4}{2} + a^4 = \frac{3a^4}{2}.$$

Task 5. Calculate mass of a solid, bounded by cylindrical area $x^2 = 2y$ and areas $y + z = 1$, $2y + z = 2$, if its density in each point is equal to the ordinate of a point.



Drawing 31

The projection of this solid (Drawing 31) on the area *Oxy* is a part of parabola $y = \frac{x^2}{2}$, cut by line $y = 1$, consequently $-\sqrt{2} \le x \le \sqrt{2}$. Then, according to the formula (5), we calculate mass of a solid:

$$m = \iiint\limits_G \gamma\, dxdydz = \iiint\limits_G y\, dxdydz =$$

$$= \int\limits_{-\sqrt{2}}^{\sqrt{2}} dx \int\limits_{x^2/2}^{1} y\, dy \int\limits_{1-y}^{2-2y} dz = \int\limits_{-\sqrt{2}}^{\sqrt{2}} dx \int\limits_{x^2/2}^{1} y(1-y)dy =$$

$$= \int\limits_{-\sqrt{2}}^{\sqrt{2}} dx \left(\frac{y^2}{2} - \frac{y^3}{3} \right)\Bigg|_{x^2/2}^{1} = \int\limits_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{1}{2} - \frac{1}{3} - \left(\frac{x^4}{8} - \frac{x^6}{24} \right) \right) dx =$$

$$= \int\limits_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{1}{6} - \frac{x^4}{8} + \frac{x^6}{24} \right) dx = \left(\frac{x}{6} - \frac{x^5}{40} + \frac{x^7}{148} \right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}} =$$

$$= 2\left(\frac{\sqrt{2}}{6} - \frac{4\sqrt{2}}{40} + \frac{8\sqrt{2}}{148} \right) = \sqrt{2}\left(\frac{1}{3} - \frac{1}{5} + \frac{2}{21} \right) = \frac{8\sqrt{2}}{35}.$$

Task 6. Calculate moment of inertia of a cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$ about its edge.

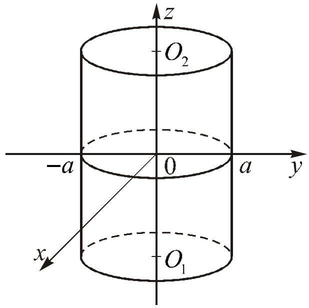
Moments of inertia of a cube with an edge equal to a about each of its edges will be equal. Let us calculate, for example, I_x, with the help of formula (6):

$$I_x = \iiint\limits_G \left(y^2 + z^2\right)\gamma\, dxdydz = \int\limits_0^a dx \int\limits_0^a dy \int\limits_0^a \left(y^2 + z^2\right)dz =$$

$$= \int\limits_0^a dx \int\limits_0^a dy \left(y^2 z + \frac{z^3}{3} \right)\Bigg|_0^a = \int\limits_0^a dx \int\limits_0^a \left(ay^2 + \frac{a^3}{3} \right) dy =$$

$$=\int_0^a dx\left(\frac{ay^3}{3}+\frac{a^3}{3}y\right)\Bigg|_0^a=\int_0^a\left(\frac{a^4}{3}+\frac{a^4}{3}\right)dx=\frac{2a^4}{3}\cdot x\Bigg|_0^a=\frac{2a^5}{3}.$$

Task 7. Calculate moment of inertia of circular cylinder with the height $O_1O_2=h$ and radius a of a base, about the axis, which is the diameter of a cylinder.



Drawing 32

The height of cylinder is $O_1O_2\in Oz$ (Drawing 32), then the diameter belongs to axis Oy. So, we calculate moment of inertia of the given solid about axis Oy:

$$I_y=\iiint_G\left(x^2+z^2\right)\gamma\,dxdydz=\iiint_{G^*}\left(\rho^2\cos^2\varphi+z^2\right)\cdot\rho\,d\rho d\varphi dz=$$

$$=\int_0^{2\pi}d\varphi\int_0^a\rho\,d\rho\int_0^h\left(\rho^2\cos^2\varphi+z^2\right)dz=$$

$$=\int_0^{2\pi}d\varphi\int_0^a\rho\,d\rho\left(\rho^2 z\cos^2\varphi+\frac{z^3}{3}\right)\Bigg|_0^h=\int_0^{2\pi}d\varphi\int_0^a\rho\,d\rho\left(\rho^2 h\cos^2\varphi+\frac{h^3}{3}\right)=$$

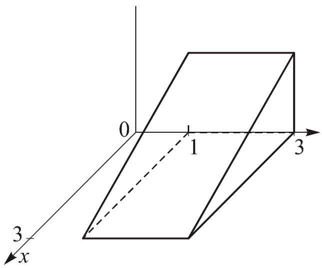
$$=\int_0^{2\pi}d\varphi\int_0^a\left(\rho^3 h\cos^2\varphi+\frac{h^3}{3}\rho\right)d\rho=\int_0^{2\pi}d\varphi\left(\frac{\rho^4}{4}h\cos^2\varphi+\frac{h^3}{3}\cdot\frac{\rho^2}{2}\right)\Bigg|_0^a=$$

$$=\int_0^{2\pi}\left(\frac{a^4}{4}h\cos^2\varphi+\frac{h^3a^2}{6}\right)d\varphi=$$

$$=\frac{a^4h}{4}\int_0^{2\pi}\frac{1+\cos 2\varphi}{2}d\varphi+\left.\frac{a^2h^3}{6}\right|_0^{2\pi}=$$

$$=\frac{a^4h}{8}\left.\left(\varphi+\frac{1}{2}\sin 2\varphi\right)\right|_0^{2\pi}+\frac{\pi a^2h^3}{3}=\frac{a^4h}{8}\cdot 2\pi+\frac{\pi a^2h^3}{3}=$$

$$=\frac{\pi a^4h}{4}+\frac{\pi a^2h^3}{3}=\frac{\pi a^2h}{12}\left(3a^2+4h^2\right).$$



Drawing 33

Task 8. Calculate coordinates of a center of mass of a homogeneous ($\gamma=1$) prismatic solid, bounded by areas $x=0$, $z=0$, $y=1$, $y=3$, $x+2z=3$ (Drawing 33).

We apply formulas (9) and (10) to calculate:

$$m=\iiint\limits_G \gamma\, dxdydz=\int_1^3 dy\int_0^{3/2}dz\int_0^{3-2z}dx=$$

$$=\left.y\right|_1^3\cdot\int_0^{3/2}(3-2z)dz=2\left.\left(3z-z^2\right)\right|_0^{3/2}=2\left(\frac{9}{2}-\frac{9}{4}\right)=2\cdot\frac{9}{4}=\frac{9}{2};$$

$$M_{yz}=\iiint\limits_G x\gamma\, dxdydz=\int_1^3 dy\int_0^{3/2}dz\int_0^{3-2z}x\,dx=2\int_0^{3/2}dz\cdot\left.\frac{x^2}{2}\right|_0^{3-2z}=$$

$$=\int_0^{3/2}\left(9-12z+z^2\right)dz=$$

$$=\left(9z-6z^2+\frac{z^3}{3}\right)\Bigg|_0^{3/2}=\frac{27}{2}-\frac{54}{4}+\frac{9}{2}=\frac{9}{2};$$

$$M_{xz}=\iiint\limits_G y\gamma\,dxdydz=\int\limits_1^3 y\,dy\int\limits_0^{3/2}dz\int\limits_0^{3-2z}dx=\frac{y^2}{2}\Bigg|_1^3\cdot\int\limits_0^{3/2}(3-2z)dz=$$

$$=\left(\frac{9}{2}-\frac{1}{2}\right)\left(3z-z^2\right)\Big|_0^{3/2}=4\left(\frac{9}{2}-\frac{9}{4}\right)=4\cdot\frac{9}{4}=9;$$

$$M_{xy}=\iiint\limits_G z\gamma\,dxdydz=\int\limits_1^3 dy\int\limits_0^{3/2}z\,dz\int\limits_0^{3-2z}dx=2\int\limits_0^{3/2}z(3-2z)dz=$$

$$=2\left(\frac{3z^2}{2}-\frac{2z^3}{3}\right)\Bigg|_0^{3/2}=2\left(\frac{27}{8}-\frac{2}{3}\cdot\frac{27}{8}\right)=\frac{27}{4}\cdot\frac{1}{3}=\frac{9}{4};$$

then $x_C=1$, $y_C=2$, $z_C=\frac{1}{2}$.

Task 9. Calculate coordinates of center of mass of a solid, bounded by areas $z^2=xy$, $x=5$, $y=5$, $z=0$.

We calculate mass and static moments of a given body:

$$m=\iiint\limits_G dxdydz=\int\limits_0^5 dx\int\limits_0^5 dy\int\limits_0^{\sqrt{xy}}dz=\int\limits_0^5\sqrt{x}\,dx\int\limits_0^5\sqrt{y}\,dy=$$

$$=\frac{x^{3/2}}{3/2}\Bigg|_0^5\cdot\frac{y^{3/2}}{3/2}\Bigg|_0^5=\frac{4}{9}\cdot 5^{3/2}\cdot 5^{3/2}=\frac{4}{9}\cdot 125;$$

$$M_{yz} = \iiint\limits_G x\,dxdydz = \int\limits_0^5 x\sqrt{x}\,dx \int\limits_0^5 \sqrt{y}\,dy = \frac{2}{5} \cdot x^{5/2}\Big|_0^5 \cdot \frac{2}{3} \cdot y^{3/2}\Big|_0^5 =$$

$$= \frac{4}{15} \cdot 5^{5/2} \cdot 5^{3/2} = \frac{4}{15} \cdot 5^4;$$

$$M_{xz} = \iiint\limits_G y\,dxdydz = \int\limits_0^5 \sqrt{x}\,dx \int\limits_0^5 y\sqrt{y}\,dy =$$

$$= \frac{2}{3} \cdot x^{3/2}\Big|_0^5 \cdot \frac{2}{5} \cdot y^{5/2}\Big|_0^5 = \frac{4}{15} \cdot 5^4;$$

$$M_{xy} = \iiint\limits_G z\,dxdydz = \int\limits_0^5 dx \int\limits_0^5 dy \cdot \frac{z^2}{2}\Big|_0^{\sqrt{xy}} = \frac{1}{2} \int\limits_0^5 x\,dx \int\limits_0^5 y\,dy =$$

$$= \frac{1}{2} \cdot \frac{x^2}{2}\Big|_0^5 \cdot \frac{y^2}{2}\Big|_0^5 = \frac{1}{8} \cdot 5^4.$$

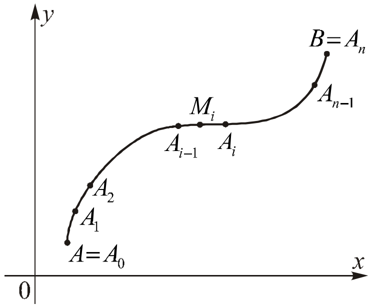
According to formulas (9) we calculate the coordinates of center of mass:

$$x_C = y_C = 3; \qquad z_C = \frac{45}{32}.$$

Lecture 5. CURVILINEAR INTEGRAL OF THE FIRST KIND: DEFINITION AND CALCULATION

A smooth curve *AB* (Drawing 34) is given in domain *Oxy*, a bounded function $f(x, y)$ is determined on this curve.

Continuous curve $x = x(t)$, $y = y(t)$ is called *smooth* on segment $\alpha \le t \le \beta$, if functions $x(t)$ and $y(t)$ have on $[\alpha;\beta]$ continuous derivatives $x'(t)$ and $y'(t)$, that are simultaneously non-zero.



Drawing 34

We divide curve *AB* with the points $A = A_0, A_1, \dots, A_{i-1}, A_i, \dots, A_{n-1}, A_n = B$ into *n* arbitrary parts. On a certain arc $\overset{\frown}{A_{i-1}A_i}$ we choose an arbitrary point $M_i(\xi_i;\eta_i)$, $i = 1, 2, \dots, n$ and calculate the sum

$$\sum_{i=1}^{n} f(\xi_i;\eta_i)\cdot \Delta l_i \,, \tag{1}$$

where Δl_i is length of an arc $\overset{\frown}{A_{i-1}A_i}$.

Sum (1) is called *integral sum* for function $f(x,y)$ on the curve $\overset{\frown}{AB}$. $\lambda = \max\limits_{1 \le i \le n} \Delta l_i$ is the biggest length of separate arcs $\overset{\frown}{A_{i-1}A_i}$.

Definition. If $\lambda \to 0$ and the integral sum (1) has limit that depends neither on the way we divide curve AB nor on the choice of points M_i, then such limit is called *curvilinear integral of the first kind (about the length of an arc)* of function $f(x,y)$ on the curve AB, i.e.

$$\int\limits_{AB} f(x,y)\,dl = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i;\eta_i)\Delta l_i \ . \qquad (2)$$

If there is limit (2), then function $f(x,y)$ is called integrable on the curve AB; AB is a contour of integration; A is the starting point of integration, and B is the finishing point of integration.

Geometrical meaning of curvilinear integral of the first kind:

$L \int\limits_{AB} = dl$ – is the length of arc AB.

Physical meaning of curvilinear integral of the first kind:

$m = \int\limits_{AB} \gamma(x,y)\,dl$ – is the mass of inhomogeneous material curve AB with linear density $\gamma(x,y)$.

Basic properties of curvilinear integral of the first kind

1. Curvilinear integral of the first kind does not depend on the direction of integration:

$$\int\limits_{AB} f(x,y)\,dl = \int\limits_{BA} f(x,y)\,dl\,;$$

2. $\int\limits_L \left(f_1(x,y) \pm f_2(x,y)\right) dl = \int\limits_L f_1(x,y)\, dl \pm \int\limits_L f_2(x,y)\, dl$;

3. $\int\limits_L C \cdot f(x,y)\, dl = C\int\limits_L f(x,y)\, dl$, $C-\text{const}$;

4. If contour of integration L is divided into two parts L_1 and L_2, then

$$\int\limits_L f(x,y)\, dl = \int\limits_{L_1} f(x,y)\, dl + \int\limits_{L_2} f(x,y)\, dl.$$

Calculation of curvilinear integrals of the first kind

1. If curve AB is represented by equation $y = y(x)$, $a \le x \le b$, where function $y(x)$ is continuous along with its derivative $y'(x)$ on $[a;b]$, then:

$$\int\limits_{AB} f(x,y)\, dl = \int\limits_a^b f(x, y(x)) \cdot \sqrt{1 + (y'(x))^2}\, dx. \qquad (3)$$

2. If curve AB is represented by equation $x = x(y)$, $c \le y \le d$ and functions $x(y)$ and $x'(y)$ are continuous on $[c;d]$, then:

$$\int\limits_{AB} f(x,y)\, dl = \int\limits_c^d f(x(y), y)\sqrt{1 + (x'(y))^2}\, dy. \qquad (4)$$

3. If curve AB is represented parametrically by equations $x = x(t)$, $y = y(t)$, $t \in [\alpha;\beta]$ and functions $x(t)$, $y(t)$, $x'(t)$, $y'(t)$ are continuous on $[\alpha;\beta]$, then:

$$\int\limits_{AB} f(x,y)\, dl = \int\limits_\alpha^\beta f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \qquad (5)$$

4. In case of spatial curve *AB*, represented by equations $x = x(t)$, $y = y(t)$, $z = z(t)$ ($\alpha \le t \le \beta$) are functions, continuous along with their first derivatives about parameter *t* on segment $[\alpha;\beta]$, then:

$$\int\limits_{AB} f(x, y, z)\, dl = \int\limits_{\alpha}^{\beta} f(x(t), y(t), z(t))\sqrt{(x_t')^2 + (y_t')^2 + (z_t')^2}\, dt . \quad (6)$$

Task 1. Calculate $\int\limits_L (x - y) dl$, where L is a segment of line from the point $A(0;0)$ to the point $B(4;3)$.

Let us work out an equation of line *AB*: $y = \frac{3}{4}x$, then $y' = \frac{3}{4}$; we use formula (3):

$$\int\limits_L (x - y) dl = \int\limits_0^4 \left(x - \frac{3}{4}x \right)\sqrt{1 + \frac{9}{16}}\, dx = \frac{5}{16}\int\limits_0^4 x\, dx = \frac{5}{32}x^2 \Big|_0^4 = \frac{5}{2} \cdot$$

Task 2. Calculate curvilinear integral $I = \int\limits_L \left(4\sqrt[3]{x} - 3\sqrt{y} \right) dl$ between the points $E(-1;0)$ and $H(0;1)$:

a) about the line *EH*;

b) about the arc of asteroid $x = \cos^3 t$, $y = \sin^3 t$.

a) Equation of line *EH*: $x = y - 1$, $x' = 1$, then according to the formula (4):

$$I = \int\limits_0^1 \left(4\sqrt[3]{y - 1} - 3\sqrt{y} \right) \cdot \sqrt{2}\, dy = \sqrt{2}\left(\frac{4(y - 1)^{4/3}}{4/3} - \frac{3y^{3/2}}{3/2} \right)\Bigg|_0^1 =$$

$$=\sqrt{2}\left(3\sqrt[3]{(y-1)^4}-2\sqrt{y^3}\right)\Big|_0^1=\sqrt{2}(-2-3)=-5\sqrt{2}.$$

b) If $x=\cos^3 t$, $y=\sin^3 t$, then $x'_t=3\cos^2 t\cdot(-\sin t)$, $y'_t=3\sin^2 t\cdot\cos t$. Then according to the formula (5) we first calculate:

$$dl=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt=3\sin t\cos t\,dt,$$

then

$$I=3\int_{\pi}^{\pi/2}\left(4\sqrt[3]{\cos^3 t}-3\sqrt{\sin^3 t}\right)\sin t\cdot\cos t\;dt=$$

$$=12\int_{\pi}^{\pi/2}\cos^2 t\cdot\sin t\;dt-9\int_{\pi}^{\pi/2}\sin^{5/2} t\cdot\cos t\;dt=$$

$$=-12\int_{\pi}^{\pi/2}\cos^2 t\,d(\cos t)-9\int_{\pi}^{\pi/2}\sin^{5/2} t\;d(\sin t)=$$

$$=-4\cos^3 t\Big|_{\pi}^{\pi/2}-9\frac{\sin^{7/2} t}{7/2}\Bigg|_{\pi}^{\pi/2}=$$

$$=-4\left(\cos^3\frac{\pi}{2}-\cos^3\pi\right)-\frac{18}{7}\left(\sin^{7/2}\frac{\pi}{2}-\sin^{7/2}\pi\right)=$$

$$=-4(0+1)-\frac{18}{7}(1-0)=-4-\frac{18}{7}=-4-2\frac{4}{7}=-6\frac{4}{7}.$$

Task 3. Calculate $\int\limits_L \left(x^2 + y^2\right)^n dl$, where L is a circle $x = a\cos t$, $y = a\sin t$.

Since $x_t' = -a\sin t$, $y_t' = a\cos t$,

$dl = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\, dt = a\,dt$, then according to the formula (5):

$$\int\limits_L \left(x^2 + y^2\right)^n dl = \int\limits_0^{2\pi} \left(a^2 \cos^2 t + a^2 \sin^2 t\right)^n a\,dt = \int\limits_0^{2\pi} \left(a^2\right)^n \cdot a\,dt =$$

$$= a^{2n+1} \int\limits_0^{2\pi} dt = a^{2n+1} \cdot t \Big|_0^{2\pi} = 2\pi \cdot a^{2n+1}.$$

Task 4. Calculate $I = \int\limits_L \frac{dl}{x^2 + y^2 + z^2}$, where L is the first turn of helical line $x = a\cos t$, $y = a\sin t$, $z = bt$.

We calculate $x_t' = -a\sin t$, $y_t' = a\cos t$, $z_t' = b$,

$dl = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2}\, dt = \sqrt{a^2 + b^2}\, dt$, then according to the formula (6):

$$I = \int\limits_0^{2\pi} \frac{\sqrt{a^2 + b^2}\, dt}{a^2 \cos^2 t + a^2 \sin^2 t + b^2 t^2} = \sqrt{a^2 + b^2} \int\limits_0^{2\pi} \frac{dt}{a^2 + b^2 t} =$$

$$= \sqrt{a^2 + b^2} \cdot \frac{1}{ab} \operatorname{arctg} \frac{bt}{a} \Big|_0^{2\pi} = \frac{\sqrt{a^2 + b^2}}{ab} \operatorname{arctg} \frac{2\pi b}{a}.$$

Task 5. Calculate curvilinear integral $\int\limits_L xy\;dl$, where L is a quarter of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, set in the first quadrant.

We express variable x from the equation of ellipse:

$$x^2=a^2\sqrt{1-\frac{y^2}{b^2}}, \qquad x=\frac{a}{b}\sqrt{b^2-y^2},$$

then

$$x'(y)=\frac{a}{b}\cdot\frac{(-2y)}{2\sqrt{b^2-y^2}}dy=-\frac{a}{b}\cdot\frac{y}{\sqrt{b^2-y^2}}dy.$$

We calculate integral according to the formula (5):

$$\int\limits_L xy\;dl=\int\limits_0^b\frac{a}{b}\sqrt{b^2-y^2}\cdot y\cdot\sqrt{1+\frac{a^2}{b^2}\cdot\frac{y^2}{b^2-y^2}}\,dy=$$

$$=\frac{a}{b}\int\limits_0^b y\cdot\sqrt{b^2-y^2}\cdot\sqrt{\frac{b^2\left(b^2-y^2\right)+a^2y^2}{b^2\cdot\left(b^2-y^2\right)}}\,dy=$$

$$=\frac{a}{b}\int\limits_0^b y\cdot\frac{\sqrt{b^4-b^2y^2+a^2y^2}}{b}dy=\frac{a}{b^2}\int\limits_0^b y\cdot\sqrt{b^4+\left(a^2-b^2\right)y^2}\,dy=$$

$$=\frac{a}{b^2}\cdot\frac{1}{2\left(a^2-b^2\right)}\int\limits_0^b 2\left(a^2-b^2\right)\cdot y\cdot\sqrt{b^4+\left(a^2-b^2\right)y^2}\,dy=$$

$$=\frac{a}{2b^2\left(a^2-b^2\right)}\cdot\left.\frac{\left(b^4+\left(a^2-b^2\right)\cdot y^2\right)^{3/2}}{3/2}\right|_0^b=$$

$$= \frac{a}{3b^2\left(a^2 - b^2\right)}\left(\left(b^4 + a^2b^2 - b^4\right)^{3/2} - \left(b^4\right)^{3/2}\right) =$$

$$= \frac{a}{3b^2\left(a^2 - b^2\right)}\left(a^3b^3 - b^6\right) = \frac{ab^3\left(a^3 - b^3\right)}{3b^2\left(a^2 - b^2\right)} =$$

$$= \frac{ab(a-b)\left(a^2 + ab + b^2\right)}{3(a-b)(a+b)} = \frac{ab\left(a^2 + ab + b^2\right)}{3(a+b)}.$$

Application of curvilinear integrals of the first kind

In plane *Oxy* is represented a smooth curve *AB*, on which a continuous function $f(x, y)$ is determined, then *area P of cylindrical surface*, represented by function $z = f(x, y)$, is calculated with the formula:

$$P = \int_{AB} f(x, y)\, dl\,, \tag{7}$$

formula for length *L* of curve *AB* is as follows:

$$L = \int_{AB} dl\,. \tag{8}$$

If lengthwise inhomogeneous curve *L* mass with linear density $\gamma(x, y)$ is spread, then *mass of curve L* is calculated according to the formula:

$$m = \int_{L} \gamma(x, y)\, dl\,. \tag{9}$$

Coordinates of center of mass of curve L are calculated with the formula:

$$x_C = \frac{M_y}{m}, \qquad y_C = \frac{M_x}{m}, \tag{10}$$

where

$$M_x = \int_L y\gamma(x,y)\,dl, \qquad M_y = \int_L x\gamma(x,y)\,dl \tag{11}$$

are *static moments of curve L* about axes *Ox* and *Oy* respectively.

Moments of inertia of curve L about axes *Ox*, *Oy* and the origin of coordinates are respectively equal to:

$$I_x = \int_L y^2\gamma(x,y)\,dl; \qquad I_y = \int_L x^2\gamma(x,y)\,dl;$$

$$I_0 = \int_L \left(x^2 + y^2\right)\gamma(x,y)\,dl. \tag{12}$$

Task 1. Calculate the area of a side surface of parabolic cylinder $y = \frac{3}{8}x^2$, bounded by planes $z = 0$, $x = 0$, $z = x$, $y = 6$.

Let us consider cylindrical surface with generatrix parallel to axis *Oz*; base – contour of integration – is arc *AB* of parabola $y = \frac{3}{8}x^2$, that connects points $A(0;0)$ and $B(4;6)$; and with heights equal to the function being integrated. So, according to the formula (7):

$$P = \int_{AB} f(x,y)\,dl = \int_0^4 x\cdot\sqrt{1+\left(\frac{3}{4}x\right)^2}\,dx = \int_0^4 x\sqrt{1+\frac{9}{16}x^2}\,dx =$$

$$= \frac{1}{4}\int_0^4 x\sqrt{16+9x^2}\,dx =$$

$$=\frac{1}{72}\int_0^4 18x\cdot\sqrt{16+9x^2}\,dx=\frac{1}{72}\int_0^4\left(16+9x^2\right)^{1/2}d\left(16+9x^2\right)=$$

$$=\frac{1}{72}\cdot\frac{2}{3}\left(16+9x^2\right)^{3/2}\Bigg|_0^4=$$

$$=\frac{1}{108}\left(\left(16+9\cdot16\right)^{3/2}-16^{3/2}\right)=\frac{1}{108}\cdot16^{3/2}\left(10^{3/2}-1\right)=$$

$$=\frac{1}{108}\cdot4^3\left(10\sqrt{10}-1\right)=\frac{64}{108}\left(10\sqrt{10}-1\right)=\frac{16}{27}\left(10\sqrt{10}-1\right).$$

Task 2. Calculate the length of cardioid $x=2a\cos t-a\cos 2t$, $y=2a\sin t-a\sin 2t$.

We calculate $x'_t=-2a\sin t+2a\sin 2t$,

$y'_t=2a\cos t-2a\cos 2t$,

then $$dl=\sqrt{\left(x'_t\right)^2+\left(y'_t\right)^2}\,dt=$$

$$=2a\sqrt{\sin^2 t-2\sin t\sin 2t+\sin^2 2t+\cos^2 t-2\cos t\cos 2t+\cos^2 2t}\,dt=$$

$$=2a\sqrt{2-2\left(\sin t\sin 2t+\cos t\cos 2t\right)}\,dt=$$

$$=2a\sqrt{2-2\cos t}=2a\sqrt{2\left(1-\cos t\right)}\,dt=$$

$$=2a\sqrt{4\sin^2\frac{t}{2}}\,dt=4a\cdot\sin\frac{t}{2}\,dt;$$

then according to the formula (8) the length of cardioids is as follows:

$$L=\int\limits_{AB} dl = 4a\int\limits_0^{2\pi}\sin\frac{t}{2}\,dt = -8a\cos\frac{t}{2}\Big|_0^{2\pi} = -8a(\cos\pi-\cos 0) = 16a\,.$$

Task 3. Calculate the length of arc AB of curve $e^{2y}\left(e^{2x}-1\right)=e^{2x}+1$; $x_A=1$, $x_B=2$.

We differentiate the equation of a given curve, having previously taken the logarithm of it:

$$\ln e^{2y}+\ln\left(e^{2x}-1\right)=\ln\left(e^{2x}+1\right);\qquad 2y=\ln\left(e^{2x}+1\right)-\ln\left(e^{2x}-1\right);$$

$$y=\frac{1}{2}\cdot\ln\left(\frac{e^{2x}+1}{e^{2x}-1}\right);$$

$$y'=\frac{1}{2}\cdot\frac{e^{2x}-1}{e^{2x}+1}\cdot\frac{2e^{2x}\left(e^{2x}-1\right)-2e^{2x}\left(e^{2x}+1\right)}{\left(e^{2x}-1\right)^2}=$$

$$=\frac{1}{2}\cdot\frac{1}{e^{2x}+1}\cdot\frac{2e^{4x}-2e^{2x}-2e^{4x}-2e^{2x}}{e^{2x}-1}=$$

$$=\frac{1}{2}\cdot\frac{-4e^{2x}}{e^{4x}-1}=-\frac{2e^{2x}}{e^{4x}-1}\,.$$

We apply formula (8):

$$L=\int\limits_{AB} dl=\int\limits_1^2\sqrt{1+\frac{4e^{4x}}{\left(e^{4x}-1\right)^2}}\,dx=\int\limits_1^2\sqrt{\frac{e^{8x}-2e^{4x}+1+4e^{4x}}{\left(e^{4x}-1\right)^2}}\,dx=$$

$$=\int\limits_1^2\sqrt{\frac{\left(e^{4x}+1\right)^2}{\left(e^{4x}-1\right)^2}}\,dx=$$

$$= \int_1^2 \frac{e^{4x}+1}{e^{4x}-1}\,dx = \int_1^2 \left(1+\frac{2}{e^{4x}-1}\right)dx = x\Big|_1^2 + I_1 = 1 + I_1;$$

$$I_1 = 2\int_1^2 \frac{dx}{e^{4x}-1} = \left[\begin{array}{c} e^{4x}-1=t;\; e^{4x}=t+1; \\ 4x=\ln(t+1);\; x=\frac{1}{4}\ln(t+1); \\ dx=\frac{dt}{4(t+1)};\; t_1=e^4-1,\; t_2=e^8-1 \end{array}\right] =$$

$$= 2\int_{e^4-1}^{e^8-1} \frac{dt}{4(t+1)\cdot t} = \frac{1}{2}\int_{e^4-1}^{e^8-1} \frac{dt}{t(t+1)};$$

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \quad \Rightarrow \quad A=1,\; B=-1 \quad \Rightarrow$$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1};$$

$$I_1 = \frac{1}{2}\int_{e^4-1}^{e^8-1} \left(\frac{1}{t}-\frac{1}{t+1}\right)dt = \frac{1}{2}\left(\ln|t| - \ln|t+1|\right)\Big|_{e^4-1}^{e^8-1} =$$

$$= \frac{1}{2}\left(\ln\left(e^8-1\right) - \ln e^8 - \ln\left(e^4-1\right) + \ln e^4\right) =$$

$$= \frac{1}{2}\left(\ln\frac{e^8-1}{e^4-1} - 8 + 4\right) = \frac{1}{2}\left(\ln\left(e^4+1\right) - 4\right).$$

We calculate the length of an arc of the curve:

$$L = 1 + \frac{1}{2}\ln\left(e^4 + 1\right) - 2 = \frac{1}{2}\ln\left(e^4 + 1\right) - 1 = \frac{1}{2}\ln\left(e^4 + 1\right) - \ln e =$$

$$= \frac{1}{2}\ln\left(e^4 + 1\right) - \frac{1}{2}\ln e^2 = \frac{1}{2}\ln\frac{e^4 + 1}{e^2} = \frac{1}{2}\ln\left(e^2 + e^{-2}\right).$$

Task 4. Calculate the mass of an arc of a circle $x = \cos t$, $y = \sin t$ ($0 \le t \le \pi$), if linear density in each point is equal to the ordinate of a point.

According to the task $\gamma(x, y) = y$, then with the help of formula (9) we calculate:

$$m = \int_L \gamma(x, y)\, dl = \int_L y\, dl = \int_0^{\pi} \sin t \cdot \sqrt{\sin^2 t + \cos^2 t}\; dt = -\cos t\Big|_0^{\pi} =$$

$$= \cos 0 - \cos \pi = 2 .$$

Task 5. Calculate the mass of an arc AB of a curve $y = \ln x$, if in every its point linear density is proportional to the square of abscissa of the point: $x_A = 1$, $x_B = 3$.

$$m = \int_L \gamma(x, y)\, dl = \int_L kx^2\, dl = k\int_1^3 x^2 \cdot \sqrt{1 + \frac{1}{x^2}}\; dx = k\int_1^3 x\sqrt{x^2 + 1}\; dx =$$

$$= \frac{k}{2}\int_1^3 \sqrt{1 + x^2}\; d\left(1 + x^2\right) = \frac{k}{2} \cdot \frac{\left(1 + x^2\right)^{3/2}}{3/2}\Bigg|_1^3 = \frac{k}{3}\left(10^{3/2} - 2^{3/2}\right) =$$

$$= \frac{k}{3}\left(10\sqrt{10} - 2\sqrt{2}\right).$$

Task 6. Calculate coordinates of center of mass of a semi-arc of cycloid $x = a\left(t - \sin t\right),\ y = a\left(1 - \cos t\right),\ 0 \le t \le \pi$.

According to the formulas (11) we calculate static moments about axes *Ox* and *Oy* respectively, having first calculated *dl*:

$$dl = \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\ dt = a\sqrt{2 - 2\cos t}\ dt =$$

$$= 2a\sqrt{\sin^2\frac{t}{2}}\ dt = 2a\sin\frac{t}{2}\ dt;$$

$$M_x = \int_L \gamma(x,y)y\ dl = a\int_0^\pi (1-\cos t)\cdot 2a\sin\frac{t}{2}\ dt =$$

$$= 2a^2\int_0^\pi\left(1 - 2\cos^2\frac{t}{2} + 1\right)\cdot\sin\frac{t}{2}\ dt =$$

$$= 4a^2\int_0^\pi \sin\frac{t}{2}\ dt - 4a^2\int_0^\pi \cos^2\frac{t}{2}\cdot\sin\frac{t}{2}\ dt =$$

$$= -8a^2\cos\frac{t}{2}\bigg|_0^\pi + 8a^2\int_0^\pi\cos^2\frac{t}{2}\ d\left(\cos\frac{t}{2}\right) =$$

$$= 8a^2 + \frac{8a^2}{3}\cdot\cos^3\frac{t}{2}\bigg|_0^\pi = 8a^2 - \frac{8}{3}a^2 = \frac{16a^2}{3};$$

$$M_y = \int_L \gamma(x,y)x\ dl = a\int_0^\pi (t - \sin t)\cdot 2a\sin\frac{t}{2}\ dt =$$

$$= 2a^2\int_0^\pi t\cdot\sin\frac{t}{2}\ dt - 4a^2\int_0^\pi \sin^2\frac{t}{2}\cos\frac{t}{2}\ dt =$$

$$= 2a^2 \cdot I_1 - 8a^2 \int_0^{\pi} \sin^2 \frac{t}{2}\, d\left(\sin\frac{t}{2}\right) = 8a^2 - 8a^2 \cdot \frac{1}{3} \sin^3 \frac{t}{2}\bigg|_0^{\pi} =$$

$$= 8a^2 - \frac{8a^2}{3} = \frac{16a^2}{3},$$

where

$$I_1 = \int_0^{\pi} t \cdot \sin\frac{t}{2}\, dt = \begin{bmatrix} t = u, & dt = du, \\ \sin\frac{t}{2} dt = dv, & v = -2\cos\frac{t}{2} \end{bmatrix} =$$

$$= -2t\cos\frac{t}{2}\bigg|_0^{\pi} + 2\int_0^{\pi} \cos\frac{t}{2}\, dt = 4\sin\frac{t}{2}\bigg|_0^{\pi} = 4.$$

Mass of the line:

$$m = \int_L \gamma(x, y)\, dl = 2a\int_0^{\pi} \sin\frac{t}{2}\, dt = -4a \cdot \cos\frac{t}{2}\bigg|_0^{\pi} =$$

$$= -4a\left(\cos\frac{\pi}{2} - \cos 0\right) = 4a.$$

Then, with the help of formulas (10), we calculate coordinates of center of mass of the line:

$$x_C = y_C = \frac{16a^2}{3 \cdot 4a} = \frac{4a}{3}.$$

Task 7. Calculate moments of inertia about coordinate axes of the first turn of helical line $x = a\cos t$, $y = a\sin t$, $z = \frac{h}{2\pi} t$.

We calculate moments of inertia according to the formulas (12), taking into consideration that

$$dl = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + \frac{h^2}{4\pi^2}}\, dt = \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi}\, dt ,$$

then

$$I_x = \int_L \gamma\left(y^2 + z^2\right) dl = \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \int_0^{2\pi} \left(a^2 \sin^2 t + \frac{h^2}{4\pi^2} t^2 \right) dt =$$

$$= \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \int_0^{2\pi} \left(\frac{a^2}{2}(1 - \cos 2t) + \frac{h^2}{4\pi^2} \cdot t^2 \right) dt =$$

$$= \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \left(\frac{a^2}{2}\left(t - \frac{1}{2}\sin 2t \right) + \frac{h^2}{4\pi^2} \cdot \frac{t^3}{3} \right)\Bigg|_0^{2\pi} =$$

$$= \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \left(\frac{a^2}{2} \cdot 2\pi + \frac{h^2}{4\pi^2} \cdot \frac{8\pi^3}{3} \right) =$$

$$= \sqrt{4\pi^2 a^2 + h^2} \cdot \left(\frac{a^2}{2} + \frac{h^2}{3} \right);$$

$$I_y = \int_L \gamma\left(x^2 + z^2\right) dl = \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \int_0^{2\pi} \left(a^2 \cos^2 t + \frac{h^2}{4\pi^2} t^2 \right) dt =$$

$$= \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \left(\frac{a^2}{2}\left(t + \frac{1}{2}\sin \frac{t}{2} \right) + \frac{h^2}{4\pi^2} \cdot \frac{t^3}{3} \right)\Bigg|_0^{2\pi} =$$

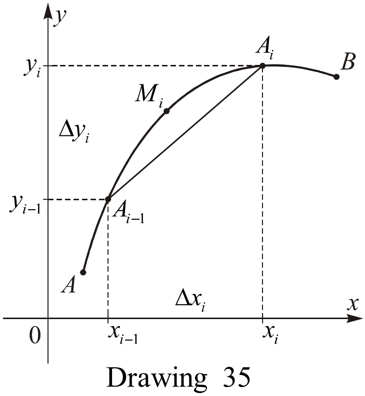
$$= \sqrt{4\pi^2 a^2 + h^2} \cdot \left(\frac{a^2}{2} + \frac{h^2}{3} \right);$$

$$I_z = \int_L \gamma\left(x^2 + y^2\right) dl = \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} \int_0^{2\pi} \left(a^2 \cos^2 t + a^2 \sin^2 t\right) dt =$$

$$= \frac{a^2 \sqrt{4\pi^2 a^2 + h^2}}{2\pi} \cdot t \Bigg|_0^{2\pi} = a^2 \cdot \sqrt{4\pi^2 a^2 + h^2} \ .$$

Lecture 6. CALCULATION AND APPLICATION OF CURVILINEAR INTEGRAL OF THE SECOND KIND. GREEN'S FORMULA. INTEGRATION OF ORDINARY DIFFERENTIAL

In the plane *Oxy* is represented a smooth curve *AB* (Drawing 35), on which a bounded function $P(x, y)$ is determined. We divide curve *AB* by points $A = A_0, A_1, \ldots, A_{i-1}, A_i, \ldots, A_{n-1}, A_n = B$ into n arbitrary parts; on each partial arc $\smile{A_{i-1}A_i}$ we choose point $M_i(\xi_i; \eta_i)$, $i = 1, 2, \ldots, n$ and calculate the sum



Drawing 35

$$\sum_{i=1}^{n} P(\xi_i; \eta_i) \cdot \Delta x_i , \qquad (1)$$

where Δx_i is a projection of a vector $\overline{A_{i-1}A_i}$ on the axis *Ox*.

If $\lambda = \max\limits_{1 \le i \le n} \Delta x_i \to 0$, and integral sum (1) has finite limit, that depends neither on the way we divide curve *AB* nor on the choice of points M_i, then this limit is called *curvilinear integral of function* $P(x, y)$ *by coordinate* x lengthwise the curve *AB*:

$$\int_{AB} P(x,y)\,dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} P(\xi_i;\eta_i)\cdot\Delta x_i\ . \qquad (2)$$

By analogy *curvilinear integral of function* $Q(x,y)$ *about coordinate* y:

$$\int_{AB} Q(x,y)\,dy = \lim_{\lambda\to 0}\sum_{i=1}^{n} Q(\xi_i;\eta_i)\cdot\Delta y_i\ , \qquad (3)$$

where Δy_i is the projection of a vector $\overline{A_{i-1}A_i}$ on the axis Oy (Drawing 35).

Sum $\int\limits_{AB} P(x,y)\,dx + Q(x,y)\,dy$ (4) is called curvilinear integral about coordinates or *curvilinear integral of the second kind.*

Calculation of curvilinear coordinates of the second kind

Let us assume that curve AB is represented by parametric equations $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$, where functions $x(t)$ and $y(t)$ on the segment $[\alpha;\beta]$ are continuous along with their derivatives $x'(t)$ and $y'(t)$; point A of a curve corresponds to the parameter α, and point B corresponds to the parameter β; functions $P(x,y)$, $Q(x,y)$ are continuous on the curve AB, then:

$$\int_{AB} P(x,y)\,dx + Q(x,y)\,dy =$$

$$= \int_{\alpha}^{\beta} \left(P(x(t),y(t))\cdot x'(t) + Q(x(t),y(t))\cdot y'(t)\right)dt\ . \qquad (5)$$

If curve AB is represented by equation $y = y(x)$, $a \le x \le b$, where function $y(x)$ and its derivative $y'(x)$ are continuous on $[a;b]$, then

$$\int_{AB} P(x,y)\,dx + Q(x,y)\,dy =$$

$$= \int_a^b \left(P(x, y(x)) + Q(x, y(x)) \cdot y'(x)\right) dx\,. \qquad (6)$$

By analogy, if curve AB is represented by equation $x = x(y)$, $c \le y \le d$, where functions $x(y)$ and $x'(y)$ are continuous on $[c;d]$, then

$$\int_{AB} P(x,y)\,dx + Q(x,y)\,dy = \int_c^d \left(P(x(y), y) \cdot x'(y) + Q(x(y), y)\right) dy\,. \quad (7)$$

The notion of curvilinear integral of the second kind may be broadened in case of spatial curves. Let us assume that functions $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ are determined and continuous on the spatial curve AB, represented by equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad \alpha \le t \le \beta,$$

where functions $x(t)$, $y(t)$, $z(t)$ and their derivatives $x'(t)$, $y'(t)$, $z'(t)$ are continuous on the segment $[\alpha;\beta]$, then the following formula is valid:

$$\int_{AB} P\,dx + Q\,dy + R\,dz =$$

$$= \int_\alpha^\beta \left(P(x(t), y(t), z(t)) \cdot x'(t) + Q(x(t), y(t), z(t)) \cdot y'(t) + \right.$$

$$+ R(x(t), y(t), z(t)) \cdot z'(t)) \, dt \, . \tag{8}$$

Curvilinear integral of the second kind depends on the direction of integration path, and reverses sign due to the change of direction:

$$\int_{AB} P\,dx + Q\,dy = -\int_{BA} P\,dx + Q\,dy \, . \tag{9}$$

Curvilinear integrals, in which the starting and the finishing points of contour of integration coincide (closed contours without points of self-intersection), are *integrals about a closed contour.* There are two directions of path tracing for a closed contour: counterclockwise (positive orientation of a contour) and clockwise (negative orientation of a contour). Curvilinear integral of the second kind about the positive-oriented contour L is denoted in the following way:

$$\oint_L P(x, y)\,dx + Q(x, y)\,dy \, .$$

Application of curvilinear integral of the second kind

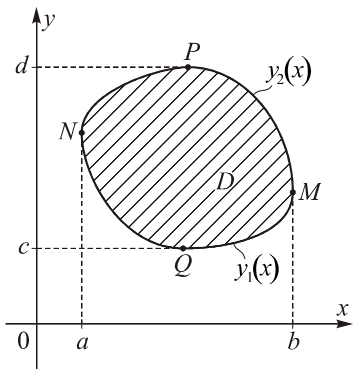
Regular domain is represented on the plane (Drawing 36):

$D = \{y_1(x) \le y \le y_2(x), \; a \le x \le b\}$.

We designate the limit of domain D, i.e. curve $PNQM$, as L, and consider it as positive oriented.

Then *area S of the regular domain D*, bounded by the curve L, is calculated with the formula:

$$S = \frac{1}{2}\oint_L x\,dy - y\,dx \, . \tag{10}$$



Drawing 36

If force $\overline{F} = P(x,y)\cdot\bar{i} + Q(x,y)\cdot\bar{j}$ performs *work A during the movement of material point* lengthwise curve L, functions $P(x,y)$ and $Q(x,y)$ are continuous on the curve L, then

$$A = \int_L P(x,y)dx + Q(x,y)dy\,. \tag{11}$$

Task 1. Calculate curvilinear integral of the second kind $\int_{AB}(x^2 - 2xy)dx + (2xy + y^2)dy$, where AB is an arc of parabola $y = x^2$ from the point $A(1;1)$ to the point $B(2;4)$.

We apply formula (6), then:

$$\int_{AB}(x^2 - 2xy)dx + (2xy + y^2)dy = \int_1^2\left((x^2 - 2x^3) + (2x^3 + x^4)\cdot 2x\right)dx =$$

$$= \int_1^2 (x^2 - 2x^3 + 4x^4 + 2x^5)dx = \left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{4x^5}{5} + \frac{x^6}{3}\right)\Bigg|_1^2 =$$

$$= \frac{8}{3} - \frac{16}{2} + \frac{128}{5} + \frac{64}{3} - \left(\frac{1}{3} - \frac{1}{2} + \frac{4}{5} + \frac{1}{3}\right) =$$

$$= \frac{70}{3} - \frac{15}{2} + \frac{124}{5} = \frac{1219}{30} = 40\frac{19}{30}\,.$$

Task 2. Calculate integral $\int_L y^2dx + x^2dy$, where L is an upper half of an ellipse $x = a\cos t$, $y = b\sin t$, the direction of path tracing is clockwise.

Since $x'(t) = -a\sin t$ and $y'(t) = b\cos t$, according to the formula (5):

$$\int\limits_L y^2dx + x^2dy = \int\limits_0^{\pi}\left(b^2\sin^2 t\cdot(-a\sin t) + a^2\cos^2 t\cdot b\cos t\right)dt =$$

$$= -ab^2\int\limits_0^{\pi}\left(1-\cos^2 t\right)\cdot\sin t\ dt + a^2b\int\limits_0^{\pi}\left(1-\sin^2 t\right)\cdot\cos t\ dt =$$

$$= -ab^2\int\limits_0^{\pi}\sin t\ dt - ab^2\int\limits_0^{\pi}\cos^2 t\ d(\cos t) +$$

$$+ a^2b\int\limits_0^{\pi}\cos t\ dt - a^2b\int\limits_0^{\pi}\sin^2 t\ d(\sin t) =$$

$$= ab^2\cos t\Big|_0^{\pi} - ab^2\cdot\frac{\cos^3 t}{3}\Big|_0^{\pi} + a^2b\cdot\sin t\Big|_0^{\pi} - a^2b\cdot\frac{\sin^3 t}{3}\Big|_0^{\pi} =$$

$$= -2ab^2 + \frac{2}{3}ab^2 = -\frac{4ab^2}{3}.$$

Taking into account that path tracing is clockwise, i.e. in negative direction, we reverse sign, that is why

$$\int\limits_L y^2dx + x^2dy = \frac{4ab^2}{3}.$$

Task 3. Calculate $\int\limits_{(0;0)}^{(\pi;2\pi)} -x\cos y\ dx + y\sin x\ dy$ lengthwise the segment that connects points $(0;0)$ and $(\pi;2\pi)$.

The equation of the line that connects points $(0;0)$ and $(\pi;2\pi)$ is as follows: $\frac{x}{\pi}=\frac{y}{2\pi}$ or $y=2x$, then $y'=2$; according to the formula (6):

$$\int_{(0;0)}^{(\pi;2\pi)} -x\cos y\ dx + y\sin x\ dy =$$

$$=\int_0^{\pi}\left(-x\cdot\cos 2x+(2x\cdot\sin x)\cdot 2\right)dx = I_1+I_2=4\pi,$$

where

$$I_1=-\int_0^{\pi} x\cdot\cos 2x\ dx=\begin{bmatrix} x=u, & dx=du, \\ \cos 2x\ dx=dv, & v=\frac{1}{2}\sin 2x \end{bmatrix}=$$

$$=-\left(\frac{x}{2}\sin 2x\Big|_0^{\pi}-\frac{1}{2}\int_0^{\pi}\sin 2x\ dx\right)=-\frac{1}{4}\cos 2x\Big|_0^{\pi}=0;$$

$$I_2=4\int_0^{\pi} x\sin x\ dx=\begin{bmatrix} x=u, & dx=du, \\ \sin x\ dx=dv, & v=-\cos x \end{bmatrix}=$$

$$=4\left(-x\cdot\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos x\ dx\right)=4\left((-\pi\cos\pi)+\sin x\Big|_0^{\pi}\right)=4\pi.$$

Task 4. Calculate curvilinear integral lengthwise the given lines: $I=\int_{(0;0)}^{(1;1)} xy\,dx+(y-x)dy$; a) $y=x$; b) $y=x^2$; c) $y^2=x$.

We apply formula (6) in assignments a) and b), and formula (8) in assignment c).

a) Since $y = x$, $dy = dx$, then: $I = \int_0^1 x^2\,dx = \frac{1}{3}$;

b) In this case $y = x^2$, $y' = 2x$, then:

$$I = \int_0^1 \left(x^3 + \left(x^2 - x\right)\cdot 2x\right)dx = \int_0^1 \left(3x^3 - 2x^2\right)dx =$$

$$= \left(\frac{3x^4}{4} - \frac{2x^3}{3}\right)\Bigg|_0^1 = \frac{1}{12};$$

c) We represent function $x = y^2$, then $dx = 2y\,dy$, so:

$$I = \int_0^1 \left(y^3 \cdot 2y + y - y^2\right)dy = \left(\frac{2y^5}{5} + \frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{2}{5} + \frac{1}{2} - \frac{1}{3} = \frac{17}{30}.$$

Task 5. Calculate $I = \int_L \frac{x^2\,dy - y^2\,dx}{x^{5/3} + y^{5/3}}$, where L is a quarter of astroid $x = R\cos^3 t$, $y = R\sin^3 t$ from the point $(R;0)$ to the point $(0;R)$.

The value of the parameter $0 \le t \le \frac{\pi}{2}$ corresponds to the given points, then taking into account that $x'(t) = -3R\cos^2 t \cdot \sin t$, $y'(t) = 3R\sin^2 t \cdot \cos t$, according to the formula (5), we calculate:

$$I=\int_0^{\pi/2}\frac{3R^3\sin^2 t\cdot\cos^7 t+3R^3\sin^7 t\cdot\cos^2 t}{\left(R\cos^3 t\right)^{5/3}+\left(R\sin^3 t\right)^{5/3}}\,dt=$$

$$=\frac{3R^3}{R^{5/3}}\int_0^{\pi/2}\frac{\sin^2 t\cos^2 t\left(\cos^5 t+\sin^5 t\right)}{\cos^5 t+\sin^5 t}\,dt=$$

$$=\frac{3R^{4/3}}{4}\int_0^{\pi/2}4\sin^2 t\cos^2 t\;dt=\frac{3}{4}R^{4/3}\int_0^{\pi/2}\sin^2 2t\;dt=$$

$$=\frac{3}{8}R^{4/3}\int_0^{\pi/2}\left(1-\cos 4t\right)\,dt=$$

$$=\frac{3}{8}R^{4/3}\left(t-\frac{1}{4}\sin 4t\right)\Bigg|_0^{\pi/2}=\frac{3}{8}R^{4/3}\cdot\frac{\pi}{2}=\frac{3}{16}\pi R\sqrt[3]{R}\,.$$

Task 6. Calculate $I=\int\limits_L x\,dx+y\,dy+\left(x+y-1\right)dz$, where L is a segment of a line from the point $\left(1;1;1\right)$ to the point $\left(2;3;4\right)$.

Let us work out an equation of the line that crosses the given points $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$; we express $y=2x-1$, then $dy=2\,dx$ and $z=3x-2$, $dz=3\,dx$; so according to the formula (8):

$$I=\int_1^2\left(x+\left(2x-1\right)\cdot 2+\left(x+2x-2\right)\cdot 3\right)dx=$$

$$=\int_1^2\left(14x-8\right)dx=\left(7x^2-8x\right)\Big|_1^2=13\,.$$

Task 7. Calculate $I=\int\limits_{OA} xy\,dx+yz\,dy+zx\,dz$, where OA is a quarter of a circle, represented by parametric equations $x=\cos t$, $y=\sin t$, $z=1$; the direction of path tracing is the growth of parameter *t*.

$$I=\int\limits_0^{\pi/2}\left(-\cos t\cdot\sin^2 t+\sin t\cdot\cos t\right)dt=\int\limits_0^{\pi/2}\left(\sin t-\sin^2 t\right)d\left(\sin t\right)=$$

$$=\left(\frac{\sin^2 t}{2}-\frac{\sin^3 t}{3}\right)\Bigg|_0^{\pi/2}=\frac{1}{2}\sin^2\frac{\pi}{2}-\frac{1}{3}\sin^3\frac{\pi}{2}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}.$$

Task 8. Calculate the area of the figure, bounded by cardioid $x=2\cos t-\cos 2t$, $y=2\sin t-\sin 2t$.

We use formula (10) to calculate the area:

$$S=\frac{1}{2}\oint\limits_L x\,dy-y\,dx=$$

$$=\frac{1}{2}\int\limits_0^{2\pi}\Big(\left(2\cos t-\cos 2t\right)\left(2\cos t-2\cos 2t\right)-$$

$$-\left(2\sin t-\sin 2t\right)\left(-2\sin t+2\sin 2t\right)\Big)dt=$$

$$=\frac{1}{2}\cdot 2\int\limits_0^{2\pi}\Big(2\cos^2 t-2\cos t\cdot\cos 2t-\cos t\cdot\cos 2t+\cos^2 2t-$$

$$-\left(-2\sin^2 t+2\sin t\sin 2t+\sin t\sin 2t-\sin^2 2t\right)\Big)dt=$$

$$= \int_0^{2\pi} \left(2\cos^2 t - 3\cos t \cos 2t + \cos^2 2t + 2\sin^2 t - 3\sin t \sin 2t + \sin^2 2t\right)dt =$$

$$= \int_0^{2\pi} \left(3 - 3(\cos t \cdot \cos 2t + \sin t \cdot \sin 2t)\right)dt =$$

$$= 3\int_0^{2\pi} (1 - \cos t)dt = 3(t - \sin t)\Big|_0^{2\pi} = 6\pi .$$

Task 9. In each point of a plane force $\overline{F}$ performs an action on a material point; the projections of $\overline{F}$ on the coordinate axes *Ox* and *Oy* are correspondingly equal to $P(x, y) = xy$ and $Q(x, y) = x + y$. Calculate the action of the force $\overline{F}$ during movement of the point from the origin of coordinates $(0;0)$ to the point $(1;1)$: a) about the line $y = x$; b) about the parabola $y = x^2$.

To calculate the action of the force $\overline{F} = P(x, y) \cdot \bar{i} + Q(x, y) \cdot \bar{j}$, we use formula (11):

a) $$A = \int_{(0;0)}^{(1;1)} xy\,dx + (x + y)dy = \int_0^1 \left(x^2 + 2x\right)dx = \left(\frac{x^3}{3} + x^2\right)\Bigg|_0^1 = \frac{4}{3};$$

b) $$A = \int_{(0;0)}^{(1;1)} xy\,dx + (x + y)dy = \int_0^1 \left(x^3 + \left(x + x^2\right) \cdot 2x\right)dx =$$

$$= \int_0^1 \left(3x^3 + 2x^2\right)dx = \left(\frac{3x^4}{4} + \frac{2x^3}{3}\right)\Bigg|_0^1 = \frac{17}{12}.$$

Task 10. Calculate the action of a force field, in each point $(x; y)$ of which the intensity (force that performs an action on a mass unit) is equal to $\overline{p} = (x + y) \cdot \overline{i} - x \cdot \overline{j}$, when a point with the mass m moves clockwise around the circle $x = a\cos t$, $y = a\sin t$.

Since the point moves clockwise around the circle, i.e. in negative orientation, then formula (11) is represented in the following way:

$$A = \oint_{-L} P(x, y)dx + Q(x, y)dy = \oint_{-L} m(x + y)dx - mx\,dy =$$

$$= \int_0^{-2\pi} \big(m(a\cos t + a\sin t)\cdot(-a\sin t) - ma\cos t \cdot a\cos t\big)dt =$$

$$= -ma^2 \int_0^{-2\pi} (1 + \sin t \cdot \cos t)dt =$$

$$= -ma^2 \int_0^{-2\pi} \left(1 + \frac{1}{2}\sin 2t\right)dt = -ma^2\left(t - \frac{1}{4}\cos 2t\right)\Bigg|_0^{-2\pi} = 2\pi ma^2.$$

Green's formula

Green's formula connects double integral about domain D with curvilinear integral about the limit L of this domain.

Let us assume that D is a certain regular domain, bounded by the closed contour L (Drawing 36), and functions $P(x, y)$ and $Q(x, y)$ are continuous along with their partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in this domain. Then Green's formula is valid:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \oint_L P\,dx + Q\,dy. \qquad (12)$$

Task 1. Calculate $I = \oint\limits_L 2\left(x^2 + y^2\right)dx + (x+y)^2\,dy$ according to the Green's formula, if L is a contour of a triangle *LMN* with vertices $L(1;1)$, $M(2;2)$, $N(1;3)$, the direction of path tracing is counterclockwise. Check the result with the help of direct integration.

We calculate the partial derivatives $\dfrac{\partial Q}{\partial x} = 2(x+y)$; $\dfrac{\partial P}{\partial y} = 4y$.

Integrating domain D is a triangle *LMN*, the equations of its sides are as follows: $y = x$ for *LM*, $y = -x + 4$ for MN, and $x = 1$ for *NL*. According to the formula (12):

$$\oint\limits_L 2\left(x^2 + y^2\right)dx + (x+y)^2\,dy = \iint\limits_D \left(2(x+y) - 4y\right)dxdy =$$

$$= 2\iint\limits_D (x-y)\,dxdy = 2\int\limits_1^2 dx \int\limits_x^{4-x} (x-y)\,dy =$$

$$= 2\int\limits_1^2 \left(xy - \frac{y^2}{2}\right)\Bigg|_x^{4-x} dx = 2\int\limits_1^2 \left(x(4-x) - \frac{1}{2}(4-x)^2 - x^2 + \frac{1}{2}x^2\right)dx =$$

$$= 4\int\limits_1^2 \left(4x - x^2 - 4\right)dx = 4\left(2x^2 - \frac{x^3}{3} - 4x\right)\Bigg|_1^2 = -\frac{4}{3}.$$

We calculate the integral directly – about the contour of the triangle *LMN*, then $I = I_1 + I_2 + I_3$:

$$I_1 = \int\limits_{LM} 2\left(x^2 + x^2\right)dx + (x+x)^2\,dx = 8\int\limits_1^2 x^2\,dx = \frac{8}{3}x^3\Bigg|_1^2 = \frac{56}{3};$$

$$I_2 = \int\limits_{MN} 2\left(x^2 + (4-x)^2\right)dx + (x - x + 4)^2(-dx) =$$

$$= \int\limits_2^1 \left(4x^2 - 16x + 16\right)dx =$$

$$= \left(\frac{4}{3}x^3 - 8x^2 + 16x\right)\Bigg|_2^1 = -\frac{4}{3};$$

$$I_3 = \int\limits_{NL} (1+y)^2\,dy = \int\limits_3^1 (1+y)^2\,dy = \frac{1}{3}(1+y)^3\Bigg|_3^1 = \frac{1}{3}(8 - 64) = -\frac{56}{3};$$

$$I = \frac{56}{3} - \frac{4}{3} - \frac{56}{3} = -\frac{4}{3}.$$

Task 2. With Green's formula, calculate curvilinear integral $\oint\limits_L -x^2y\,dx + xy^2dy$, where L is a circle $x^2 + y^2 = R^2$; the direction of path tracing is counterclockwise.

We calculate $\dfrac{\partial Q}{\partial x} = y^2$, $\dfrac{\partial P}{\partial y} = -x^2$, then according to the formula (12):

$$\oint\limits_L -x^2y\,dx + xy^2dy = \iint\limits_D \left(x^2 + y^2\right)dxdy = \iint\limits_{D^*} \rho^2 \cdot \rho\,d\rho d\varphi =$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_0^R \rho^3 d\rho = \varphi\Bigg|_0^{2\pi} \cdot \frac{\rho^4}{4}\Bigg|_0^R = \frac{\pi R^4}{2}.$$

Task 3. With the Green's formula, calculate the integral $\oint\limits_L \sqrt{x^2+y^2}\,dx + y\left(xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right)dy$, where L is a contour of a rectangle $1 \le x \le 4$, $0 \le y \le 2$.

Since $$\frac{\partial Q}{\partial x} = y\left(y + \frac{1}{x+\sqrt{x^2+y^2}}\cdot\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\right) =$$

$$= y\left(y + \frac{1}{x+\sqrt{x^2+y^2}}\cdot\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}}\right) =$$

$$= y\left(y+\frac{1}{\sqrt{x^2+y^2}}\right) = y^2 + \frac{y}{\sqrt{x^2+y^2}}$$

and $\dfrac{\partial P}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$,

then

$$\oint\limits_L \sqrt{x^2+y^2}\,dx + y\left(xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right)dy =$$

$$= \iint\limits_D \left(y^2 + \frac{y}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\right)dxdy =$$

$$= \iint\limits_D y^2 dxdy = \int\limits_1^4 dx \int\limits_0^2 y^2 dy = x\Big|_1^4 \cdot \frac{y^3}{3}\Big|_0^2 = 3\cdot\frac{8}{3} = 8.$$

Conditions of independence of curvilinear integral from the shape of integration path

The value of a curvilinear integral, as demonstrated, depends on the curve that connects extreme points of integration path. However, there are integrals, the value of which does not depend on the shape of integration path. Let us observe the conditions, under which the value of integral remains constant about all the possible curves that connect extreme points of integration curve.

Theorem. Functions $P(x,y)$ and $Q(x,y)$ are determined and continuous along with their partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in a certain closed singly connected domain *D*. Then the following four conditions are equivalent, i.e. the fulfillment of one of them implies the other three:

1) for an arbitrary closed smooth curve in domain *D*,

$$\oint_L P\,dx + Q\,dy = 0;$$

2) for two arbitrary points *M* and *N* of domain *D* the value of the integral

$$\int_{MN} P\,dx + Q\,dy$$

does not depend on the shape of integration path, that lays in domain *D*;

3) expression $P\,dx + Q\,dy$ is an ordinary differential of a certain function, determined in domain *D*, i.e. there is a certain function $F(x,y)$, determined in domain *D*, that $dF = P\,dx + Q\,dy$;

4) in all points of domain *D* the following equality is valid

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{13}$$

It follows from the equivalence of conditions 1–4 of the theorem, that conditions 3 and 4 are necessary and sufficient, under which curvilinear integral does not depend on the shape of integration path. However, equality (13) is the most suitable necessary and sufficient condition.

The theorem is valid in the case of curvilinear integral of second kind lengthwise spatial curves.

Task. Calculate curvilinear integral $\int\limits_{AB}\left(x^2-y^2\right)dx-2xy\,dy$ about the arbitrary curve that connects points $A(0;0)$ and $B(1;1)$.

Since equality (13) is valid for the given integral:

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=-2y,$$

then the value of integral does not depend on the shape of integration path, but only on the points *A* and *B*. We calculate integral about the line $y=x$ that connects these points, then

$$\int\limits_{AB}\left(x^2-y^2\right)dx-2xy\,dy=\int\limits_0^1\left(\left(x^2-x^2\right)-2x^2\right)dx=$$

$$=-2\int\limits_0^1 x^2\,dx=-\frac{2}{3}x^3\Big|_0^1=-\frac{2}{3}.$$

Integration of ordinary differential

In a certain singly connected domain *D* functions $P(x,y)$ and $Q(x,y)$, and their partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous, and besides $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. We choose point $M(x_0; y_0)$ and consider function

$$F(x, y) = \int_{(x_0; y_0)}^{(x; y)} P\,dx + Q\,dy\,,$$

then the ordinary differential of this function is as follows:

$$dF = P\,dx + Q\,dy\,. \tag{14}$$

However, there is a limitless quantity of functions of two variables, for which (14) is an ordinary differential. All the functions of this kind are determined by the formula $u(x, y) + C$, where $u(x, y)$ is any of the antiderivatives, and C – const.

Each of these functions is called *antiderivative* of an ordinary differential (14):

$$\int_{(x_0; y_0)}^{(x; y)} P\,dx + Q\,dy = u(x, y) + C\,.$$

We assume that $x = x_0$, $y = y_0$, and calculate $C = u(x_0; y_0)$, so

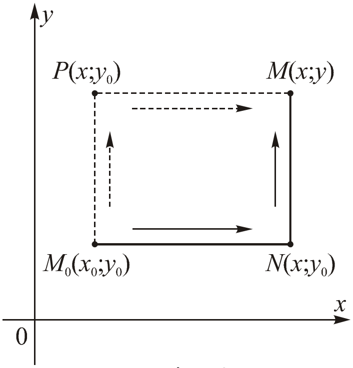
$$\int_{(x_0; y_0)}^{(x; y)} P\,dx + Q\,dy = u(x, y) - u(x_0, y_0)\,.$$

In particular, if $x = x_1$, $y = y_1$, then

$$\int_{(x_0; y_0)}^{(x_1; y_1)} P\,dx + Q\,dy = u(x_1, y_1) - u(x_0, y_0)\,. \tag{15}$$

Expression (15) is called *Newton-Leibniz formula for curvilinear integral of ordinary differential.*

Since curvilinear integral does not depend on the shape of integrating path, then to calculate antiderivative $u(x, y)$ it is sufficient to calculate integral about an arbitrary line that connects points M_0 and M. However, the most suitable way is to integrate about the polygonal line that connects points M_0 and M in such a way that the sides of the polygonal line are parallel to the coordinate axes (Drawing 37).



Drawing 37

Then on the segment M_0N $y = y_0$, $dy = 0$, and on the segment NM $x = \text{const}$, $dx = 0$, so

$$u(x, y) = \int_{x_0}^{x} P(x, y_0)dx + \int_{y_0}^{y} Q(x, y)dy + C, \qquad (16)$$

where the first determined integral is calculated under the constant value $y = y_0$, and the second is calculated under the constant value x.

By analogy while integrating about polygonal line M_0PM (Drawing 37):

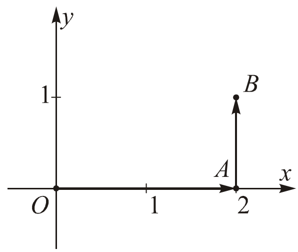
$$u(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy + C. \qquad (17)$$

Formulas (16) and (17) are applied to calculate antiderivative; the starting point $(x_0; y_0)$ should be chosen in such a way that expressions being integrated are the most simplified.

Task 1. Calculate the integral

$$I = \int_{(0;0)}^{(2;1)} \left(x^2 + 2xy - y^2\right)dx + \left(x^2 - 2xy + y^2\right)dy .$$

Since equality (13) is valid for the given integral, i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x - 2y$, then it does not depend on the integration path. We integrate about the polygonal line *OAB* (Drawing 38).



Drawing 38

On the segment *OA*:

$$y = 0,\ dy = 0,\ 0 \le x \le 2;$$

on the segment *AB*: $x = 2$, $dx = 0$, $0 \le y \le 1$, then:

$$I = \int_0^2 x^2 dx + \int_0^1 \left(4 - 4y + y^2\right)dy = \left.\frac{x^3}{3}\right|_0^2 + \left.\left(4y - 2y^2 + \frac{y^3}{3}\right)\right|_0^1 =$$

$$= \frac{8}{3} + 4 - 2 + \frac{1}{3} = 5 .$$

Task 2. Calculate $I = \int_{(1;1)}^{(2;3)} (x + 3y)dx + (y + 3x)dy$.

The given integral does not depend on the integration path, because $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 3$. We integrate about the polygonal line, the segments of which are parallel to the coordinate axes; on the first segment $y = 1$, $dy = 0$, $1 \le x \le 2$; on the second segment $x = 2$, $dx = 0$, $1 \le y \le 3$, so

$$I=\int_1^2 (x+3)dx+\int_1^3 (y+6)dy=\left(\frac{x^2}{2}+3x\right)\Bigg|_1^2+\left(\frac{y^2}{2}+6y\right)\Bigg|_1^3=$$

$$=2+6-\frac{1}{2}-3+\frac{9}{2}+18-\frac{1}{2}-6=20\frac{1}{2}.$$

Task 3. Calculate antiderivative function $u(x,y)$, if its ordinary differential is as follows:

a) $du=(4x+2y)dx+(2x-6y)dy$;

b) $du=(y+\ln(x+1))dx+(x+1-e^y)dy$;

c) $du=\left(\frac{1}{x}+\frac{1}{y}\right)dx+\left(\frac{2}{y}-\frac{x}{y^2}\right)dy$.

a) Condition (13) is fulfilled: $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=2$; we take $x_0=0$, $y_0=0$, then

$$u(x,y)=\int_0^x 4x\,dx+\int_0^y (2x-6y)dy+C=2x^2+2xy-3y^2+C$$

or

$$u(x,y)=\int_0^y -6y\,dy+\int_0^x (4x+2y)dx+C=-3y^2+2x^2+2xy+C.$$

b) Since $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=1$, then let us assume that $x_0=0$, $y_0=0$, and antiderivative function is equal to:

$$u(x,y)=\int_0^x \ln(x+1)dx+\int_0^y \left(x+1-e^y\right)dy=$$

$$=\left(x\cdot\ln(x+1)-x+\ln(x+1)\right)\Big|_0^x+\left(xy+y-e^y\right)\Big|_0^y=$$

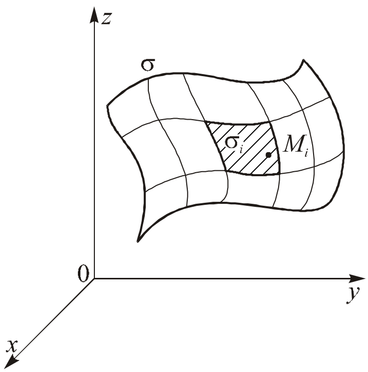
$$=(x+1)\ln(x+1)-x+xy+y-e^y+1+C\,.$$

c) We calculate $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=-\frac{1}{y^2}$. We cannot take the origin of coordinates $(0;0)$ as the starting point, because functions $P(x,y)$ and $Q(x,y)$ are not determined in this point, and that is why let us take point $(1;1)$ as the starting point, then antiderivative is equal to:

$$u(x,y)=\int_1^x\left(\frac{1}{x}+1\right)dx+\int_1^y\left(\frac{2}{y}-\frac{x}{y^2}\right)dy=\ln x+2\ln y+\frac{x}{y}-1+C\,.$$

Lecture 7. SURFACE INTEGRAL. OSTROGRADSKY-GAUSS FORMULA. STOKES FORMULA

Let us assume that a closed function $f(M)=f(x,y,z)$ is determined in the points of a certain smooth surface σ. We divide surface σ into n arbitrary parts σ_i without common inner points (Drawing 39). $\Delta\sigma_i$ is an area, and $\text{diam}(\sigma_i)$ is a diameter of a part of surface σ_i, $i=1, 2,\dots, n$. In each part σ_i we choose an arbitrary point $M_i(\xi_i;\eta_i;\mu_i)$ and calculate the sum



Drawing 39

$$\sum_{i=1}^{n} f(\xi_i;\eta_i;\mu_i)\cdot\Delta\sigma_i\,, \qquad (1)$$

which we will call *the integral sum for function* $f(x,y,z)$ *about surface* σ.

If there is a finite limit of the integral sum (1) when $\lambda=\max\limits_{1\le i\le n}\text{diam}(\sigma_i)\to 0$, that does not depend neither on the way we divide surface σ into parts σ_i, nor on the choice of points M_i, then this limit is called *the surface integral of the first kind:*

$$\iint\limits_{\sigma} f(x,y,z)d\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i;\eta_i;\mu_i)\cdot\Delta\sigma_i\ . \qquad (2)$$

Function $f(x,y,z)$ is called integrable about the surface σ, and surface σ is called the integrating domain. If function $f(x,y,z)$ is continuous on the surface σ, then it is integrable about σ.

Smooth surface σ, represented by equation $z = z(x,y)$ is projected on the plane *Oxy* into the domain *D*. Let us assume that function $f(x,y,z)$ is continuous on the surface σ, and functions $z(x,y)$, $z'_x(x,y)$, $z'_y(x,y)$ are continuous in domain *D*, then the surface integral (2) is calculated with the formula:

$$\iint\limits_{\sigma} f(x,y,z)d\sigma = \iint\limits_{D_{xy}} f(x,y,z(x,y))\cdot\sqrt{1+(z'_x)^2+(z'_y)^2}\,dxdy\ . \qquad (3)$$

Formula (3) expresses the surface integral of the first kind through the double integral about the projection of surface σ on the plane *Oxy*.

By analogy, if surface σ is represented by equation $y = y(x,z)$ or $x = x(y,z)$, then

$$\iint\limits_{\sigma} f(x,y,z)d\sigma = \iint\limits_{D_{xz}} f(x,y(x,z),z)\cdot\sqrt{1+(y'_x)^2+(y'_z)^2}\,dxdz\ ,$$

or

$$\iint\limits_{\sigma} f(x,y,z)d\sigma = \iint\limits_{D_{yz}} f(x(y,z),y,z)\cdot\sqrt{1+(x'_y)^2+(x'_z)^2}\,dydz\ ,$$

where D_{xz}, D_{yz} are respectively the projections of surface σ on the coordinate planes *Oxz* and *Oyz*.

Let us assume that in formula (2) $f(x, y, z) = 1$ on the surface σ, then integral $P = \iint\limits_{\sigma} d\sigma$ (4) expresses *the area of surface σ.*

If the mass with surface density $\gamma = \gamma(x, y, z)$ is spread on the smooth surface σ, then:

a) *mass of material surface*

$$m = \iint\limits_{\sigma} \gamma(x, y, z) d\sigma ; \tag{5}$$

b) *coordinates of center of mass of the surface:*

$$x_C = \frac{M_{yz}}{m}, \quad y_C = \frac{M_{xz}}{m}, \quad z_C = \frac{M_{xy}}{m}, \tag{6}$$

where

$$M_{yz} = \iint\limits_{\sigma} x\gamma \, d\sigma , \; M_{xz} = \iint\limits_{\sigma} y\gamma \, d\sigma , \; M_{xy} = \iint\limits_{\sigma} z\gamma \, d\sigma \tag{7}$$

are respectively *the static moments of surface σ* about coordinate planes *Oyz*, *Oxz*, *Oxy*;

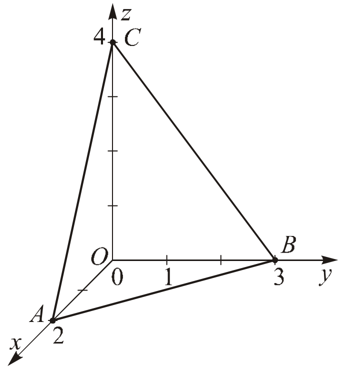
c) *moments of inertia of surface* about coordinate axes and the origin of coordinates:

$$I_x = \iint\limits_{\sigma} \left(y^2 + z^2\right)\gamma \, d\sigma , \; I_y = \iint\limits_{\sigma} \left(x^2 + z^2\right)\gamma \, d\sigma ,$$

$$I_z = \iint\limits_{\sigma} \left(x^2 + y^2\right)\gamma \, d\sigma , \; I_O = \iint\limits_{\sigma} \left(x^2 + y^2 + z^2\right)\gamma \, d\sigma . \tag{8}$$

Task 1. Calculate $\iint\limits_{\sigma} \left(z + 2x + \frac{4}{3} y \right) d\sigma$, σ is a part of the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$, set in the first octant.

ΔABC (Drawing 40) is the integrating surface, the projection of which on the plane Oxy is ΔAOB: $0 \le y \le 3\left(1-\dfrac{x}{2}\right),\ 0 \le x \le 2$. Let us calculate $d\sigma$:



Drawing 40

$$z = 4\left(1-\frac{x}{2}-\frac{y}{3}\right) \quad \Rightarrow \quad z'_x = -2,$$

$$z'_y = -\frac{4}{3};$$

$$d\sigma = \sqrt{1+4+\frac{16}{9}}\,dxdy = \frac{\sqrt{61}}{3}\,dxdy\,.$$

We apply formula (3):

$$\iint\limits_{\sigma}\left(z+2x+\frac{4}{3}y\right)d\sigma = \frac{\sqrt{61}}{3}\iint\limits_{D_{xy}}\left(4-2x-\frac{4}{3}y+2x+\frac{4}{3}y\right)dxdy =$$

$$= \frac{4\sqrt{61}}{3}\iint\limits_{D_{xy}}dxdy = \frac{4\sqrt{61}}{3}\int_0^2 dx\int_0^{3\left(1-\frac{x}{2}\right)}dy = \frac{4\sqrt{61}}{3}\int_0^2 3\left(1-\frac{x}{2}\right)dx =$$

$$= 4\sqrt{61}\left(x-\frac{x^2}{4}\right)\Bigg|_0^2 = 4\sqrt{61}(2-1) = 4\sqrt{61}\,.$$

Task 2. Calculate $\iint\limits_{\sigma} xyz\,d\sigma$, where σ is a part of the plane $x+y+z=1$, set in the first octant.

Equation of the surface is $z=1-x-y$, we express $z'_x=z'_y=-1$, then $d\sigma=\sqrt{1+1+1}\,dxdy=\sqrt{3}\,dxdy$, consequently

$$\iint\limits_{\sigma} xyz\,d\sigma=\sqrt{3}\iint\limits_{D_{xy}} xy(1-x-y)\,dxdy=$$

$$=\sqrt{3}\int\limits_0^1 x\,dx\int\limits_0^{1-x}\left((1-x)y-y^2\right)dy=$$

$$=\sqrt{3}\int\limits_0^1 x\,dx\left((1-x)\cdot\frac{y^2}{2}-\frac{y^3}{3}\right)\Bigg|_0^{1-x}=$$

$$=\sqrt{3}\int\limits_0^1 x\left(\frac{1}{2}(1-x)^3-\frac{1}{3}(1-x)^3\right)dx=\frac{\sqrt{3}}{6}\int\limits_0^1 x(1-x)^3\,dx=$$

$$=\frac{\sqrt{3}}{6}\int\limits_0^1\left(x-3x^2+3x^3-x^4\right)dx=\frac{\sqrt{3}}{6}\left(\frac{x^2}{2}-x^3+\frac{3x^4}{4}-\frac{x^5}{5}\right)\Bigg|_0^1=$$

$$=\frac{\sqrt{3}}{6}\left(\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}\right)=\frac{\sqrt{3}}{6}\cdot\frac{1}{20}=\frac{\sqrt{3}}{120}.$$

Task 3. Calculate $\iint\limits_{\sigma}\left(x^2+y^2\right)d\sigma$, where σ is a part of conic surface $z^2=x^2+y^2$, set between the planes $z=0$ and $z=1$.

Since the equation of the surface is $z = \sqrt{x^2 + y^2}$, then

$$z'_x = \frac{x}{\sqrt{x^2 + y^2}},\ z'_y = \frac{y}{\sqrt{x^2 + y^2}},$$

and $d\sigma = \sqrt{1 + \dfrac{x^2}{x^2 + y^2} + \dfrac{y^2}{x^2 + y^2}}\ dxdy = \sqrt{2}\ dxdy$, so the surface integral is equal to:

$$\iint_{\sigma} \left(x^2 + y^2\right) d\sigma = \iint_{D_{xy}} \left(x^2 + y^2\right)\sqrt{2}\ dxdy = \sqrt{2} \iint_{D^*_{xy}} \rho^3\, d\rho d\varphi = \sqrt{2} \int_0^{2\pi} d\varphi \int_0^1 \rho^3\, d\rho =$$

$$= \sqrt{2} \cdot \varphi\Big|_0^{2\pi} \cdot \frac{\rho^4}{4}\Big|_0^1 = \frac{\sqrt{2}}{2}\pi .$$

Task 4. Calculate $\iint_{\sigma} x\ d\sigma$, where σ is a part of the sphere $x^2 + y^2 + z^2 = R^2$, set in the first octant.

Out of the equation of the sphere $z = \sqrt{R^2 - x^2 - y^2}$ we calculate $z'_x = \dfrac{-x}{\sqrt{R^2 - x^2 - y^2}}$, $z'_y = \dfrac{-y}{\sqrt{R^2 - x^2 - y^2}}$, then

$$d\sigma = \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\ dxdy = \frac{R\, dxdy}{\sqrt{R^2 - x^2 - y^2}};$$

consequently

$$\iint_{\sigma} x\ d\sigma = R \iint_{D_{xy}} \frac{x\, dxdy}{\sqrt{R^2 - x^2 - y^2}} = R \iint_{D^*_{xy}} \frac{\rho\cos\varphi \cdot \rho\, d\rho d\varphi}{\sqrt{R^2 - \rho^2}} =$$

$$= R\int_0^{\pi/2}\cos\varphi\, d\varphi\int_0^R \frac{\rho^2\, d\rho}{\sqrt{R^2-\rho^2}} =$$

$$= \begin{bmatrix} \rho = R\sin z, & z = \arcsin\dfrac{\rho}{R}, \\ d\rho = R\cos z\, dz, & z_1 = 0, \quad z_2 = \pi/2 \end{bmatrix} =$$

$$= R\cdot\sin\varphi\Big|_0^{\pi/2}\cdot\int_0^{\pi/2}\frac{R^2\sin^2 z\cdot R\cos z\, dz}{\sqrt{R^2-R^2\sin^2 z}} =$$

$$= R\cdot R^2\int_0^{\pi/2}\sin^2 z\; dz = \frac{R^3}{2}\int_0^{\pi/2}(1-\cos 2z)\; dz =$$

$$= \frac{R^3}{2}\left(z-\frac{1}{2}\sin 2z\right)\Bigg|_0^{\pi/2} = \frac{R^3}{2}\cdot\frac{\pi}{2} = \frac{\pi R^3}{4}.$$

Task 5. Calculate the moment of inertia of hemisphere $z=\sqrt{a^2-x^2-y^2}$ about axis Oz.

According to the formulas (8):

$$I_z = \iint_\sigma \left(x^2+y^2\right) d\sigma =$$

$$= \iint_{D_{xy}} \left(x^2+y^2\right)\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\; dxdy =$$

$$= a\iint\limits_{D_{xy}} \frac{\left(x^2+y^2\right)}{\sqrt{a^2-x^2-y^2}}\,dxdy = a\iint\limits_{D^*_{xy}} \frac{\rho^3\,d\rho d\varphi}{\sqrt{a^2-\rho^2}} = a\int_0^{2\pi} d\varphi \int_0^a \frac{\rho^3\,d\rho}{\sqrt{a^2-\rho^2}} =$$

$$= \left[\begin{array}{ll} \sqrt{a^2-\rho^2} = t, & d\rho = \dfrac{-t}{\sqrt{a^2-t^2}}\,dt, \\ \rho = \sqrt{a^2-t^2}, & t_1 = a, \quad t_2 = 0 \end{array}\right] =$$

$$= 2\pi a \int_a^0 \frac{\left(a^2-t^2\right)\sqrt{a^2-t^2}\cdot(-t)\,dt}{t\cdot\sqrt{a^2-t^2}} =$$

$$= 2\pi a \int_0^a \left(a^2-t^2\right)dt = 2\pi a\left(a^2 t - \frac{t^3}{3}\right)\Bigg|_0^a = 2\pi a\left(a^3-\frac{a^3}{3}\right) = \frac{4}{3}\pi a^4 .$$

Task 6. Calculate coordinates of center of mass of a part of the plane $z = x$, bounded by the planes $x + y = 1$, $y = 0$, $x = 0$.

We apply formulas (6) and (7):

$$z = x \quad \Rightarrow \quad z'_x = 1, \quad z'_y = 0, \quad d\sigma = \sqrt{2}\,dxdy,$$

then

$$m = \iint\limits_{\sigma} d\sigma = \iint\limits_{D_{xy}} \sqrt{2}\,dxdy = \sqrt{2}\int_0^1 dx \int_0^{1-x} dy = \sqrt{2}\int_0^1 (1-x)dx =$$

$$= \sqrt{2}\left(x - \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{\sqrt{2}}{2};$$

$$M_{yz} = \iint\limits_{\sigma} x\,d\sigma = \sqrt{2}\int_0^1 x\,dx \int_0^{1-x} dy = \sqrt{2}\int_0^1 x(1-x)dx =$$

$$=\sqrt{2}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^1=\frac{\sqrt{2}}{6};$$

$$M_{xz}=\iint_\sigma y\,d\sigma=\sqrt{2}\int_0^1 dx\int_0^{1-x} y\,dy=\sqrt{2}\int_0^1 dx\cdot\frac{y^2}{2}\Bigg|_0^{1-x}=$$

$$=\frac{\sqrt{2}}{2}\int_0^1\left(1-2x+x^2\right)dx=\frac{\sqrt{2}}{2}\left(x-x^2+\frac{x^3}{3}\right)\Bigg|_0^1=\frac{\sqrt{2}}{6};$$

$$M_{xy}=\iint_\sigma z\,d\sigma=\sqrt{2}\iint_\sigma x\,d\sigma=\frac{\sqrt{2}}{6}.$$

As a result, $x_C=y_C=z_C=\dfrac{1}{3}$.

Task 7. Calculate center of mass of homogeneous surface of the cone $a^2z^2=b^2\left(x^2+y^2\right)$, $0\le z\le b$.

Cone $z=\dfrac{b}{a}\sqrt{x^2+y^2}$ is projected on the plane *Oxy* in the circle with radius a:

$$\begin{cases} z=\dfrac{b}{a}\sqrt{x^2+y^2}, \\ z=b \end{cases}\Rightarrow\quad\frac{b}{a}\sqrt{x^2+y^2}=b\quad\Rightarrow\quad\sqrt{x^2+y^2}=a;$$

$$z'_x=\frac{b}{a}\cdot\frac{x}{\sqrt{x^2+y^2}},\quad z'_y=\frac{b}{a}\cdot\frac{y}{\sqrt{x^2+y^2}},$$

then

$$d\sigma = \sqrt{1 + \frac{b^2}{a^2} \cdot \frac{x^2}{x^2 + y^2} + \frac{b^2}{a^2} \cdot \frac{y^2}{x^2 + y^2}}\; dxdy =$$

$$= \frac{1}{a}\sqrt{\frac{\left(a^2 + b^2\right)\left(x^2 + y^2\right)}{x^2 + y^2}}\; dxdy = \frac{\sqrt{a^2 + b^2}}{a}\; dxdy\;;$$

$$m = \iint\limits_{\sigma} d\sigma = \iint\limits_{D_{xy}} \frac{\sqrt{a^2 + b^2}}{a}\, dxdy = \frac{\sqrt{a^2 + b^2}}{a} \iint\limits_{D_{xy}^*} \rho\, d\rho d\varphi =$$

$$= \frac{\sqrt{a^2 + b^2}}{a} \int_0^{2\pi} d\varphi \int_0^a \rho\, d\rho =$$

$$= \frac{\sqrt{a^2 + b^2}}{a} \cdot \varphi\Big|_0^{2\pi} \cdot \frac{\rho^2}{2}\Big|_0^a = \frac{\sqrt{a^2 + b^2}}{a} \cdot 2\pi \cdot \frac{a^2}{2} = \pi a \sqrt{a^2 + b^2}\;;$$

$$M_{yz} = \iint\limits_{\sigma} x\, d\sigma = \frac{\sqrt{a^2 + b^2}}{a} \iint\limits_{D_{xy}^*} \rho^2 \cos\varphi\; d\rho d\varphi =$$

$$= \frac{\sqrt{a^2 + b^2}}{a} \int_0^{2\pi} \cos\varphi\, d\varphi \int_0^a \rho^2\, d\rho = 0\,;$$

$$M_{xz} = \iint\limits_{\sigma} y\, d\sigma = \frac{\sqrt{a^2 + b^2}}{a} \iint\limits_{D_{xy}^*} \rho^2 \sin\varphi\, d\rho d\varphi =$$

$$= \frac{\sqrt{a^2 + b^2}}{a} \int_0^{2\pi} \sin\varphi\, d\varphi \int_0^a \rho^2\, d\rho = 0\,;$$

$$M_{xy} = \iint_{\sigma} z\,d\sigma = \frac{\sqrt{a^2+b^2}}{a} \iint_{D_{xy}} \frac{b}{a}\sqrt{x^2+y^2}\,dxdy =$$

$$= \frac{b\sqrt{a^2+b^2}}{a^2} \iint_{D_{xy}^*} \rho^2\,d\rho d\varphi =$$

$$= \frac{b\sqrt{a^2+b^2}}{a^2} \int_0^{2\pi} d\varphi \int_0^{a} \rho^2\,d\rho = \frac{b\sqrt{a^2+b^2}}{a^2} \cdot 2\pi \cdot \frac{a^3}{3} = \frac{2\pi}{3} \cdot ab\sqrt{a^2+b^2};$$

$$x_C = y_C = 0; \quad z_C = \frac{2}{3}b.$$

Task 8. Calculate the moment of inertia of a part of a side surface of the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le h$ about axis *Oz*.

According to the task $z = \sqrt{x^2+y^2}$, then $z'_x = \dfrac{x}{\sqrt{x^2+y^2}}$,

$z'_y = \dfrac{y}{\sqrt{x^2+y^2}}$, so

$$d\sigma = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\,dxdy = \sqrt{2}\,dxdy.$$

The required moment of inertia is equal to:

$$I_z = \iint_{\sigma} \left(x^2+y^2\right) d\sigma = \sqrt{2} \iint_{D_{xy}} \left(x^2+y^2\right) dxdy = \sqrt{2} \iint_{D_{xy}^*} \rho^3\,d\rho d\varphi =$$

$$= \sqrt{2} \int_0^{2\pi} d\varphi \int_0^{h} \rho^3\,d\rho = = \sqrt{2} \cdot \varphi\Big|_0^{2\pi} \cdot \frac{\rho^4}{4}\Big|_0^{h} = \sqrt{2} \cdot 2\pi \cdot \frac{h^4}{4} = \frac{\sqrt{2}}{2}\pi h^4.$$

Task 9. Calculate the area of a part of the surface of the cone $z^2 = 2xy$, set in the first octant between the planes $x = 2$, $y = 4$.

We apply formula (4):

$$P = \iint_{\sigma} d\sigma = \iint_{D_{xy}} \sqrt{1 + (z'_x)^2 + (z'_y)^2}\, dxdy =$$

$$= \iint_{D_{xy}} \sqrt{1 + \frac{y^2}{2xy} + \frac{x^2}{2xy}}\, dxdy = \iint_{D_{xy}} \frac{x+y}{\sqrt{2xy}}\, dxdy =$$

$$= \frac{1}{\sqrt{2}} \int_0^2 dx \int_0^4 \left(\frac{x}{\sqrt{xy}} + \frac{y}{\sqrt{xy}} \right) dy = \frac{1}{\sqrt{2}} \int_0^2 dx \int_0^4 \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) dy =$$

$$= \frac{1}{\sqrt{2}} \int_0^2 dx \cdot \left(\sqrt{x} \cdot 2\sqrt{y} + \frac{1}{\sqrt{x}} \cdot \frac{y^{3/2}}{3/2} \right) \Bigg|_0^4 =$$

$$= \frac{1}{\sqrt{2}} \int_0^2 \left(4\sqrt{x} + \frac{16}{3\sqrt{x}} \right) dx = \frac{1}{\sqrt{2}} \left(4 \cdot \frac{x^{3/2}}{3/2} + \frac{16}{3} \cdot 2\sqrt{x} \right) \Bigg|_0^2 =$$

$$= \frac{1}{\sqrt{2}} \left(\frac{8}{3} x^{3/2} + \frac{32}{3} \sqrt{x} \right) \Bigg|_0^2 = \frac{1}{\sqrt{2}} \left(\frac{16}{3}\sqrt{2} + \frac{32}{3}\sqrt{2} \right) = \frac{48}{3} = 16 \text{ sq.un.}$$

Task 10. Calculate the mass of a hemisphere, if in each its point the surface density is equal to the distance from the point to the radius, perpendicular to the base of a hemisphere.

The equation of the hemisphere is $z = \sqrt{R^2 - x^2 - y^2}$, we express $z'_x = \dfrac{-x}{\sqrt{R^2 - x^2 - y^2}}$, $z'_y = \dfrac{-y}{\sqrt{R^2 - x^2 - y^2}}$, then

$$d\sigma = \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\, dxdy = \frac{R\, dxdy}{\sqrt{R^2 - x^2 - y^2}}.$$

We calculate the mass with the formula (5):

$m = \iint\limits_{\sigma} \gamma\, d\sigma$, where density is $\gamma = \sqrt{x^2 + y^2}$, so

$$m = \iint\limits_{\sigma} \sqrt{x^2 + y^2}\, d\sigma = R \iint\limits_{D_{xy}} \frac{\sqrt{x^2 + y^2}\, dxdy}{\sqrt{R^2 - x^2 - y^2}} = R \iint\limits_{D_{xy}^*} \frac{\rho^2\, d\rho d\varphi}{\sqrt{R^2 - \rho^2}} =$$

$$= R \int_0^{2\pi} d\varphi \int_0^R \frac{\rho^2\, d\rho}{\sqrt{R^2 - \rho^2}} = \frac{\pi^2 R^3}{2},$$

then

$$\int_0^R \frac{\rho^2\, d\rho}{\sqrt{R^2 - \rho^2}} = \left[\begin{array}{ll} \rho = R \sin t, & t = \arcsin \dfrac{\rho}{R}, \\ d\rho = R \cos t\, dt, & t_1 = 0, \quad t_2 = \dfrac{\pi}{2} \end{array} \right] =$$

$$= \int_0^{\pi/2} \frac{R^2 \sin^2 t \cdot R \cos t\; dt}{\sqrt{R^2 - R^2 \sin^2 t}} =$$

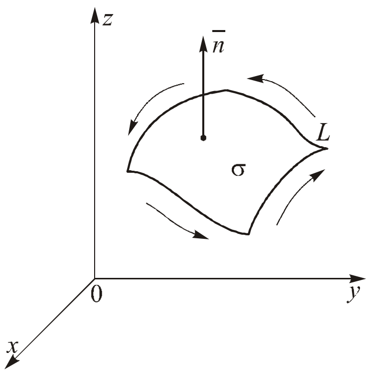
$$= R^2 \int_0^{\pi/2} \frac{1 - \cos 2t}{2}\, dt = \frac{R^2}{2} \left(t - \frac{1}{2} \sin 2t \right) \Bigg|_0^{\pi/2} = \frac{\pi R^2}{4}.$$

Surface integrals of the second kind

Let us consider the notion of the side of surface. On a smooth surface σ we choose an arbitrary point M, draw in it normal $\overline{n}$ with a certain direction, and consider on the surface σ an arbitrary closed contour that leaves point M and comes back to the point M without crossing the limits of surface σ. If we come back to the arbitrary point M of surface σ with the initial direction of normal $\overline{n}$, after path tracing of arbitrary closed contour on the surface σ, that does not cross its limit, then this surface is called *bilateral*; if the direction of a normal reverses, then a surface is called *unilateral*.

Bilateral surface is called *oriented*, and the choice of its certain side is called *the orientation of a surface*. Later on we will consider bilateral surfaces.

Let us assume that σ is an oriented surface, bounded by contour L that does not have any points of self-intersection. The *positive* direction of path tracing of contour L is that one when the direction of a normal coincides with the direction from the feet to the head of the observer in movement, who leaves the surface back on the left side (Drawing 41). The opposite direction of path tracing is called *negative*. If the orientation of the surface is changed, then the directions of path tracing of contour L reverse as well.



Drawing 41

Let us assume that σ is a smooth surface, represented by equation $z = f(x, y)$, and $R(x, y, z)$ is a bounded function, determined in the points of surface σ. We divide σ into n arbitrary parts. We denote D_i as a projection of a part i of surface σ_i on the plane Oxy, and ΔS_i

is an area of D_i with positive sign «+» if the outer side of surface σ is chosen, and negative sign «–», if the inner side is chosen.

In every part σ_i we choose an arbitrary point $M_i(\xi_i;\eta_i;\mu_i)$ and calculate the sum

$$\sum_{i=1}^{n} R(\xi_i;\eta_i;\mu_i)\Delta S_i\ , \tag{9}$$

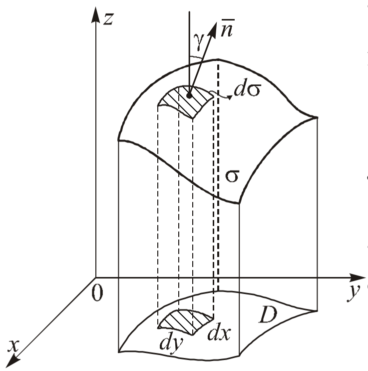
which we will call the integral sum. Let us assume that $\lambda = \max\limits_{1\le i\le n} d(\sigma_i)$ is the maximum diameter of surfaces σ_i.

If $\lambda \to 0$, and there is the limit of integral sums (9) that does not depend neither on the way we divide surface σ, nor on the choice of points M_i, then this limit is called *the surface integral of the second kind.*

$$\iint\limits_{\sigma} R(x,y,z)dxdy = \lim_{\lambda\to 0}\sum_{i=1}^{n} R(\xi_i;\eta_i;\mu_i)\ \Delta S_i\ . \tag{10}$$

Integral reverses sign when a side of a surface is reversed, because ΔS_i reverses sign.

If surface σ is projected on the coordinate planes *Oxz* and *Oyz*, then we get two more surface integrals $\iint\limits_{\sigma} P(x,y,z)\ dxdz$ and $\iint\limits_{\sigma} Q(x,y,z)\ dydz$, where $P(x,y,z)$ and $Q(x,y,z)$ are the functions, determined in the points of surface σ.



Drawing 42

Since $dxdy = \cos\gamma\ d\sigma$, $dxdz = \cos\beta\ d\sigma$, $dydz = \cos\alpha\ d\sigma$, (Drawing 42) where $d\sigma$ is an element of the area of surface σ; α, β, γ are the angles between a normal to surface σ and the axes Ox, Oy, Oz respectively, then the following formulas are valid:

$$\iint_{\sigma} P(x,y,z)\,dydz = \iint_{\sigma} P(x,y,z)\cos\alpha\,d\sigma,$$

$$\iint_{\sigma} Q(x,y,z)\,dxdz = \iint_{\sigma} Q(x,y,z)\cos\beta\,d\sigma,$$

$$\iint_{\sigma} R(x,y,z)\,dxdy = \iint_{\sigma} R(x,y,z)\cos\gamma\,d\sigma.$$

The following formula is also frequently applied in practice:

$$\iint_{\sigma} P\,dydz + Q\,dxdz + R\,dxdy =$$

$$= \iint_{\sigma} \left(P\cos\alpha + Q\cos\beta + R\cos\gamma\right) d\sigma. \qquad (11)$$

To calculate general integral (11) surface σ is projected on all the three coordinate planes, then the following formula is applied:

$$\iint_{\sigma} P(x,y,z)\,dydz + Q(x,y,z)\,dxdz + R(x,y,z)\,dxdy =$$

$$= \pm\iint_{D_{yz}} P(x(y,z),y,z)\,dydz \pm \iint_{D_{xz}} Q(x,y(x,z),z)\,dxdz \pm$$

$$\pm\iint_{D_{xy}} R(x,y,z(x,y))\,dxdy, \qquad (12)$$

where D_{yz}, D_{xz}, D_{xy} are respectively the projections of surface σ on coordinate planes Oyz, Oxz, Oxy. Sign «+» in formula (12) is taken when a normal to the surface makes acute angle with axis Ox, Oy, Oz

respectively; sign «–» is taken when angle is obtuse. The correct choice of signs of double integrals in formula (12) can be checked with the help of the following formula:

$$\bar{n}_0 = \pm \frac{\varphi'_x \cdot \bar{i} + \varphi'_y \cdot \bar{j} + \varphi'_z \cdot \bar{k}}{\sqrt{(\varphi'_x)^2 + (\varphi'_y)^2 + (\varphi'_z)^2}},$$

that is applied to calculate unit normal vector to the surface $\varphi(x, y, z) = 0$. Double sign in this formula corresponds to the two sides of surface σ. It follows from formula (11) that a sign of a double integral coincides with a sign of a corresponding direction cosine of normal $\bar{n}$:

$$\cos\alpha = \cos\left(\bar{n} \,\hat{,}\, \overline{Ox}\right) = \bar{n}_0 \cdot \bar{i}; \quad \cos\beta = \cos\left(\bar{n} \,\hat{,}\, \overline{Oy}\right) = \bar{n}_0 \cdot \bar{j};$$

$$\cos\gamma = \cos\left(\bar{n} \,\hat{,}\, \overline{Oz}\right) = \bar{n}_0 \cdot \bar{k}.$$

Task 1. Calculate the surface integral $\iint\limits_{\sigma} \left(y^2 + z^2\right) dydz$, where σ is an outer side of a paraboloid $x = a^2 - y^2 - z^2$, cut off by the plane *Oyz*.

The projection of paraboloid on the plane *Oyz* is a circle $y^2 + z^2 = a^2$ or $\rho = a$, then

$$\iint\limits_{\sigma} \left(y^2 + z^2\right) dydz = \iint\limits_{D_{yz}} \left(y^2 + z^2\right) dydz = \iint\limits_{D^*_{yz}} \rho^3 \, d\rho d\varphi =$$

$$= \int_0^{2\pi} d\varphi \int_0^a \rho^3 \, d\rho = \varphi \Big|_0^{2\pi} \cdot \frac{\rho^4}{4} \Big|_0^a = 2\pi \cdot \frac{a^4}{4} = \frac{\pi a^4}{2}.$$

Task 2. Calculate $I = \iint\limits_{\sigma} x^2 \, dydz + y^2 \, dzdx + z^2 \, dxdy$, where σ is an outer side of a surface of a hemisphere $x^2 + y^2 + z^2 = a^2$ ($z \geq 0$).

Since the equation of the plane *Oxy* is $z = 0$, then $\iint\limits_{\sigma} z^2 \, dxdy = \iint\limits_{D_{xy}} 0 \, dxdy = 0$; the projections of the hemisphere on the planes *Oyz* and *Oxz* are the semicircles with radius *a*, then

$$\iint\limits_{\sigma} x^2 \, dydz = \iint\limits_{D_{yz}} \left(a^2 - y^2 - z^2\right) dydz = \iint\limits_{D_{yz}^*} \left(a^2 - \rho^2\right) \rho \, d\rho d\varphi =$$

$$= \int_0^{\pi} d\varphi \int_0^a \left(a^2 \rho - \rho^3\right) d\rho = \pi \cdot \left(\frac{a^2 \rho^2}{2} - \frac{\rho^4}{4} \right) \Bigg|_0^a = \pi \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{\pi a^4}{4};$$

$$\iint\limits_{\sigma} y^2 \, dzdx = \iint\limits_{D_{zx}} \left(a^2 - x^2 - z^2\right) dzdx = \iint\limits_{D_{zx}^*} \left(a^2 - \rho^2\right) \rho \, d\rho d\varphi = \frac{\pi a^4}{4};$$

consequently $I = \dfrac{\pi a^4}{4} + \dfrac{\pi a^4}{4} + 0 = \dfrac{\pi a^4}{2}$.

Task 3. Calculate $I = \iint\limits_{\sigma} 2 \, dxdy + y \, dxdz - x^2 z \, dydz$, where σ is an outer side of a part of an ellipsoid $4x^2 + y^2 + 4z^2 = 4$, set in the first octant.

We calculate the projections of the part of the ellipsoid on the coordinate planes:

$$D_{xy} = \left\{ 0 \le y \le \sqrt{4-4x^2};\ 0 \le x \le 1 \right\},$$

$$D_{xz} = D^*_{xz} = \left\{ 0 \le \rho \le 1;\ 0 \le \varphi \le \frac{\pi}{2} \right\},$$

$$D_{yz} = \left\{ 0 \le y \le \sqrt{4-4z^2};\ 0 \le z \le 1 \right\},$$

then

$$I_1 = \iint_\sigma 2\ dxdy = 2\int_0^1 dx \int_0^{\sqrt{4-4x^2}} dy = 4\int_0^1 \sqrt{1-x^2}\ dx =$$

$$= \left[\begin{array}{ll} x = \sin t, & t = \arcsin x, \\ dx = \cos t\ dt, & t_1 = 0,\ t_2 = \dfrac{\pi}{2} \end{array} \right] =$$

$$= 4\int_0^{\pi/2} \sqrt{1-\sin^2 t} \cdot \cos t\ dt = 4\int_0^{\pi/2} \cos^2 t\ dt = 4\int_0^{\pi/2} \frac{1+\cos 2t}{2}\ dt =$$

$$= 2\int_0^{\pi/2} (1+\cos 2t)\ dt\ = 2\left(t + \frac{1}{2}\sin 2t \right)\Bigg|_0^{\pi/2} = 2 \cdot \frac{\pi}{2} = \pi\,;$$

$$I_2 = \iint_\sigma y\ dxdz = \iint_{D_{xz}} \sqrt{4-4x^2-4z^2}\ dxdz = 2\iint_{D^*_{xz}} \sqrt{1-\rho^2}\ \rho\ d\rho d\varphi =$$

$$= -\int_0^{\pi/2} d\varphi \int_0^1 (-2\rho)\sqrt{1-\rho^2}\ d\rho = -\varphi\Big|_0^{\pi/2} \cdot \frac{\left(1-\rho^2\right)^{3/2}}{3/2}\Bigg|_0^1 = -\frac{\pi}{2} \cdot \frac{2}{3}(0-1) = \frac{\pi}{3}\ ;$$

$$I_3 = \iint_{\sigma} x^2 z\ dydz = \iint_{D_{yz}} \left(1 - \frac{y^2}{4} - z^2\right) dydz =$$

$$= \int_0^1 z\ dz \left(\left(1 - z^2\right) y - \frac{y^3}{12}\right)\Bigg|_0^{\sqrt{4-4z^2}} =$$

$$= \int_0^1 z\ dz \left(2\left(1 - z^2\right)\sqrt{1 - z^2} - \frac{1}{12}\left(2\sqrt{1 - z^2}\right)^3\right) =$$

$$= \int_0^1 z\ dz \left(2\left(1 - z^2\right)^{3/2} - \frac{8}{12}\left(1 - z^2\right)^{3/2}\right) =$$

$$= \frac{4}{3}\int_0^1 z\left(1 - z^2\right)^{3/2} dz = -\frac{2}{3}\int_0^1 (-2z)\left(1 - z^2\right)^{3/2} dz =$$

$$= -\frac{2}{3} \cdot \frac{\left(1 - z^2\right)^{5/2}}{5/2}\Bigg|_0^1 = -\frac{4}{15}.$$

So, integral $I = I_1 + I_2 - I_3 = \pi + \frac{\pi}{4} - \frac{4}{15} = \frac{4\pi}{3} - \frac{4}{15}$.

Task 4. Calculate $I = \iint_{\sigma} x\ dydz + y\ dxdz + z\ dxdy$, where σ is a positive side of a cube, bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

Three cube faces are the coordinate planes *Oxy*, *Oxz*, *Oyz* the equations of which are $z = 0$, $y = 0$, $x = 0$; $dz = 0$, $dy = 0$,

$dx = 0$, so the surface integral about these three cube faces is equal to zero. We calculate integral about three cube faces: $x = 1$ ($dx = 0$, $y = 0$, $z = 0$); $y = 1$ ($dy = 0$, $x = 0$, $z = 0$); $z = 1$ ($dz = 0$, $x = 0$, $y = 0$), then

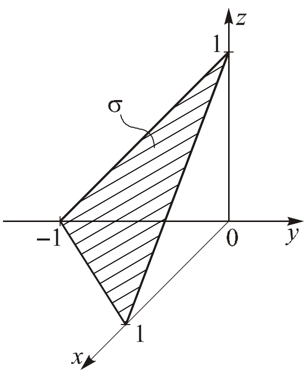
$$I_1 = \iint_{\sigma} x\ dydz = \iint_{D_{yz}} dydz = \int_0^1 dy \int_0^1 dz = 1, \text{ by analogy}$$

$$I_2 = \iint_{\sigma} y\ dxdz = \iint_{D_{xz}} dxdz = 1, \qquad I_3 = \iint_{\sigma} z\ dxdy = \iint_{D_{xy}} dxdy = 1.$$

Consequently $I = 1 + 1 + 1 = 3$.

Task 5. Calculate $I = \iint_{\sigma} z\ dxdy + x\ dxdz + y\ dydz$, where σ is an outer side of a triangle, built as the intersection of a plane $x - y + z = 1$ with the coordinate planes.

The integrating domain σ and its projections on the coordinate planes *Oxy*, *Oxz*, *Oyz* are depicted respectively on Drawing 43 and Drawing 44 a-c.



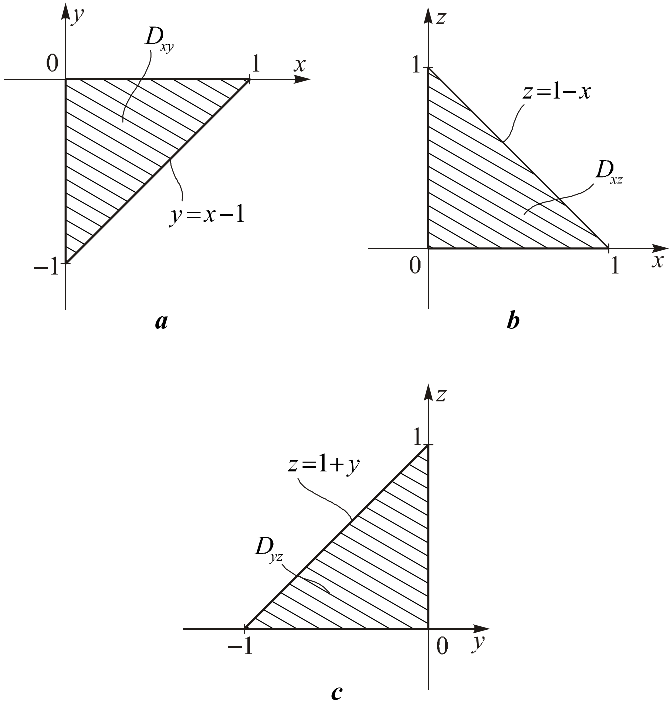
Drawing 43

We calculate a normal vector $\overline{n}_0$ to the surface σ:

$$\varphi(x, y, z) = x - y + z - 1; \quad \varphi'_x = 1,$$

$$\varphi'_y = -1, \quad \varphi'_z = 1;$$

$$\overline{n}_0 = \frac{1}{\sqrt{3}}\overline{i} - \frac{1}{\sqrt{3}}\overline{j} + \frac{1}{\sqrt{3}}\overline{k}.$$



Drawing 44

Since

$$\cos\alpha = \cos\left(\overline{n}\,\widehat{,}\,\overline{Ox}\right) = \overline{n}_0 \cdot \overline{i} = \frac{1}{\sqrt{3}} > 0,$$

$$\cos\beta = \cos\left(\overline{n}\,\widehat{,}\,\overline{Oy}\right) = \overline{n}_0 \cdot \overline{j} = -\frac{1}{\sqrt{3}} < 0,$$

$$\cos\gamma = \cos\left(\overline{n}\,\widehat{,}\,\overline{Oz}\right) = \overline{n}_0 \cdot \overline{k} = \frac{1}{\sqrt{3}} > 0,$$

then the signs of the first and the third integrals are «+», and of sign the second integral is «–», i.e. $I = I_1 - I_2 + I_3$, where

$$I_1 = \iint_{\sigma} z\ dxdy = \iint_{D_{xy}} (x-y-1)\ dxdy = \int_0^1 dx \int_{x-1}^{0} (x-1-y)\ dy =$$

$$= \int_0^1 dx \left((x-1)y - \frac{y^2}{2} \right)\Bigg|_{x-1}^{0} =$$

$$= -\frac{1}{2}\int_0^1 (x-1)^2\ dx = -\frac{1}{2}\cdot\frac{(x-1)^3}{3}\Bigg|_0^1 = -\frac{1}{2}\cdot\frac{1}{3} = -\frac{1}{6};$$

$$I_2 = \iint_{\sigma} x\ dxdz = \iint_{D_{xz}} x\ dxdz = \int_0^1 x\ dx \int_0^{1-x} dz = \int_0^1 x(1-x)\ dx =$$

$$= \int_0^1 \left(x - x^2\right) dx = \left(\frac{x^2}{2} - \frac{x^3}{3} \right)\Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6};$$

$$I_3 = \iint_{\sigma} y\ dydz = \iint_{D_{yz}} y\ dydz = \int_{-1}^{0} y\ dy \int_0^{1+y} dz = \int_{-1}^{0} y(1+y)\ dy =$$

$$= \int_{-1}^{0} \left(y + y^2\right) dy = \left(\frac{y^2}{2} + \frac{y^3}{3} \right)\Bigg|_{-1}^{0} = -\left(\frac{1}{2} - \frac{1}{3}\right) = -\frac{1}{6}.$$

So $I = -\frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{2}.$

Ostrogradsky-Gauss formula

Ostrogradsky-Gauss formula connects surface integral about a closed surface and triple integral about the spatial domain, bounded by this surface.

Let us assume that σ is a closed smooth surface that bounds volume G, and functions $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ are continuous along with their partial first-order derivatives in a closed domain G, then *Ostrogradsky-Gauss formula* is valid:

$$\iint_{\sigma} P\,dydz + Q\,dxdz + R\,dxdy = \iiint_{G} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz. \quad (13)$$

Calculate the following surface integrals according to Ostrogradsky-Gauss formula:

Task 1. $I = \iint_{\sigma} x^2 dydz + y^2 dxdz + z^2 dxdy$, where σ is an outer side of a cube surface $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Here we have $P = x^2$, $Q = y^2$, $R = z^2$, then $\frac{\partial P}{\partial x} = 2x$, $\frac{\partial Q}{\partial y} = 2y$, $\frac{\partial R}{\partial z} = 2z$.

We apply formula (13):

$$I = \iiint_{G} (2x + 2y + 2z)\,dxdydz = 2\int_0^a dx \int_0^a dy \int_0^a (x + y + z)\,dz =$$

$$= 2\int_0^a dx \int_0^a dy \left((x + y)z + \frac{z^2}{2} \right) \Bigg|_0^a =$$

$$= 2\int_0^a dx \int_0^a \left(a(x+y) + \frac{a^2}{2} \right) dy = 2\int_0^a dx \left(\left(ax + \frac{a^2}{2} \right) y + \frac{ay^2}{2} \right) \Bigg|_0^a =$$

$$= 2\int_0^a \left(a^2 x + \frac{a^3}{2} + \frac{a^3}{2} \right) dx = 2\left(\frac{a^2 x^2}{2} + a^3 x \right) \Bigg|_0^a = 2\left(\frac{a^4}{2} + a^4 \right) = 3a^4.$$

Task 2. $I = \iint\limits_{\sigma} x\,dydz + y\,dzdx + z\,dxdy$, σ is an outer side of the pyramid, bounded by the planes $x + y + z = a$, $x = 0$, $y = 0$, $z = 0$.

Since $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial R}{\partial z} = 1$, then according to the formula (13):

$$I = 3\iiint\limits_G dxdydz = 3\int_0^a dx \int_0^{a-x} dy \int_0^{a-x-y} dz = 3\int_0^a dx \int_0^{a-x} (a - x - y)\,dy =$$

$$= 3\int_0^a dx \left((a-x)y - \frac{y^2}{2} \right) \Bigg|_0^{a-x} =$$

$$= 3\int_0^a \left((a-x)^2 - \frac{(a-x)^2}{2} \right) dx = \frac{3}{2}\int_0^a (a-x)^2\,dx =$$

$$= -\frac{3}{2} \cdot \frac{(a-x)^3}{3} \Bigg|_0^a = -\frac{1}{2}\left(0 - a^3\right) = \frac{a^3}{2}.$$

Task 3. $I = \iint\limits_{\sigma} x^3 dydz + y^3 dzdx + z^3 dxdy$, where σ is an outer part of a sphere $x^2 + y^2 + z^2 = a^2$.

Since $\dfrac{\partial P}{\partial x} = 3x^2$, $\dfrac{\partial Q}{\partial y} = 3y^2$, $\dfrac{\partial R}{\partial z} = 3z^2$, then according to the formula (13):

$$I = 3\iiint\limits_{G} \left(x^2 + y^2 + z^2\right) dxdydz = 3\iiint\limits_{G^*} \rho^4 \sin\theta \, d\rho d\varphi d\theta =$$

$$= 3\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} \sin\theta \, d\theta \int\limits_0^{a} \rho^4 d\rho =$$

$$= 3 \cdot \varphi\Big|_0^{2\pi} \cdot \left(-\cos\theta\right)\Big|_0^{\pi} \cdot \frac{\rho^5}{5}\Big|_0^{a} = 3 \cdot 2\pi \cdot \left(\cos 0 - \cos\pi\right) \cdot \frac{a^5}{5} = \frac{12\pi a^5}{5}.$$

Task 4. $I = \iint\limits_{\sigma} x\, dydz + y\, dxdz + z\, dxdy$, where σ is an outer part of a surface of a cylinder $x^2 + y^2 = a^2$, $-h \le z \le h$.

According to the formula (13):

$$I = 3\iiint\limits_{G} dxdydz = 3\int\limits_0^{2\pi} d\varphi \int\limits_0^{a} \rho \, d\rho \int\limits_{-h}^{h} dz = 3 \cdot \varphi\Big|_0^{2\pi} \cdot \frac{\rho^2}{2}\Big|_0^{a} \cdot z\Big|_{-h}^{h} =$$

$$= 3 \cdot 2\pi \cdot \frac{a^2}{2} \cdot 2h = 6\pi a^2 h.$$

Stokes formula

Stokes formula connects surface integrals and curvilinear integrals. Let us assume that σ is a surface, represented by equation $z = z(x, y)$, functions $z(x, y)$, $z'_x(x, y)$, $z'_y(x, y)$, continuous in domain *D*, are projections of surface σ on the plane *Oxy*; *L* is a contour that bounds σ, *l* is a projection of contour *L* on the plane *Oxy*, i.e. *l* is a limit of domain *D*.

$$\oint_L P\,dx + Q\,dy + R\,dz =$$

$$= \iint_\sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dxdz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \,. \quad (14)$$

Formula (14) gives opportunity to calculate curvilinear integrals on closed contours with the help of surface integrals.

It follows from Stokes formula that under the condition of fulfillment of the following equalities

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

curvilinear integral about an arbitrary spatial closed contour *L* is equal to zero: $\oint_L P\,dx + Q\,dy + R\,dz = 0$, i.e. curvilinear integral does not depend on the shape of integration contour, and expression $Pdx + Qdy + Rdz$ is an ordinary differential of a certain function $u(x, y, z)$, that may be calculated with the formula:

$$u(x, y, z) = \int_{x_0}^{x} P(x, y_0, z_0)\,dx + \int_{y_0}^{y} Q(x, y, y_0)\,dy +$$

$$+\int_{z_0}^{z} R(x,y,z)dz\,. \qquad (15)$$

Calculate the following integrals according to Stokes formula:

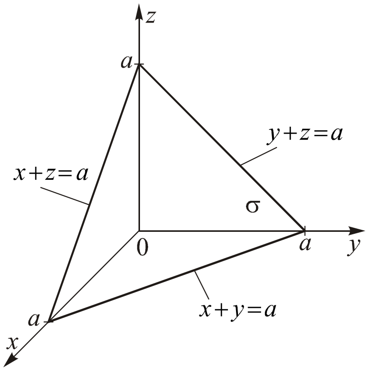
Task 1. $I=\oint\limits_{ABCA} y^2dx+z^2dy+x^2dz$, where $ABCA$ is a contour of ΔABC with the vertices $A(a;0;0)$, $B(0;a;0)$, $C(0;0;a)$.

Here we observe $P=y^2$, $Q=z^2$, $R=x^2$, then

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=0-2z=-2z,$$

$$\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=0-2x=-2x,$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0-2y=-2y.$$



Drawing 45

We apply formula (14):

$$I=\iint\limits_{\sigma}-2z\,dydz-2xdxdz-2ydxdy=$$

$$=-2\iint\limits_{D_{yz}} z\,dydz-2\iint\limits_{D_{xz}} x\,dxdz-2\iint\limits_{D_{xy}} y\,dxdy,$$

where D_{yz}, D_{xz}, D_{xy} are respectively the projections of surface σ on the coordinate planes Oyz, Oxz, Oxy (Drawing 45). Then

$$I=-2\int_0^2 dy\int_0^{a-y} z\,dz-2\int_0^a x\,dx\int_0^{a-x} dz-2\int_0^a dx\int_0^{a-x} y\,dy=$$

$$=-2\int_0^a dy\cdot\left.\frac{z^2}{2}\right|_a^{a-y}-2\int_0^a x(a-x)dx-$$

$$-2\int_0^a dx\cdot\left.\frac{y^2}{2}\right|_0^{a-x}=-\int_0^a (a-y)^2\,dy-2\int_0^a\left(ax-x^2\right)dx-\int_0^a (a-x)^2\,dx=$$

$$=\left.\frac{(a-y)^3}{3}\right|_0^a-2\left.\left(\frac{ax^2}{2}-\frac{x^3}{3}\right)\right|_0^a+$$

$$+\left.\frac{(a-x)^3}{3}\right|_0^a=\frac{1}{3}\left(0-a^3\right)-2\left(\frac{a^3}{2}-\frac{a^3}{3}\right)+\frac{1}{3}\left(0-a^3\right)=$$

$$=-\frac{2a^3}{3}-2\cdot\frac{a^3}{6}=-\frac{2}{3}a^3-\frac{a^3}{3}=-a^3\,.$$

Task 2. $I=\oint\limits_L x^2y^3dx+dy+z\,dz$, where L is a circle

$x^2+y^2=r^2,\ z=0$.

Since $P=x^2y^3$, $Q=1$, $R=z$, then

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=0,\qquad \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=0,\qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-3x^2y^2\,.$$

According to the formula (14):

$$I = -3\iint\limits_{\sigma} x^2 y^2 dxdy = -3\iint\limits_{D_{xy}} \rho^2 \cos^2\varphi \cdot \rho^2 \sin^2\varphi \cdot \rho\, d\rho d\varphi =$$

$$= -\frac{3}{4}\int_0^{2\pi} 4\sin^2\varphi \cos^2\varphi\, d\varphi \int_0^r \rho^5 d\rho =$$

$$= -\frac{3}{4}\int_0^{2\pi} \sin^2 2\varphi\, d\varphi \cdot \left.\frac{\rho^6}{6}\right|_0^r = -\frac{r^6}{8}\cdot\int_0^{2\pi}\frac{1-\cos 4\varphi}{2}d\varphi =$$

$$= -\frac{r^6}{16}\left.\left(\varphi - \frac{1}{4}\sin 4\varphi\right)\right|_0^{2\pi} = -\frac{r^6}{16}\cdot 2\pi = -\frac{\pi r^6}{8}.$$

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